HAESE MATHEMATICS

Mathematics

for the international student

Further Mathematics HL: Linear algebra and Geometry

FM Topic 1

FM Topic 2

Catherine Quinn

Robert Haese

Michael Haese

for use with

IB Diploma Programme

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Specialists in mathematics publishing

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MATHEMATICS FOR THE INTERNATIONAL STUDENT
Further Mathematics HL: Linear Algebra and Geometry

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FOREWORD

Further Mathematics HL: Linear Algebra and Geometry has been written to provide students and teachers with appropriate coverage of these two Further Mathematics HL Topics, to be first examined in 2014.

The Linear Algebra topic introduces students to matrices, vector spaces, and linear transformations. Useful preparation for this topic including the introduction to matrices is provided online with our MYP5 Extended book. The Principle of Mathematical Induction is useful for proofs in this topic, but is not essential in its preliminary study.

The Geometry topic aims to develop students' geometric intuition and deductive reasoning, particularly in plane Euclidean geometry. Most of this topic can be done using prior knowledge from the MYP5 Extended course. However, the final sections on conic sections require calculus from the HL Core course.

Detailed explanations and key facts are highlighted throughout the text. Each sub-topic contains numerous Worked Examples, highlighting each step necessary to reach the answer for that example.

Theory of Knowledge is a core requirement in the International Baccalaureate Diploma Programme, whereby students are encouraged to think critically and challenge the assumptions of knowledge. Discussion topics for Theory of Knowledge have been included on pages 124 and 129. These aim to help students discover and express their views on knowledge issues.

Graphics calculator instructions for Casio fx-9860G Plus, Casio fx-CG20, TI-84 Plus and TI-nspire are available from icons in the book.

Fully worked solutions are provided at the back of the text. However, students are encouraged to attempt each question before referring to the solution.

It is not our intention to define the course. Teachers are encouraged to use other resources. We have developed this book independently of the International Baccalaureate Organization (IBO) in consultation with experienced teachers of IB Mathematics. The text is not endorsed by the IBO.

In this changing world of mathematics education, we believe that the contextual approach shown in this book, with associated use of technology, will enhance the student's understanding, knowledge and appreciation of mathematics and its universal applications.

We welcome your feedback.
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CTQ PJB
RCH PMH

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- Geometry packages
- Demonstrations
- Printable pages
- Calculator instructions

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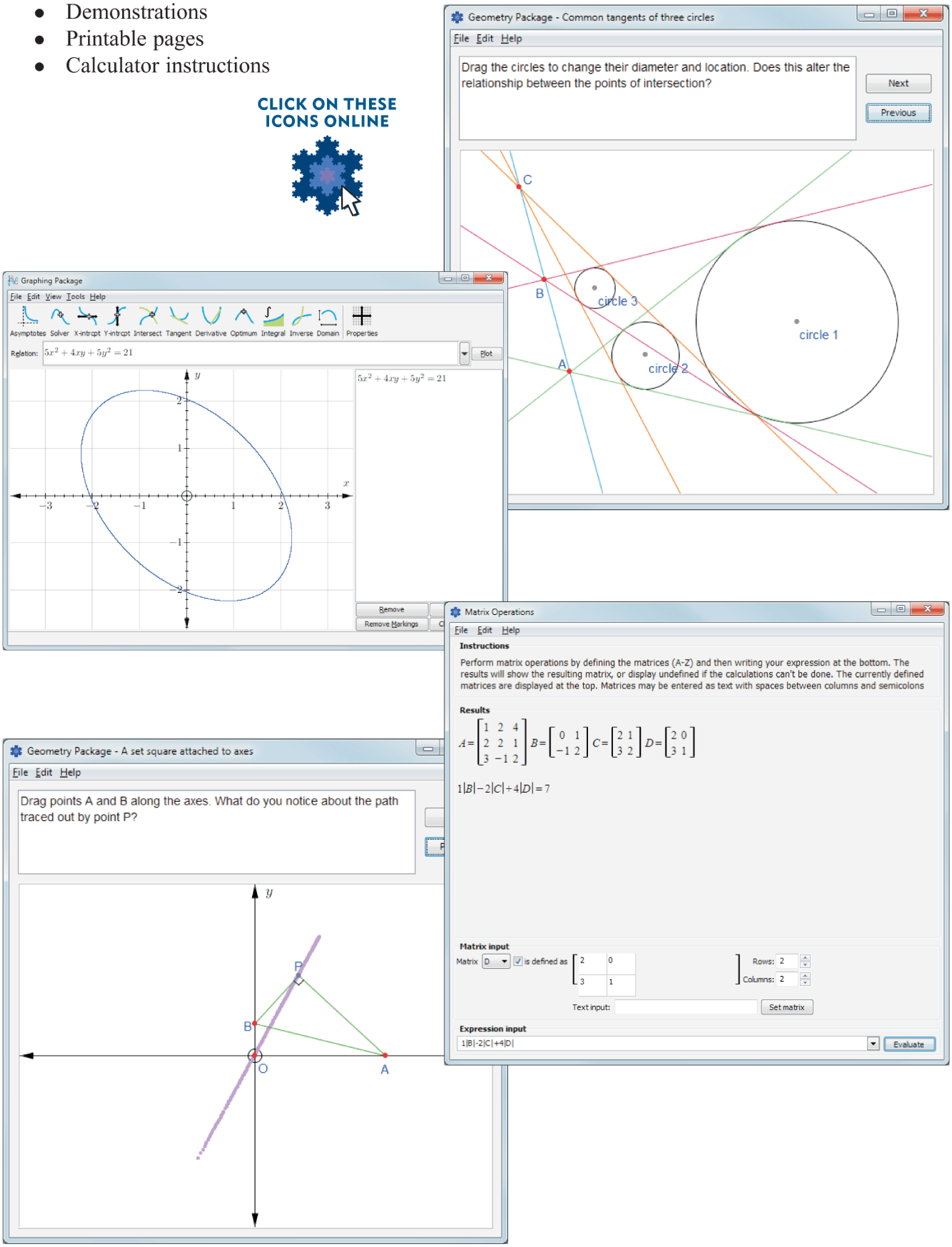

Geometry Package - Common tangents of three circles
File Edit Help
Next
Previous
C
B
A
circle 3
circle 2
circle 1
Graphing Package
File Edit View Tools Help
Asymptotes Solver X-intrcpt Y-intrcpt Intersect Tangent Derivative Optimum Integral Inverse Domain Properties
Relation:
Plot
Remove
Remove Markings
x
y
−3
−2
−1
1
2
3
Matrix Operations
File Edit Help
Geometry Package - A set square attached to axes
File Edit Help
P
B
O
A
x
y

Drag the circles to change their diameter and location. Does this alter the relationship between the points of intersection?

Relation: $5x^2 + 4xy + 5y^2 = 21$

$5x^2 + 4xy + 5y^2 = 21$

Instructions

Perform matrix operations by defining the matrices (A-Z) and then writing your expression at the bottom. The results will show the resulting matrix, or display undefined if the calculations can't be done. The currently defined matrices are displayed at the top. Matrices may be entered as text with spaces between columns and semicolons

Results

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 2 & 1\\ 3 & -1 & 2\end{bmatrix}\ B=\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}\ C=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\ D=\begin{bmatrix}2 & 0\\ 3 & 1\end{bmatrix}$$

$1|B|-2|C|+4|D|=7$

Matrix input

Matrix D ☑ is defined as $\begin{bmatrix}2 & 0\\ 3 & 1\end{bmatrix}$ Rows: 2 Columns: 2

Text input: Set matrix

Expression input

1|B|-2|C|+4|D| Evaluate

Drag points A and B along the axes. What do you notice about the path traced out by point P?

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SYMBOLS AND NOTATION USED IN THIS BOOK

$\approx$	is approximately equal to
$>$	is greater than
$\geqslant$	is greater than or equal to
$<$	is less than
$\leqslant$	is less than or equal to
$\{......\}$	the set of all elements ......
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$\in$	is an element of
$\notin$	is not an element of
$\mathbb{N}$	the set of all natural numbers $\{0, 1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers $\{0, \pm 1, \pm 2, \pm 3, ...\}$
$\mathbb{Z}^+$	the set of positive integers $\{1, 2, 3, ...\}$
$\mathbb{R}$	the set of real numbers
$\cup$	union
$\cap$	intersection
$\subset$	is a proper subset of
$\subseteq$	is a subset of
$\Rightarrow$	implies that
$\not\Rightarrow$	does not imply that
$\Leftrightarrow$	if and only if
$\dfrac{dy}{dx}$	the derivative of y with respect to x
$f(x)$	the image of x under the function f
$f'(x)$	the derivative of $f(x)$ with respect to x
$\sum\limits_{i=1}^{n} u_i$	$u_1 + u_2 + u_3 + ... + u_n$
sin, cos, tan	the circular functions
arcsin, arccos, arctan	the inverse circular functions
$\parallel$	is parallel to
$\perp$	is perpendicular to

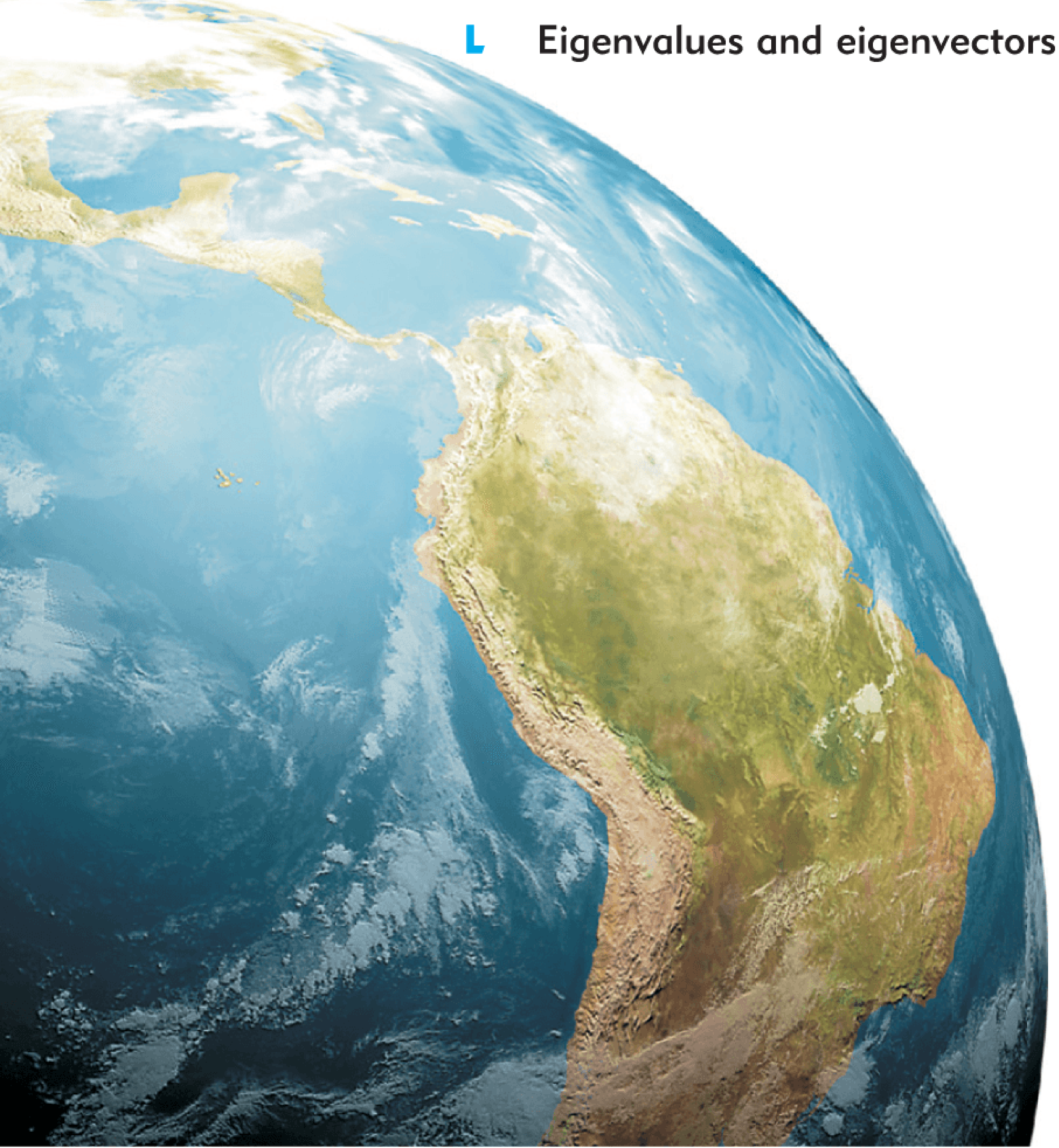
$\mathrm{A}(x, y)$	the point A in the plane with Cartesian coordinates x and y
[AB]	the line segment with end points A and B
AB	the length of [AB]
AX.AY	AX × AY
(AB)	the line containing points A and B
$\mathrm{C\hat{A}B}$	the angle between [CA] and [AB]
$\triangle\mathrm{ABC}$	the triangle whose vertices are A, B, and C
$\mathbf{v}$	the vector $\mathbf{v}$
$\mathbf{0}$	the zero vector
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the Cartesian coordinate axes
$\mathbf{v} \bullet \mathbf{w}$	the vector dot product or scalar product of $\mathbf{v}$ and $\mathbf{w}$
$\mathbf{v} \times \mathbf{w}$	the vector cross-product of $\mathbf{v}$ and $\mathbf{w}$
a_{ij}	the element in the ith row and jth column of matrix $\mathbf{A}$
$\mathbf{A}^{-1}$	the inverse of the non-singular matrix $\mathbf{A}$
$\mathbf{A}^{\mathrm{T}}$	the transpose of the matrix $\mathbf{A}$
$\det \mathbf{A}$, $\lvert \mathbf{A} \rvert$	the determinant of the square matrix $\mathbf{A}$
$\mathrm{tr}(\mathbf{A})$	the trace of the square matrix $\mathbf{A}$
$\mathbf{I}$	the identity matrix
$\mathbf{O}$	the zero matrix
$\sim$	which has the same solution as
$\ker(\mathrm{T})$	the kernel of the linear transformation T
$\mathcal{R}(\mathrm{T})$	the range of the linear transformation T

Linear Algebra

1

Contents:

A Systems of linear equations
B Gaussian elimination
C Matrix structure and operations
D Matrix multiplication
E Matrix transpose
F Matrix determinant and inverse
G Solving systems of linear equations using matrices
H Elementary matrices
I Vector spaces
J Linear transformations
K Geometric transformations
L Eigenvalues and eigenvectors



We are all familiar with linear equations of the form $y = mx + c$ or $ax + by + c = 0$ where x and y are variables and the other letters represent constants. These equations represent straight lines on the 2-dimensional Cartesian plane.

In this course we consider linear equations in higher dimensions, and how they can be represented using **matrices**. We will study the properties of matrices, their use in solving systems of linear equations, and the concept of **vector spaces**. These things are all part of the branch of mathematics called **linear algebra**.

Linear algebra is fundamental to all modern technology, being the basis for all computer science. Its applications extend to engineering, physics, natural sciences, analytic geometry, and economics.

HISTORICAL NOTE

CARL GAUSS 1777 - 1855

Carl Gauss

Carl Friedrich Gauss was born in 1777 in Brunswick, now a part of Germany. His parents were very poor. His mother could not read or write, so she did not record the date of Gauss' birth. However, Gauss himself worked out his birthday from his mother remembering that he was born on a Wednesday, 8 days before the Feast of the Ascension, a Christian celebration which occurs 40 days after Easter. At the same time, Gauss found a method for finding the date of Easter in both past and future years.

Gauss was a child prodigy. At the age of ten, Gauss' teacher set his class the task of adding up the numbers between one and a hundred. Expecting the students to take a long time, the teacher was surprised when Gauss derived a formula to solve the problem, and presented his correct solution in mere minutes. Gauss started university at the age of 15, and attended for three years. His tuition was paid for by the Duke of Brunswick, who had heard of Gauss' mathematical talents. His greatest work, *Disquisitiones Arithmeticae*, was completed when he was 21 years old. When it was published three years later, it was dedicated to the Duke.

Gauss made many mathematical discoveries during his doctoral studies. These include the Fundamental Theorem of Algebra (which he proved in four different ways) and the matrix method for solving systems of linear equations which we now call Gaussian elimination.

In the late 18th century, the dwarf planet Ceres was tracked by a number of Italian astronomers until it was 'lost' behind the sun. Gauss was able to predict a position for Ceres in December 1801, which turned out to be accurate within a half-degree when it was rediscovered by German astronomers. In 1807, Gauss was appointed Professor of Astronomy at Göttingen University, and Director of the astronomical observatory in Göttingen, a post he held for the rest of his life.

A SYSTEMS OF LINEAR EQUATIONS

$a_1x_1 + a_2x_2 + ... + a_nx_n = b$ is a **linear equation** in the variables $x_1, x_2, ..., x_n$ where $a_1, a_2, ..., a_n, b$ are constants.

For example, $2x = 3$, $4y = 7$, and $3x - y = 4$ are all linear equations.

$x_1 = c_1$, $x_2 = c_2$, ..., $x_n = c_n$ is a **solution** to the linear equation $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ if these values of the variables together satisfy the equation.

For example, consider the linear equation $x_1 + 2x_2 - 3x_3 = 8$.

- $x_1 = 3$, $x_2 = 4$, $x_3 = 1$ is a solution to the equation since $3 + 2(4) - 3(1) = 8$.
- $x_1 = -1$, $x_2 = 3$, $x_3 = -1$ is a solution to the equation since $-1 + 2(3) - 3(-1) = 8$.
- There are actually an infinite number of solutions of the equation. They have the form $x_1 = 8 - 2s + 3t$, $x_2 = s$, $x_3 = t$ where $s, t \in \mathbb{R}$.

The set of all solutions of a linear equation is called the **solution set**.

The solution set for a linear equation can often be written in many different forms.

For example, the solution set for $x_1 + 2x_2 - 3x_3 = 8$ can also be written as $x_1 = s$, $x_2 = t$, $x_3 = \dfrac{s + 2t - 8}{3}$ where $s, t \in \mathbb{R}$.

Example 1

Find the solution set of:

a $x - 3y = 1$

b $2x_1 - x_2 + 4x_3 = 11$

a Let $y = t$

$\therefore\ x - 3t = 1$

$\therefore\ x = 1 + 3t$

$\therefore$ the solution set is $x = 1 + 3t$, $y = t$, $t \in \mathbb{R}$.

b Let $x_1 = s$ and $x_3 = t$

$\therefore\ 2s - x_2 + 4t = 11$

$\therefore\ 2s + 4t - 11 = x_2$

$\therefore$ the solution set is $x_1 = s$, $x_2 = 2s + 4t - 11$, $x_3 = t$ where $s, t \in \mathbb{R}$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \ \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases}$$

is an $m \times n$ **system of linear equations** which consists of m equations in the n unknowns $x_1, x_2, ..., x_n$.

$x_1 = c_1$, $x_2 = c_2$, ..., $x_n = c_n$ is a **solution** of the system of linear equations if these values satisfy *all* m equations simultaneously.

For example, consider the system $\begin{cases} 2x_1 - x_2 + 4x_3 = 11 & \text{.... (1)} \\ x_1 + x_2 - 3x_3 = 2 & \text{.... (2)} \end{cases}$

- $x_1 = 3,\ x_2 = -1,\ x_3 = 1$ satisfies (1) since $2(3) - (-1) + 4(1) = 11$
  but does not satisfy (2) since $(3) + (-1) - 3(1) = -1 \neq 2$

  $\therefore$ $x_1 = 3,\ x_2 = -1,\ x_3 = 1$ is not a solution of the system.

- $x_1 = \frac{11}{3},\ x_2 = \frac{13}{3},\ x_3 = 2$ satisfies (1) since $2(\frac{11}{3}) - (\frac{13}{3}) + 4(2) = 11$
  and satisfies (2) since $(\frac{11}{3}) + (\frac{13}{3}) - 3(2) = 2$

  $\therefore$ $x_1 = \frac{11}{3},\ x_2 = \frac{13}{3},\ x_3 = 2$ is a solution to the system.

An $m \times n$ system of linear equations is:

- **inconsistent** if it has no solution
- **consistent** if it has at least one solution.

For example, consider the system $\begin{cases} x + 2y = 3 & \text{.... (1)} \\ 2x + 4y = 7 & \text{.... (2)} \end{cases}$

If we divide equation (2) by 2, we get $x + 2y = 3.5$. This is inconsistent with equation (1), so there are no solutions.

An $m \times n$ system of linear equations is **homogeneous** if $b_i = 0$ for all $i = 1, 2, ..., n$.

If a system is homogeneous then $x_1 = x_2 = ... = x_n = 0$ is always a solution. It is not necessarily the only solution.

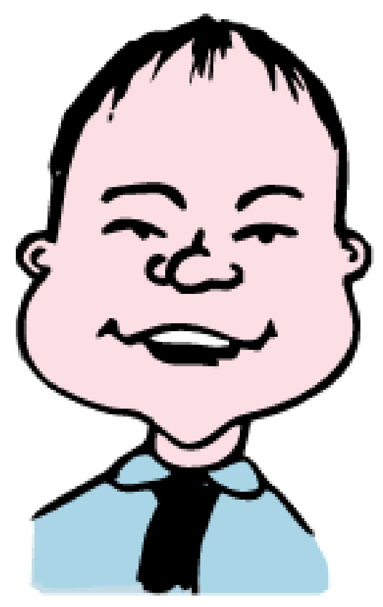
For example, $\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_2 + x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \end{cases}$ is a homogeneous system of linear equations.

An $m \times n$ system of linear equations is:

- **underspecified** if it has more unknowns than equations $(n > m)$
- **overspecified** if it has more equations than unknowns $(m > n)$.

For example, $\begin{cases} x + y + 2z = 2 \\ 2x + y - z = 4 \end{cases}$ is an underspecified system of linear equations, since there are 2 equations in 3 unknowns.

If an underspecified system of linear equations is consistent then it will have infinitely many solutions.



AUGMENTED MATRICES

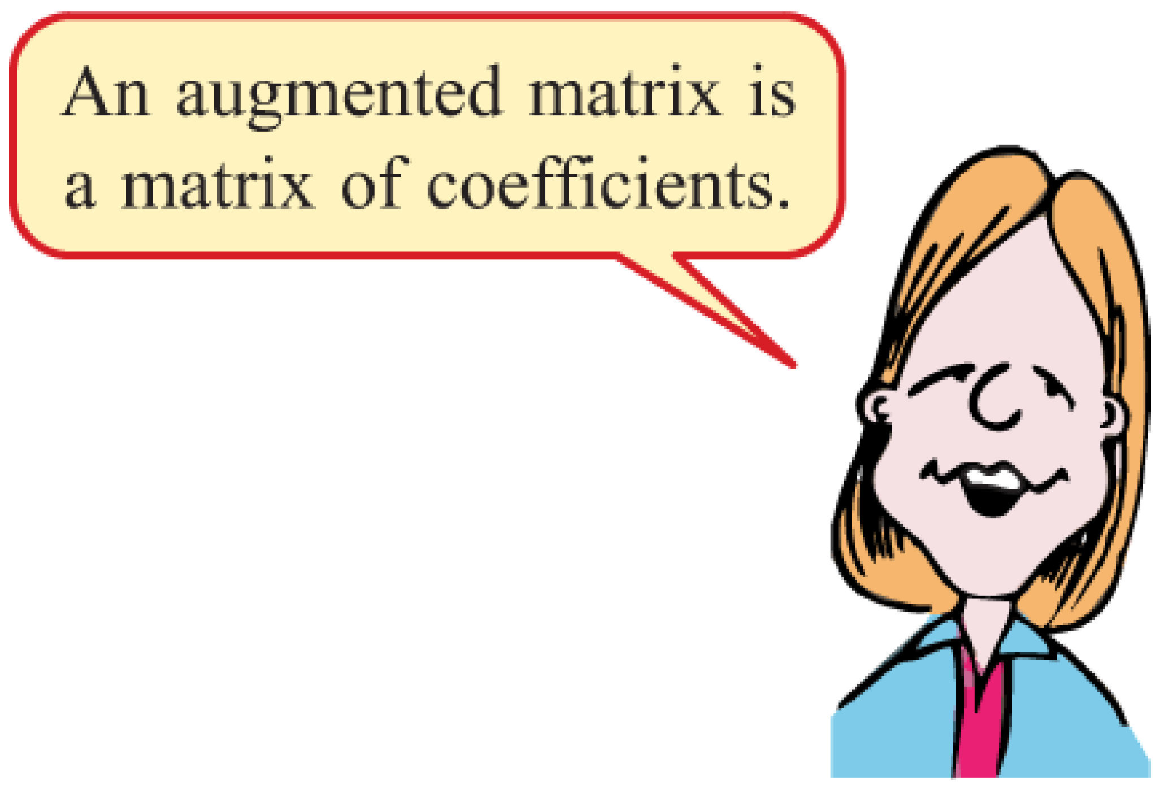
A system of $m \times n$ linear equations can be written as a rectangular array of m by $(n+1)$ numbers by leaving out the $+$ and $=$ signs, and the variables. We call this an **augmented matrix** (AM).

The general system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases} \quad \text{has AM} \quad \left(\begin{array}{cccc|c} a_{11} & a_{12} & .... & a_{1n} & b_1 \\ a_{21} & a_{22} & .... & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & .... & a_{mn} & b_m \end{array}\right).$$

For example, the system

$$\begin{cases} 2x_1 + 3x_2 \qquad\quad - x_4 = 5 \\ x_1 - x_2 + x_3 + 2x_4 = 10 \\ x_1 - 2x_2 + 3x_3 - x_4 = 1 \end{cases} \quad \text{has AM} \quad \left(\begin{array}{cccc|c} 2 & 3 & 0 & -1 & 5 \\ 1 & -1 & 1 & 2 & 10 \\ 1 & -2 & 3 & -1 & 1 \end{array}\right).$$


An augmented matrix is a matrix of coefficients.

EXERCISE 1A

1 Explain why each of the following is not a linear equation:

a $2x_1 + x_2 + x_3x_4 = 3$ **b** $x_1 - x_2 - 2x_3{}^2 = 0$ **c** $x_1 = 7 - \sqrt{x_2}$

2 Find the solution set for:

a $8x - y = 3$ **b** $x_1 - 2x_2 + x_3 = 10$ **c** $x_1 + x_2 - 2x_3 + x_4 = -2$

3 Write down the system of linear equations corresponding to the augmented matrix, and state if the system is underspecified or overspecified:

a $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 2 & 1 & -1 \end{array}\right)$ **b** $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 4 \\ 2 & 1 & 3 & -1 & 3 \end{array}\right)$ **c** $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 1 & 6 \\ 0 & 0 & 2 & 0 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$

4 Write down the augmented matrix for the system of equations:

a $\begin{cases} x_1 + x_2 - x_3 = 4 \\ x_1 - x_2 + x_3 = 8 \\ 2x_1 + x_2 - 3x_3 = 0 \end{cases}$ **b** $\begin{cases} x_1 + x_2 - 2x_3 = 7 \\ 3x_1 \qquad + x_3 = 2 \end{cases}$ **c** $\begin{cases} x_1 + x_2 - x_3 - x_4 = 5 \\ 3x_2 + x_3 + x_4 = 1 \\ 4x_3 - x_4 = 6 \end{cases}$

5 For what value(s) of $a \in \mathbb{R}$ does the system $\begin{cases} x + y = 4 \\ 3x + 3y = a \end{cases}$ have:

a no solutions **b** infinitely many solutions **c** exactly one solution?

6 For what value(s) of $k \in \mathbb{R}$ is the system $\begin{cases} x - 2y = k \\ 2x - 4y = 8 \end{cases}$ consistent?

7 Find the relationship between p, q, and r given that the system $\begin{cases} x+y+\ \ z=p \\ x \qquad\ +2z=q \\ 2x+y+3z=r \end{cases}$ is consistent.

8 **a** Explain why the system $\begin{cases} x+y=3 \\ 2x-y=6 \\ 3x+y=8 \end{cases}$ is inconsistent.

b Is this system underspecified or overspecified?

9 **a** Determine whether the system $\begin{cases} x_1+x_2-\ \ x_3-7=0 \\ x_1-x_2+2x_3-9=0 \end{cases}$ is homogeneous.

b Under what conditions is the system $\begin{cases} x_1+x_2-x_3 \qquad\ =a \\ 2x_1-x_2+x_3-8=b \end{cases}$ homogeneous?

B GAUSSIAN ELIMINATION

In previous years, we have solved 2×2 systems of linear equations by **elimination**.

Consider the system $\begin{cases} 2x + y = -1 \\ x - 3y = 17. \end{cases}$

From using the method of elimination, we know we can:

- interchange the equations, called **swapping**

  $$\begin{cases} 2x + y = -1 \\ x - 3y = 17 \end{cases} \quad \text{has the same solution as} \quad \begin{cases} x - 3y = 17 \\ 2x + y = -1 \end{cases}$$

- replace an equation by any non-zero multiple of itself, called **scaling**

  $$\begin{cases} 2x + y = -1 \\ x - 3y = 17 \end{cases} \quad \text{has the same solution as} \quad \begin{cases} -6x - 3y = 3 \\ \quad x - 3y = 17 \end{cases} \qquad \{\text{multiplying by } -3\}$$

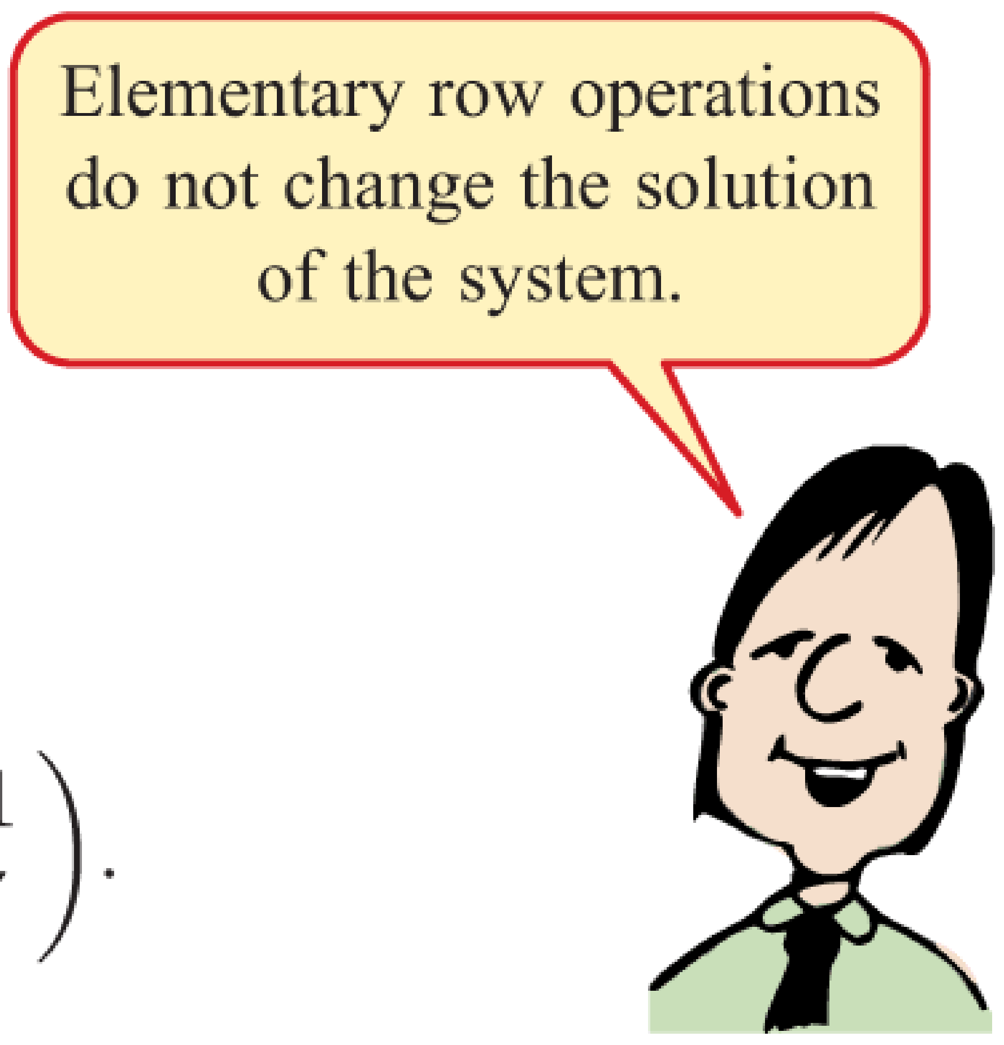
- replace an equation by a multiple of itself plus a multiple of another equation, called **pivoting**.
  If we replace the second equation by "twice the second equation, minus the first equation", we have

  $$\begin{array}{r} 2x - 6y = 34 \\ -(2x + y = -1) \\ \hline -7y = 35 \end{array}$$

  $$\therefore \quad \begin{cases} 2x + y = -1 \\ x - 3y = 17 \end{cases} \quad \text{has the same solution as} \quad \begin{cases} 2x + y = -1 \\ \quad -7y = 35\,. \end{cases}$$

The principles of swapping, scaling, and pivoting are applied to augmented matrices as **elementary row operations**. We can hence:

- interchange rows
- replace any row by a non-zero multiple of itself
- replace any row by itself plus a multiple of another row.


Elementary row operations do not change the solution of the system.

For example, the system $\begin{cases} 2x + y = -1 \\ x - 3y = 17 \end{cases}$ has AM $\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & -3 & 17 \end{array}\right)$.

- If we interchanged rows 1 and 2, we would write:

  $$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & -3 & 17 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & 17 \\ 2 & 1 & -1 \end{array}\right) \quad R_1 \leftrightarrow R_2$$

  means "which has the same solution as"

  indicates rows 1 and 2 have been interchanged

- If we multiplied row 1 by -3, we would write:

  $$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & -3 & 17 \end{array}\right) \sim \left(\begin{array}{cc|c} -6 & -3 & 3 \\ 1 & -3 & 17 \end{array}\right) \quad -3R_1 \to R_1$$

  indicates row 1 has been replaced by "$-3 \times$ row 1"

- If we replaced row 2 by "twice row 2 minus row 1", we would write:

$$\left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & -3 & 17 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 1 & -1 \\ 0 & -7 & 35 \end{array}\right) \quad \boxed{2R_2 - R_1 \to R_2}$$

indicates row 2 has been replaced by "twice row 2 minus row 1"

In the process of **row reduction**, we use elementary row operations to eliminate variables from selected rows of an augmented matrix. This allows us to systematically solve the corresponding system of linear equations.

SOLVING 2×2 SYSTEMS OF LINEAR EQUATIONS

To solve a 2×2 system of linear equations by row reduction, we aim to obtain a 0 in the bottom left corner of the augmented matrix. This is equivalent to eliminating x_1 from the corresponding equation.

Example 2

Use elementary row operations to solve: $\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17 \end{cases}$

In augmented matrix form, the system is $\left(\begin{array}{cc|c} 2 & 3 & 4 \\ 5 & 4 & 17 \end{array}\right)$

$$\sim \left(\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & -7 & 14 \end{array}\right) \quad 2R_2 - 5R_1 \to R_2$$

Using row 2, $-7y = 14$

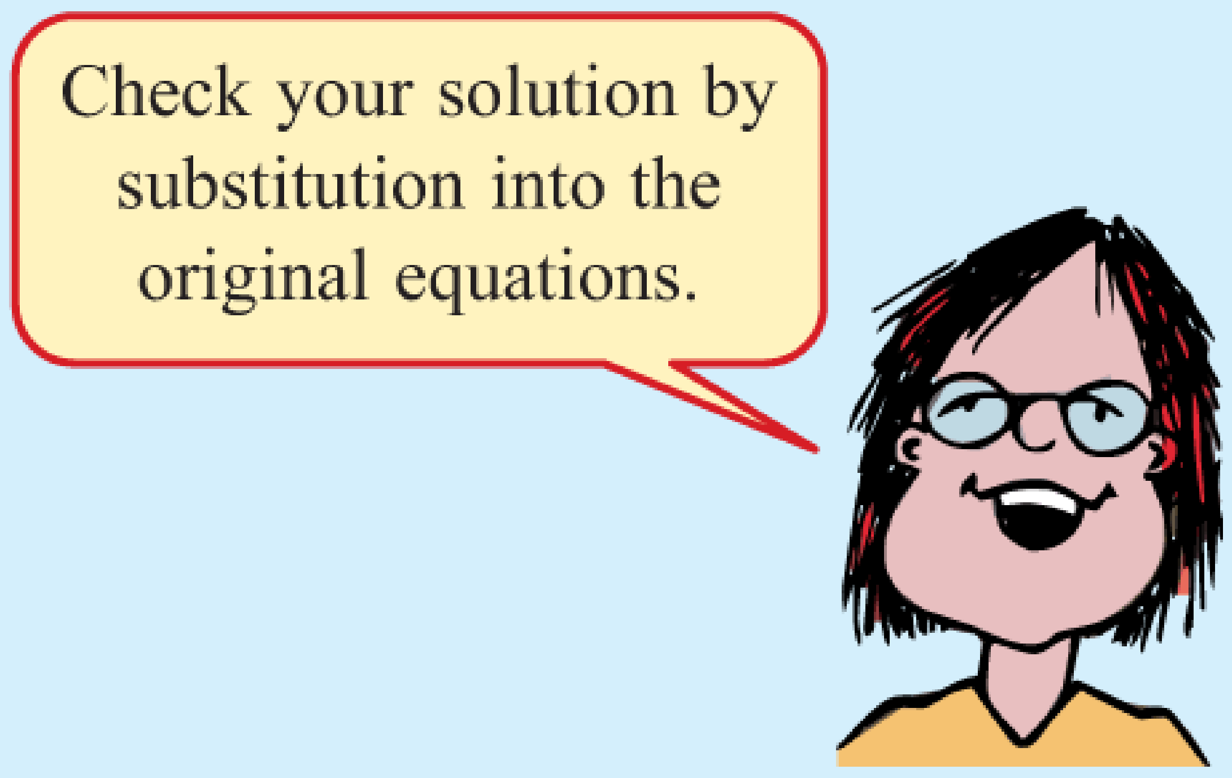
$\therefore \; y = -2$

Substituting into row 1, $2x + 3(-2) = 4$

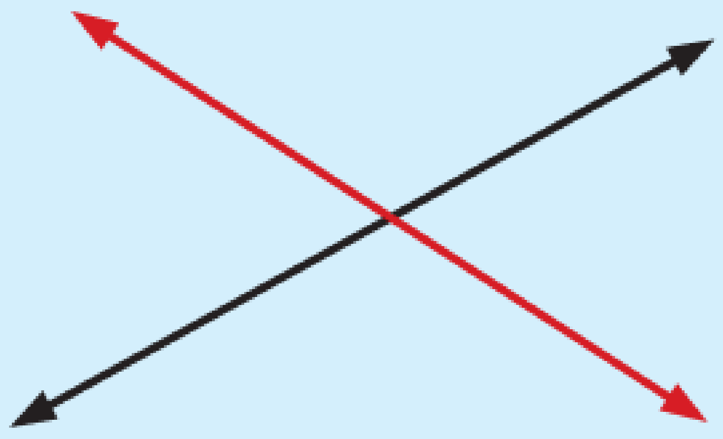
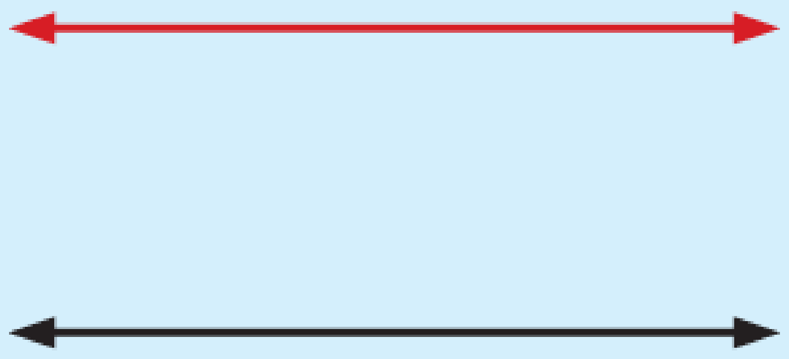
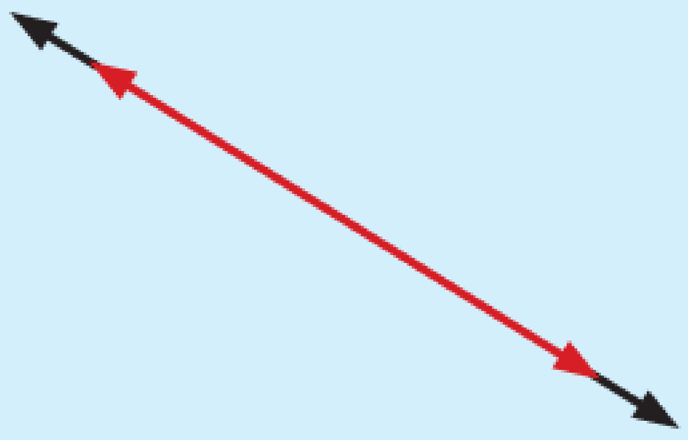
$\therefore \; 2x = 10$

$\therefore \; x = 5$

$\therefore$ the solution is $x = 5$, $y = -2$.


Check your solution by substitution into the original equations.

In previous courses, you should have seen that $ax + by = c$ where a, b, c are constants, is a line in the Cartesian plane. Given two such lines, there are three possible cases which may occur:

Intersecting lines	**Parallel lines**	**Coincident lines**
		
one point of intersection a unique simultaneous solution For example: $\begin{cases} 2x + 3y = 1 \\ x - 2y = 8 \end{cases}$	no points of intersection no simultaneous solutions For example: $\begin{cases} 2x + 3y = 1 \\ 2x + 3y = 7 \end{cases}$	infinitely many points of intersection infinitely many simultaneous solutions For example: $\begin{cases} 2x + 3y = 1 \\ 4x + 6y = 2 \end{cases}$

EXERCISE 1B.1

1 Consider the system of linear equations $\begin{cases} x - 3y = 2 \\ 2x + y = -3 \end{cases}$

- **a** Write the system as an augmented matrix.
- **b** Replace the second row with "the second row minus twice the first row".
- **c** Hence solve the system.

2 By inspection, decide whether the pair of lines is intersecting, parallel, or coincident, and state the number of solutions to the system.

- **a** $\begin{cases} x + 2y = 1 \\ 3x + 6y = 3 \end{cases}$
- **b** $\begin{cases} 2x - y = -1 \\ x + 4y = 13 \end{cases}$
- **c** $\begin{cases} x - 5y = 8 \\ 2x = 10y + 14 \end{cases}$
- **d** $\begin{cases} x + y = 4 \\ x + y = a, \quad a \in \mathbb{R} \end{cases}$

3 Use elementary row operations to solve:

- **a** $\begin{cases} x - 3y = -8 \\ 4x + 5y = 19 \end{cases}$
- **b** $\begin{cases} x + 7y = -17 \\ 2x - y = 11 \end{cases}$
- **c** $\begin{cases} 2x + 3y = -8 \\ x + 4y = -9 \end{cases}$
- **d** $\begin{cases} 3x - y = 9 \\ 4x + 3y = -1 \end{cases}$

4 Consider the system $\begin{cases} x + 3y = 4 \\ 2x + 6y = 8. \end{cases}$

- **a** Explain why there are infinitely many solutions.
- **b** Try to solve the system using elementary row operations. Explain what happens.
- **c** Let $y = t, \quad t \in \mathbb{R}$. Solve the system in terms of t.
- **d** Let $x = s, \quad s \in \mathbb{R}$. Solve the system in terms of s.
- **e** Explain why your solutions in **c** and **d** are equivalent.

5 Consider the system $\begin{cases} x - 5y = 8 \\ 2x - 10y = a \end{cases}$ where $a \in \mathbb{R}$.

- **a** Write the system as an augmented matrix, and perform an elementary row operation to make the bottom left corner element 0.
- **b** Explain what the second row means for the cases where $a \neq 16$. How many solutions does the system have in this case?
- **c** Find all solutions for the case where $a = 16$.

6 Discuss the solutions to $\begin{cases} x + 3y = 4 \\ 2x + ay = b \end{cases}$ for all $a, b \in \mathbb{R}$.

SOLVING 3×3 SYSTEMS OF LINEAR EQUATIONS

The general 3×3 systems of linear equations $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_3 \end{cases}$ can be written as the augmented matrix $\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & d_1 \\ a_{21} & a_{22} & a_{23} & d_2 \\ a_{31} & a_{32} & a_{33} & d_3 \end{array}\right)$.

We can use elementary row operations to reduce the matrix to the form $\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right)$ in which there is a triangle of zeros in the bottom left corner. We call this **row echelon form**.

From the row echelon form, we can see that:

- If $h \neq 0$, the third row means $hx_3 = i$. We can therefore solve for x_3, and hence for x_2 and x_1 using rows 2 and 1 respectively. The system has a **unique solution**.

Example 3

Solve using elementary row operations: $\begin{cases} x + 3y - z = 15 \\ 2x + y + z = 7 \\ x - y - 2z = 0 \end{cases}$

The system has AM $\left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -2 & 0 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 0 & -5 & 3 & -23 \\ 0 & -4 & -1 & -15 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 15 \\ 0 & -5 & 3 & -23 \\ 0 & 0 & -17 & 17 \end{array}\right) \quad 5R_3 - 4R_2 \to R_3$$

Using row 3, $-17z = 17$

$\therefore \ z = -1$

Substituting into row 2, $-5y + 3(-1) = -23$

$\therefore \ -5y = -20$

$\therefore \ y = 4$

Substituting into row 1, $x + 3(4) - (-1) = 15$

$\therefore \ x = 2$

$\therefore$ the solution is $x = 2$, $y = 4$, $z = -1$.

- If $h = 0$ and $i \neq 0$, the third row means $0x_1 + 0x_2 + 0x_3 = i$ where $i \neq 0$. In this case there is **no solution** and the system is **inconsistent**.

Example 4

Solve using elementary row operations: $\begin{cases} x + 2y + z = 3 \\ 2x - y + z = 8 \\ 3x - 4y + z = 18 \end{cases}$

The system has AM $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 1 & 8 \\ 3 & -4 & 1 & 18 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & -1 & 2 \\ 0 & -10 & -2 & 9 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & -1 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right) \quad R_3 - 2R_2 \to R_3$$

Row 3 means that $0x + 0y + 0z = 5$, which is absurd.

$\therefore$ there is no solution, and the system is inconsistent.

- If $h = 0$ and $i = 0$, the last row is all zeros. In this case the system has **infinitely many solutions**. We let $x_3 = t$ where $t \in \mathbb{R}$ and write x_1 and x_2 in terms of t. In this case the solution is a **parametric representation** with parameter t. We call x_1 and x_2 **basic variables** and x_3 a **free variable**.

Example 5

Solve using elementary row operations: $\begin{cases} 2x - y + z = 5 \\ x + y - z = 2 \\ 3x - 3y + 3z = 8 \end{cases}$

The system has AM $\left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & 1 & -1 & 2 \\ 3 & -3 & 3 & 8 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 3 & -3 & -1 \\ 0 & -3 & 3 & 1 \end{array}\right) \quad \begin{array}{l} 2R_2 - R_1 \to R_2 \\ 2R_3 - 3R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_2 + R_3 \to R_3$$

Row 3 indicates there are infinitely many solutions.

If we let $z = t$, then using row 2, $3y - 3t = -1$

$$\therefore\ 3y = 3t - 1$$

$$\therefore\ y = t - \tfrac{1}{3}$$

Substituting into row 1, $2x-(t-\frac{1}{3})+t=5$

$$\therefore\ 2x=\tfrac{14}{3}$$

$$\therefore\ x=\tfrac{7}{3}$$

$\therefore$ the solutions have the form $x=\frac{7}{3},\ y=t-\frac{1}{3},\ z=t,\ t\in\mathbb{R}$.

EXERCISE 1B.2

1 Solve each system of linear equations by row reduction to echelon form:

a $\begin{cases} x+4y+11z=7 \\ x+6y+17z=9 \\ x+4y+8z=4 \end{cases}$ **b** $\begin{cases} 2x-y+3z=17 \\ 2x-2y-5z=4 \\ 3x+2y+2z=10 \end{cases}$ **c** $\begin{cases} 2x+3y+4z=1 \\ 5x+6y+7z=2 \\ 8x+9y+10z=4 \end{cases}$

d $\begin{cases} x-2y+5z=1 \\ 2x-y+8z=2 \\ -3x\quad -11z=-3 \end{cases}$ **e** $\begin{cases} x+2y-z=4 \\ 3x+2y+z=7 \\ 5x+2y+3z=11 \end{cases}$ **f** $\begin{cases} 2x+4y+z=1 \\ 3x-5y-3z=19 \\ 5x+13y+7z=1 \end{cases}$

Example 6

Consider the system $\begin{cases} x-2y-z=-1 \\ 2x+y+3z=13 \\ x+8y+9z=a \end{cases}$ where $a\in\mathbb{R}$.

a Row reduce the system to echelon form.

b For what values of a does the system have no solutions?

c Under what conditions does the system have infinitely many solutions? Find the solutions in this case.

a The system has AM $\left(\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 2 & 1 & 3 & 13 \\ 1 & 8 & 9 & a \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 0 & 5 & 5 & 15 \\ 0 & 10 & 10 & a+1 \end{array}\right) \quad \begin{array}{l} R_2-2R_1\to R_2 \\ R_3-R_1\to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 0 & 5 & 5 & 15 \\ 0 & 0 & 0 & a-29 \end{array}\right) \quad R_3-2R_2\to R_3$$

b Using row 3, the system has no solutions if $a\neq 29$.

c The system has infinitely many solutions if the last row is all zeros. This occurs when $a=29$.
In this case we let $z=t$.

$\therefore$ using row 2, $5y+5t=15$

$$\therefore\ 5y=15-5t$$

$$\therefore\ y=3-t$$

Substituting into row 1, $x-2(3-t)-t=-1$

$$\therefore\ x-6+t=-1$$

$$\therefore\ x=5-t$$

$\therefore$ the solutions have the form $x=5-t,\ y=3-t,\ z=t,\ t\in\mathbb{R}$.

2 Consider the system $\left\{\begin{array}{r} x+2y+z=3 \\ 2x-y+4z=1 \\ x+7y-z=k \end{array}\right.$ where $k \in \mathbb{R}$.

- **a** Row reduce the system to echelon form.
- **b** For what values of k does the system have no solutions?
- **c** Under what condition does the system have infinitely many solutions? Find the solutions in this case.
- **d** Explain why the system never has a unique solution.

3 Consider the system $\left\{\begin{array}{r} x+2y-2z=5 \\ x-y+3z=-1 \\ x-7y+kz=-k \end{array}\right.$ where $k \in \mathbb{R}$.

- **a** Row reduce the system to echelon form.
- **b** Show that for one value of k, the system has infinitely many solutions. Find the solutions in this case.
- **c** Show that there is a unique solution for all other values of k. Find this solution in terms of k.

4 Consider the system $\left\{\begin{array}{r} x+3y+3z=a-1 \\ 2x-y+z=7 \\ 3x-5y+az=16 \end{array}\right.$ where $a \in \mathbb{R}$.

- **a** Row reduce the system to echelon form.
- **b** Show that for one value of a, the system has infinitely many solutions. Find the solutions in this case.
- **c** Show that there is a unique solution for all other values of a. Find the solution in terms of a.

REDUCED ROW ECHELON FORM

We have seen how elementary row operations can be used to reduce augmented matrices to **echelon form**.

We can use further row operations to convert the augmented matrix into a form from which the solution can be read by inspection.

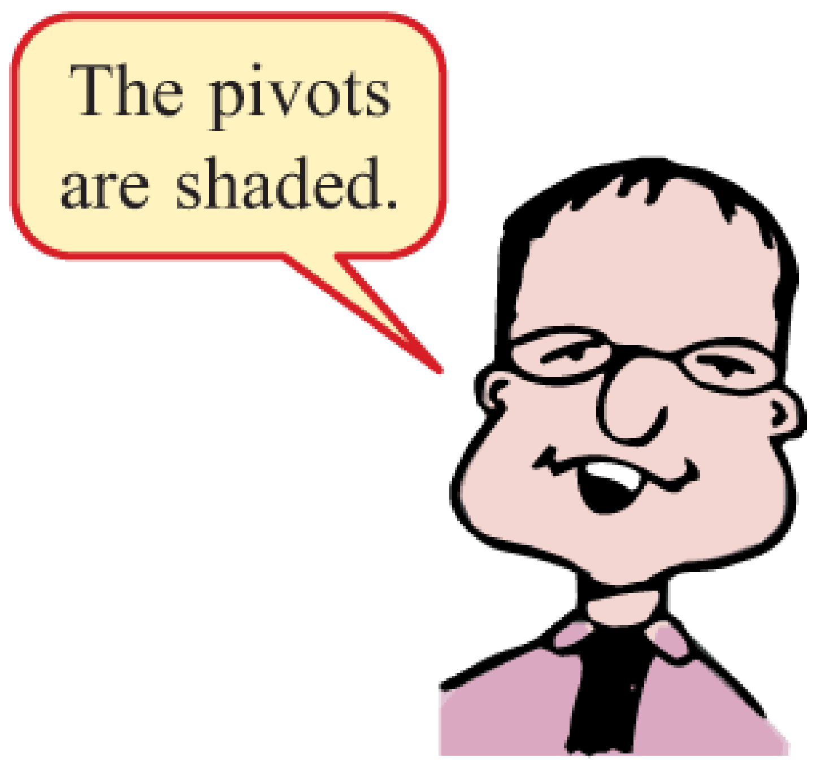
An augmented matrix is said to be in **reduced row echelon form** if:

- any row containing all zeros is placed at the bottom
- for every other row, the first or leading non-zero element is 1
- the rows which contain non-zero elements are ordered according to the positions of the leading 1s
- every column containing a leading 1, has zeros elsewhere.

For example:

- these matrices are in row echelon form:

$$\begin{pmatrix} \boxed{1} & 2 & 4 \\ 0 & \boxed{1} & 3 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & -1 & 4 & 5 \\ 0 & \boxed{1} & 0 & 6 \end{pmatrix}, \quad \begin{pmatrix} 0 & \boxed{1} & 2 & 6 & 3 \\ 0 & 0 & 0 & \boxed{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$


The pivots are shaded.

- these matrices are in reduced row echelon form:

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & -4 \\ 0 & 0 & \boxed{1} & 5 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 3 & 0 & 0 & 2 & -1 \\ 0 & 0 & \boxed{1} & 0 & 1 & 4 \\ 0 & 0 & 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The leading 1s in each row are the **pivots** for the row operations, and the variables corresponding to the columns which they are in are **basic variables**. The remaining variables are **free variables**, and must be allocated **parameters**.

The systematic procedure by which a system of linear equations is written as an augmented matrix in reduced row echelon form and hence solved, is called **Gaussian Elimination**.

We generally use a calculator for this task, since it can take a long time by hand.

Click on the icon to obtain instructions for your graphics calculator. You should be able to enter an augmented matrix, then reduce it to row echelon form or reduced row echelon form.

GRAPHICS CALCULATOR INSTRUCTIONS

Example 7

Solve the system of linear equations whose augmented matrix in reduced row echelon form is:

a $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \end{array}\right)$ **b** $\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

a By inspection, the system has the unique solution $x_1 = -3$, $x_2 = 4$, $x_3 = 7$.

b The basic variables are x_1 and x_3, and the free variables are x_2 and x_4.

Let $x_2 = r$ and $x_4 = s$.

Using row 2, $x_3 + s = 4$

$\therefore\ x_3 = 4 - s$

Using row 1, $x_1 + 3r + 2s = -1$

$\therefore\ x_1 = -1 - 3r - 2s$

So, the solutions have the form $x_1 = -1 - 3r - 2s$, $x_2 = r$, $x_3 = 4 - s$, $x_4 = s$, where $r, s \in \mathbb{R}$.

EXERCISE 1B.3

1 Which of the following augmented matrices are in reduced row echelon form?

a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

2 Solve the system whose augmented matrix in reduced row echelon form is:

a $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right)$ **b** $\left(\begin{array}{ccc|c} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ **c** $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right)$

3 Solve each system of linear equations using Gauss elimination on your calculator to write the system in reduced row echelon form.

a $$\left\{\begin{array}{l} 3x_1 + x_2 - x_3 = 12 \\ x_1 - x_2 + x_3 = -8 \\ 4x_1 - 2x_2 + x_3 = -8 \end{array}\right.$$

b $$\left\{\begin{array}{l} x_2 + 2x_4 = 4 \\ x_1 + x_2 + 4x_4 = 9 \\ x_2 - x_3 + x_4 = -2 \end{array}\right.$$

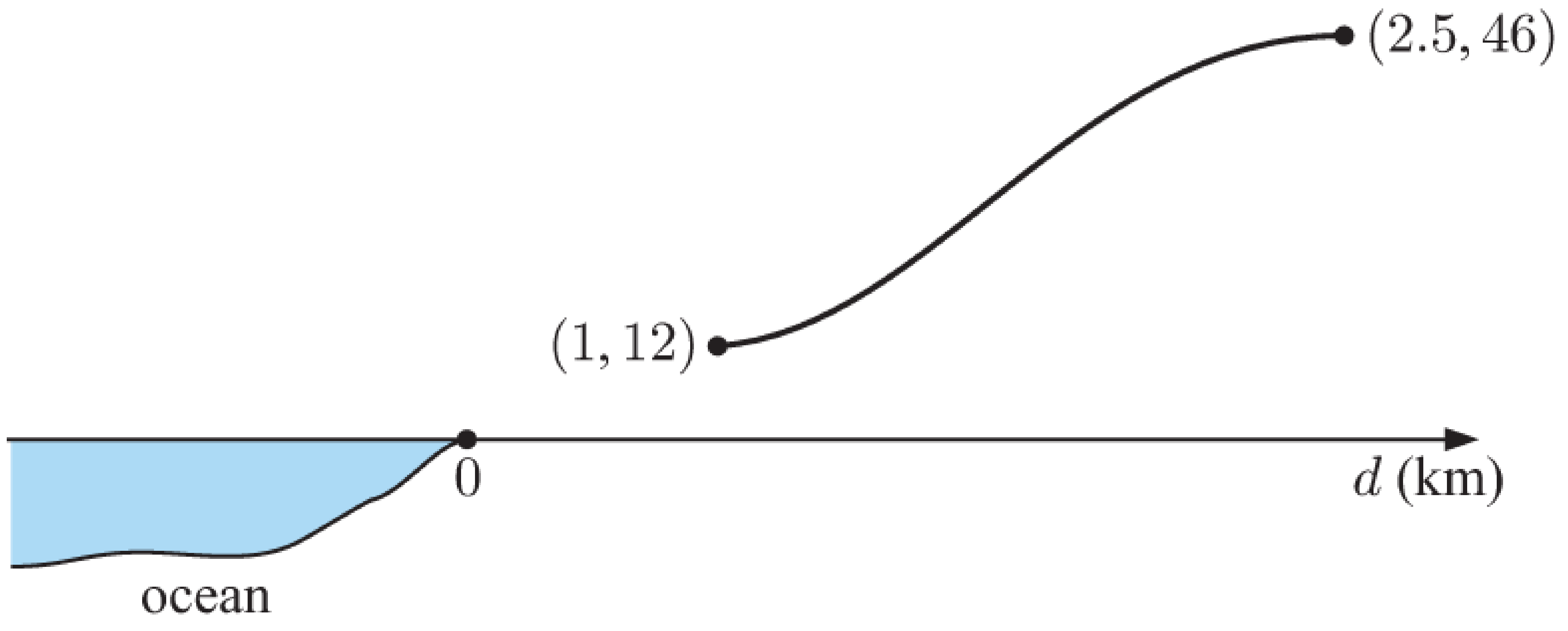
c $$\left\{\begin{array}{l} x_1 + 2x_2 + 3x_3 = 4 \\ x_1 - x_2 + 4x_3 = 7 \\ 3x_1 + 3x_2 + 10x_3 = 15 \\ 6x_1 + 9x_2 + 19x_3 = 9 \end{array}\right.$$

d $$\left\{\begin{array}{l} x_1 + x_2 - x_3 - 4x_4 = 1 \\ x_1 + 7x_2 + 3x_3 + 2x_4 = 2 \\ x_1 + 13x_2 + 7x_3 + 8x_4 = 3 \end{array}\right.$$

e $$\left\{\begin{array}{l} x_1 + x_2 + x_3 - 2x_4 + 3x_5 = 1 \\ 3x_1 - 3x_2 + 2x_3 - 4x_4 - 9x_5 = 3 \\ 2x_1 + 2x_2 - x_3 + 2x_4 + 6x_5 = 2 \end{array}\right.$$

f $$\left\{\begin{array}{l} x_1 + x_2 + x_3 + 2x_4 + x_5 = 2 \\ x_1 - x_2 + x_3 - x_4 + x_5 = 3 \\ 3x_1 + x_2 + 3x_3 + 3x_4 + 3x_5 = 7 \\ 2x_1 + 2x_3 + x_4 + 2x_5 = 5 \end{array}\right.$$

4 A cubic function $h(d) = x_1d^3 + x_2d^2 + x_3d + x_4$, $1 \leqslant d \leqslant 2.5$, is used to model the height h of a hill in metres above sea level, at a distance d km from the ocean.
At the point $(1, 12)$, the gradient of the hill is 0.1.
The point $(2.5, 46)$ is the top of the hill, at which the gradient is zero.



a Use the points $(1, 12)$ and $(2.5, 46)$ to write two equations in the unknowns x_1, x_2, x_3, x_4.

b The gradient of the hill is modelled by the function $h'(d) = 3x_1d^2 + 2x_2d + x_3$.

  i If you have already studied calculus, explain why this is so.

  ii Use the gradients of the hill at the given points to write two more linear equations.

c Solve the system of linear equations to find x_1, x_2, x_3, x_4.

d Hence estimate the height of the hill at the point 2 km from the ocean.

HOMOGENEOUS EQUATIONS

We have seen that a **homogeneous** system of linear equations has all constant terms zero.
It has the form

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0. \end{array}\right.$$

All homogeneous systems have the **trivial solution** $x_1 = x_2 = ... = x_n = 0$.

If a homogeneous system of linear equations is under-specified (so it has more unknowns than equations) then it has infinitely many solutions.

Example 8

Solve the homogeneous system:

a $\begin{cases} x_1 + x_2 + x_3 + 3x_4 = 0 \\ x_1 - x_2 + x_3 - 5x_4 = 0 \end{cases}$

b $\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ 2x_1 + 3x_2 - x_3 = 0 \\ 3x_1 - x_2 - 2x_3 = 0 \end{cases}$

a The system has AM $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 1 & -1 & 1 & -5 & 0 \end{array}\right)$

$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 4 & 0 \end{array}\right)$ {using technology}

Let $x_3 = s$ and $x_4 = t$.

$\therefore$ $x_2 + 4t = 0$ and $x_1 + s - t = 0$

$\therefore$ $x_1 = -s + t,\ x_2 = -4t,\ x_3 = s,\ x_4 = t,\ s, t \in \mathbb{R}$

b The system has AM $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & -1 & -2 & 0 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ {using technology}

$\therefore$ the only solution is the trivial solution $x_1 = x_2 = x_3 = 0$.

EXERCISE 1B.4

1 Briefly explain why the following systems of homogeneous equations have non-trivial solutions:

a $\begin{cases} x + 2y = 0 \\ 2x + 4y = 0 \end{cases}$ **b** $\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 + 3x_2 + 5x_3 = 0 \end{cases}$

2 Solve the following homogeneous systems of linear equations:

a $\begin{cases} x + 3y - z = 0 \\ 2x - y + 5z = 0 \end{cases}$ **b** $\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{cases}$ **c** $\begin{cases} x_1 - x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 - x_3 - 2x_4 = 0 \\ 3x_1 - x_2 + 2x_3 + x_4 = 0 \end{cases}$

3 The system $\begin{cases} (p-2)x + y = 0 \\ x + (p-2)y = 0 \end{cases}$ has a non-trivial solution. Find p.

4 The system of equations $\begin{cases} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{cases}$ has one solution $x = x_1,\ y = y_1$.

a Show that $x = cx_1,\ y = cy_1$ is a solution for all $c \in \mathbb{R}$.

b If $x = x_2,\ y = y_2$ is also a solution, show that $x = x_1 + x_2,\ y = y_1 + y_2$ is also a solution.

C MATRIX STRUCTURE AND OPERATIONS

A **matrix** is a rectangular array of numbers arranged in rows and columns.

The numbers are called the **elements** of the matrix.

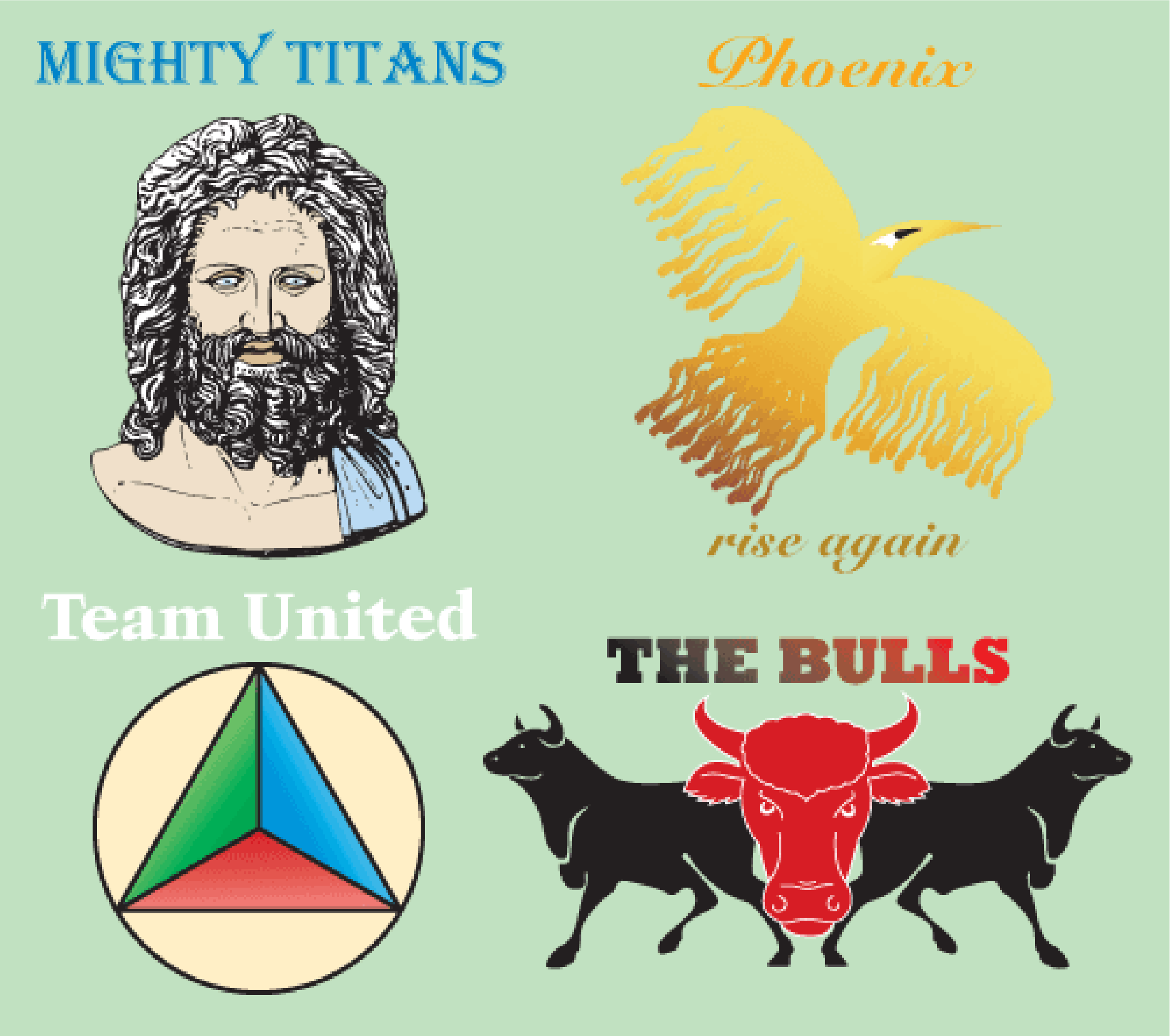
We can convert any table of numbers into a matrix.

	Won	*Lost*	*Drawn*	*Points*
Titans	10	2	3	23
Phoenix	7	6	2	16
United	3	7	5	11
Bulls	3	8	4	10

For example, the premiership table opposite can be written as the matrix

$$\begin{array}{cc} & \begin{array}{cccc} \mathrm{W} & \mathrm{L} & \mathrm{D} & \mathrm{P} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{P} \\ \mathrm{U} \\ \mathrm{B} \end{array} & \begin{pmatrix} 10 & 2 & 3 & 23 \\ 7 & 6 & 2 & 16 \\ 3 & 7 & 5 & 11 \\ 3 & 8 & 4 & 10 \end{pmatrix} \end{array}$$

or simply $\begin{pmatrix} 10 & 2 & 3 & 23 \\ 7 & 6 & 2 & 16 \\ 3 & 7 & 5 & 11 \\ 3 & 8 & 4 & 10 \end{pmatrix}$.



Notice that

- every number within a matrix has a particular meaning
- the organisation of the data is maintained in matrix form.

MATRIX ORDER

Consider the following examples:

$\begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix}$ has 4 rows and 1 column, and we say that this is a 4×1 **column matrix** or **column vector**.

column 2

row 3 $\begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & \textcircled{3} & 4 \end{pmatrix}$

this element is in row 3, column 2

has 3 rows and 3 columns, and is called a 3×3 **square matrix**.

$\begin{pmatrix} 3 & 0 & -1 & 2 \end{pmatrix}$ has 1 row and 4 columns, and is called a 1×4 **row matrix** or **row vector**.

An $m \times n$ matrix has m rows and n columns.

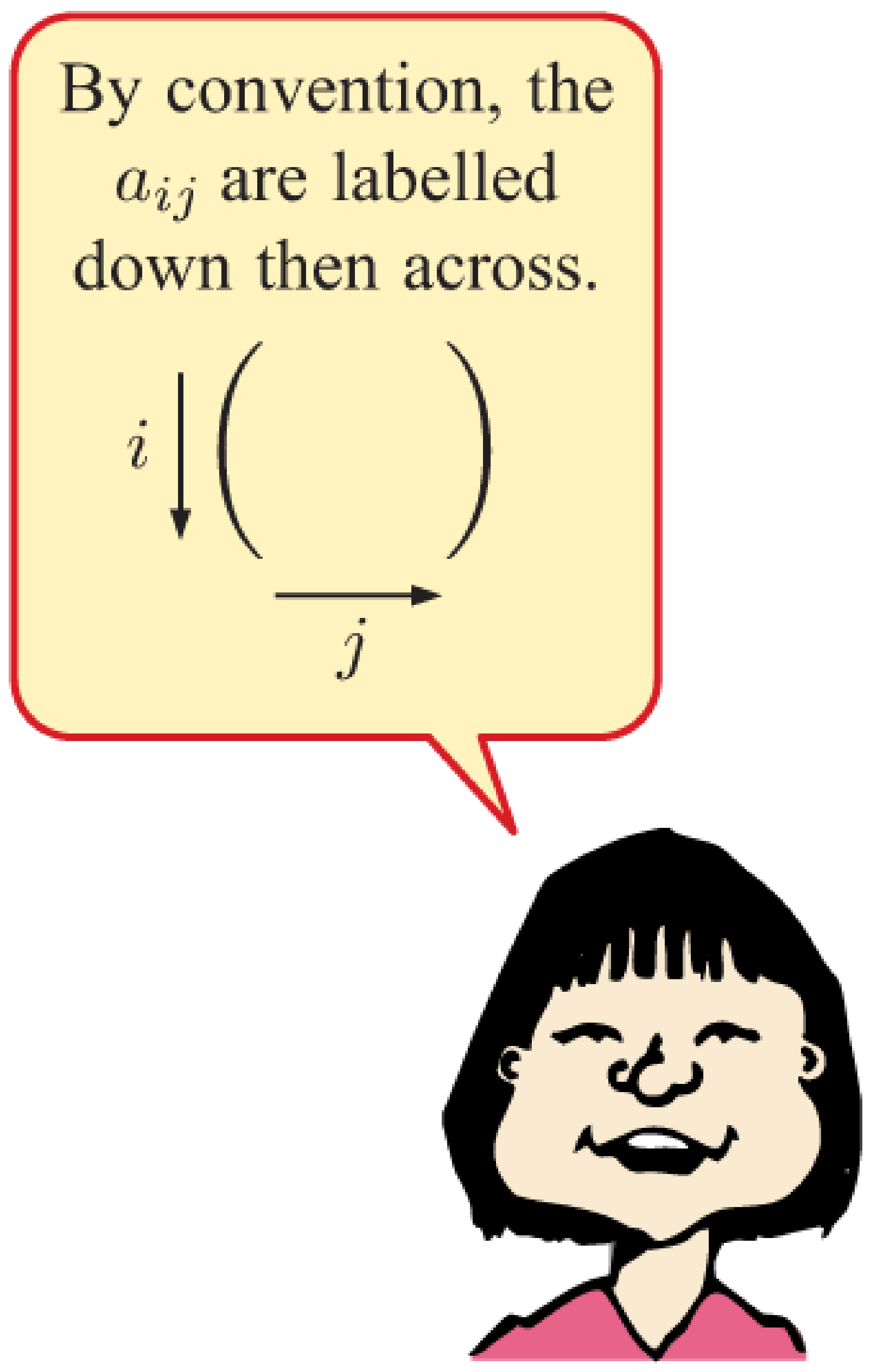
$m \times n$ specifies the **order** of a matrix.

MATRIX TERMINOLOGY

Consider a matrix **A** with order $m \times n$. We can write

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}.$$

$\mathbf{A} = (a_{ij})$ where $i = 1, 2, 3, \dots, m$
$j = 1, 2, 3, \dots, n$
and a_{ij} is the element in the ith row, jth column.



By convention, the a_{ij} are labelled down then across.

For example, a_{23} is the number in row 2 and column 3 of matrix **A**.

The ith row of **A** is $\begin{pmatrix} a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \end{pmatrix}$ and

the jth column of **A** is $\begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{pmatrix}$.

The elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ form the **main diagonal** of an $n \times n$ matrix.

Some special names used to describe matrices are:

Example

- A **zero matrix O** has all elements 0 and can be of any size. $a_{ij} = 0$ for all i, j.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- A **square matrix** has the same number of rows as columns.

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

- An **identity matrix I** is a square matrix with all main diagonal elements 1, and zeros everywhere else.
  $a_{ii} = 1$ and $a_{ij} = 0$ for all $i \neq j$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- A **diagonal matrix** is a square matrix where $a_{ij} = 0$ for all $i \neq j$, and at least one diagonal element is non-zero.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

- An **upper triangular matrix** is a square matrix in which $a_{ij} = 0$ for $i > j$.

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

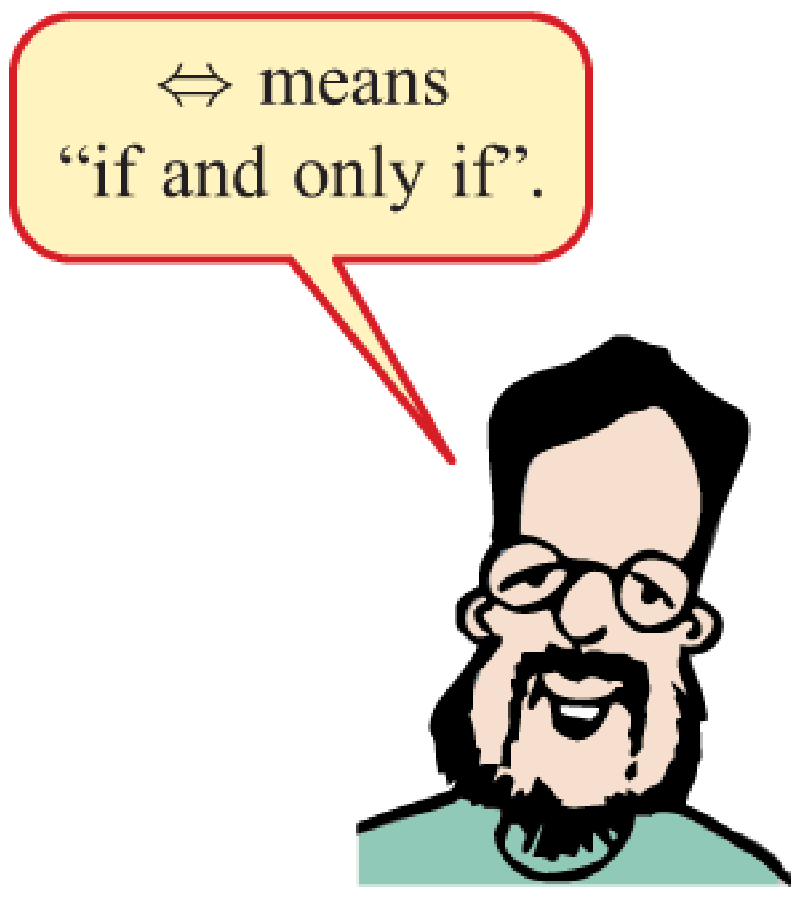
- A **lower triangular matrix** is a square matrix in which $a_{ij} = 0$ for $i < j$.

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 3 & -1 & 1 \end{pmatrix}$$

EQUALITY OF MATRICES

Two matrices are **equal** if they have the **same order** *and* the elements in corresponding positions are equal.

$$\mathbf{A} = \mathbf{B} \quad \Leftrightarrow \quad a_{ij} = b_{ij} \quad \text{for all } i,\ j.$$



For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \ \Leftrightarrow\ a = w,\ \ b = x,\ \ c = y,\ \text{ and }\ d = z.$$

MATRIX ADDITION

Carla has three boxes of sports equipment: A, B, and C. The boxes contain bats, balls, and cones according to the matrix shown.

$$\begin{array}{c c}
 & \text{Box} \\
 & \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\
 & \begin{pmatrix} 12 & 15 & 11 \\ 32 & 25 & 21 \\ 26 & 28 & 20 \end{pmatrix}
\end{array}
\begin{array}{l} \\ \\ \text{bats} \\ \text{balls} \\ \text{cones} \end{array}$$

Carla has ordered more equipment for the boxes. 10 bats, 20 balls, and 15 cones will be added to each. The new equipment is given by the matrix shown.

$$\begin{pmatrix} 10 & 10 & 10 \\ 20 & 20 & 20 \\ 15 & 15 & 15 \end{pmatrix}$$

When the new equipment is added to the boxes, we have the matrix addition:

$$\begin{pmatrix} 12 & 15 & 11 \\ 32 & 25 & 21 \\ 26 & 28 & 20 \end{pmatrix} + \begin{pmatrix} 10 & 10 & 10 \\ 20 & 20 & 20 \\ 15 & 15 & 15 \end{pmatrix} = \begin{pmatrix} 22 & 25 & 21 \\ 52 & 45 & 41 \\ 41 & 43 & 35 \end{pmatrix}$$

To **add** two matrices, they must be of the **same order**, and we **add corresponding elements**.

$$\mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

MULTIPLES OF MATRICES

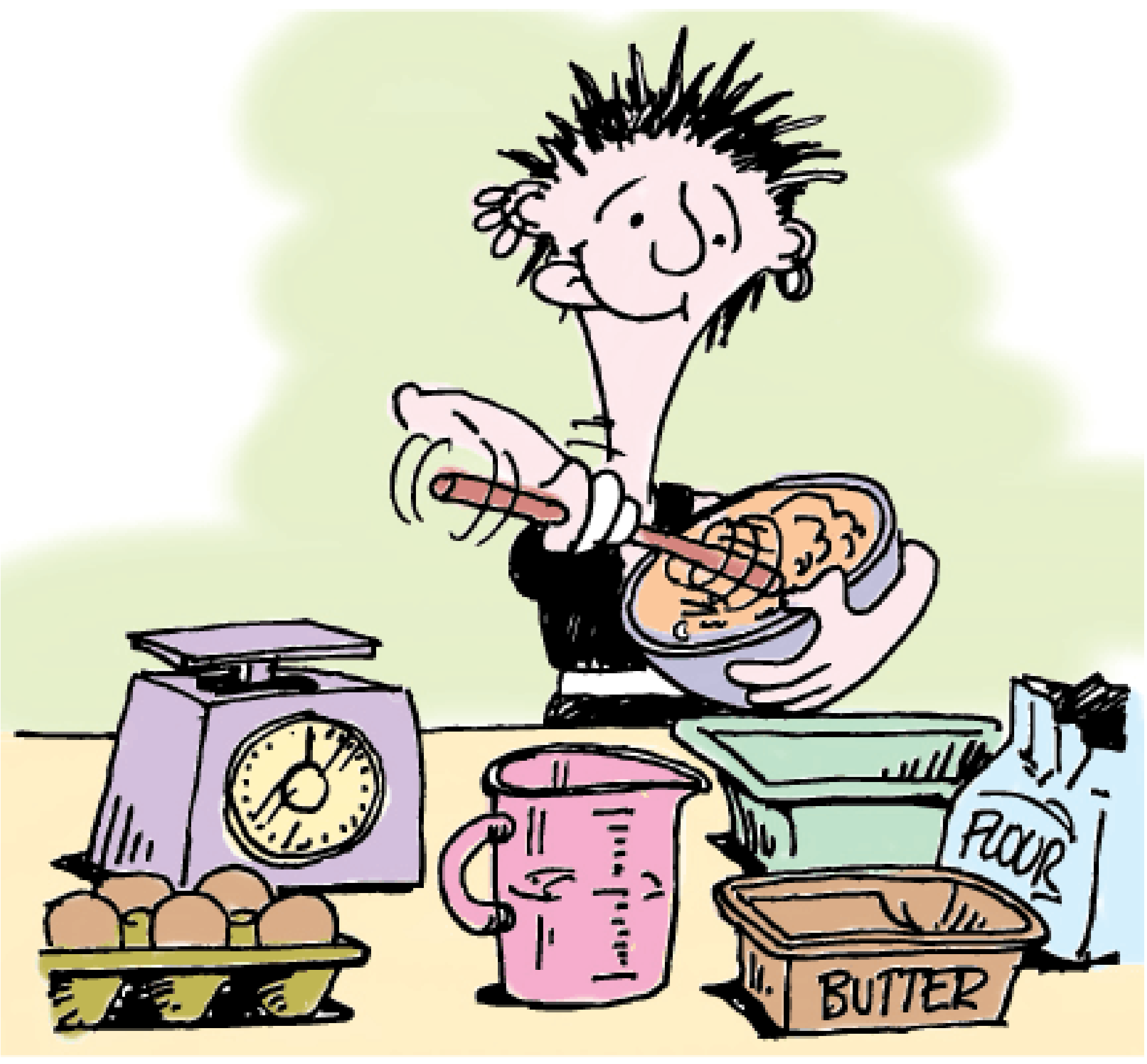
A cake recipe requires 3 cups of flour, 2 cups of sugar, and 6 eggs.

We can represent these ingredients using the matrix $\mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$.

If we make two cakes using this recipe, we will need 6 cups of flour, 4 cups of sugar, and 12 eggs.

We can represent this using the matrix $2\mathbf{C} = \mathbf{C} + \mathbf{C} = \begin{pmatrix} 6 \\ 4 \\ 12 \end{pmatrix}$.

Notice that to get $2\mathbf{C}$ from $\mathbf{C}$, we multiply each element of $\mathbf{C}$ by 2.



Similarly, to make three cakes using the recipe, the ingredients needed are given by

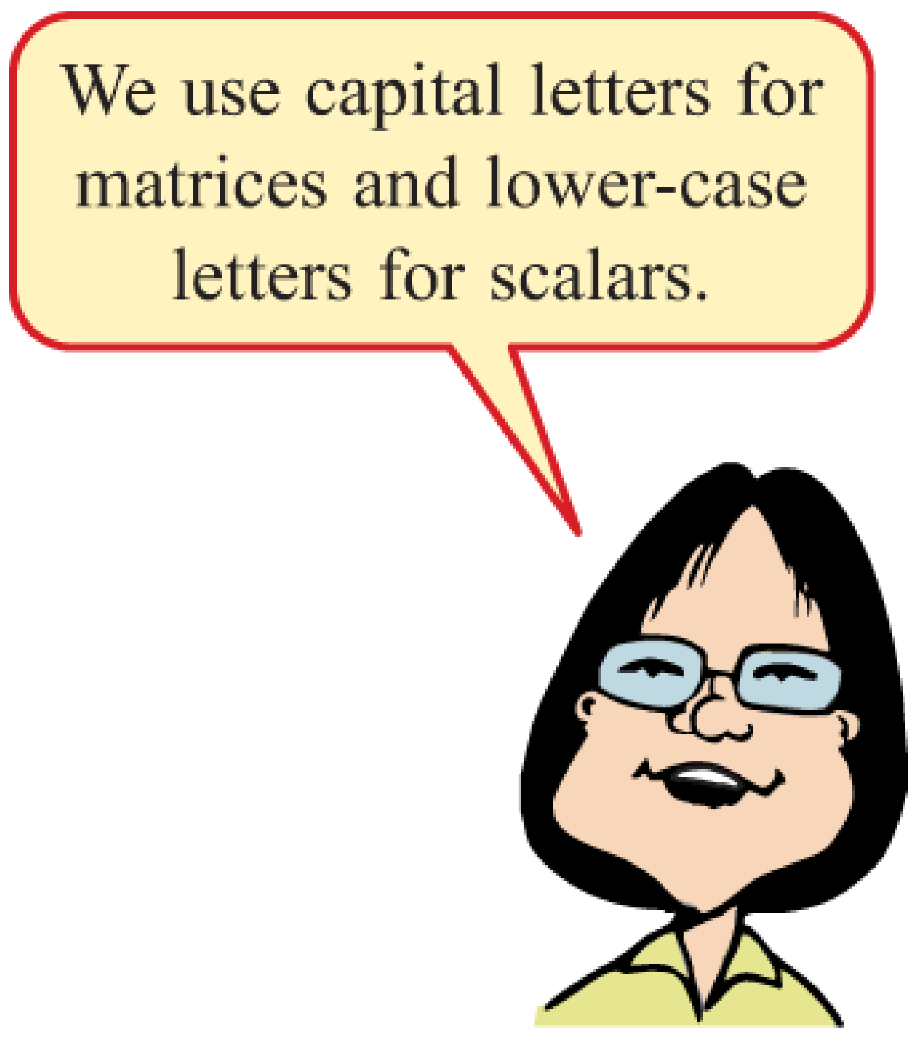
$$3\mathbf{C} = \begin{pmatrix} 3\times 3 \\ 3\times 2 \\ 3\times 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 18 \end{pmatrix}.$$

If we made a cake using only half the ingredients, we would need $\frac{1}{2}\mathbf{C} = \begin{pmatrix} \frac{1}{2}\times 3 \\ \frac{1}{2}\times 2 \\ \frac{1}{2}\times 6 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 3 \end{pmatrix}$.

If $\mathbf{A} = (a_{ij})$ has order $m \times n$, and k is a scalar, then $k\mathbf{A} = (ka_{ij})$.

So, to find $k\mathbf{A}$, we multiply each element in $\mathbf{A}$ by k.

The result is another matrix of order $m \times n$.


We use capital letters for matrices and lower-case letters for scalars.

NEGATIVE MATRICES

The **negative** matrix $\mathbf{A}$, denoted $-\mathbf{A}$, is actually $-1\mathbf{A}$.

$$-\mathbf{A} = (-1 \times a_{ij}) = (-a_{ij})$$

$-\mathbf{A}$ is obtained from $\mathbf{A}$ by reversing the sign of each element of $\mathbf{A}$.

For example, if $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$, then $-\mathbf{A} = \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix}$.

MATRIX SUBTRACTION

To subtract two matrices, we define $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$.

For example, suppose that over the next 6 months, Carla's sports equipment is lost or broken according to the matrix $\begin{pmatrix} 3 & 2 & 4 \\ 15 & 12 & 7 \\ 4 & 0 & 3 \end{pmatrix}$.

The equipment remaining is given by

$$\begin{pmatrix} 22 & 25 & 21 \\ 52 & 45 & 41 \\ 41 & 43 & 35 \end{pmatrix} - \begin{pmatrix} 3 & 2 & 4 \\ 15 & 12 & 7 \\ 4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 19 & 23 & 17 \\ 37 & 33 & 34 \\ 37 & 43 & 32 \end{pmatrix}.$$

To **subtract** matrices, they must be of the **same order**, and we **subtract corresponding elements**. That is, $\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})$.

Example 9

Let $\mathbf{A}=\begin{pmatrix}3&2&1\\4&5&6\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2&0&-3\\1&7&-1\end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix}1&3\\4&2\end{pmatrix}$. Find:

a $\mathbf{A}+\mathbf{B}$
b $\mathbf{A}+\mathbf{C}$
c $\mathbf{A}-\mathbf{B}$
d $3\mathbf{A}$
e $-\frac{1}{2}\mathbf{B}$
f $2\mathbf{A}-3\mathbf{B}$

a $\mathbf{A}+\mathbf{B}$

$$=\begin{pmatrix}3&2&1\\4&5&6\end{pmatrix}+\begin{pmatrix}2&0&-3\\1&7&-1\end{pmatrix}$$

$$=\begin{pmatrix}5&2&-2\\5&12&5\end{pmatrix}$$

b $\mathbf{A}+\mathbf{C}$ cannot be found as $\mathbf{A}$ and $\mathbf{C}$ have different orders.

c $\mathbf{A}-\mathbf{B}$

$$=\begin{pmatrix}3&2&1\\4&5&6\end{pmatrix}-\begin{pmatrix}2&0&-3\\1&7&-1\end{pmatrix}$$

$$=\begin{pmatrix}1&2&4\\3&-2&7\end{pmatrix}$$

d $3\mathbf{A}$

$$=\begin{pmatrix}9&6&3\\12&15&18\end{pmatrix}$$

e $-\frac{1}{2}\mathbf{B}$

$$=\begin{pmatrix}-1&0&\frac{3}{2}\\-\frac{1}{2}&-\frac{7}{2}&\frac{1}{2}\end{pmatrix}$$

f $2\mathbf{A}-3\mathbf{B}$

$$=\begin{pmatrix}6&4&2\\8&10&12\end{pmatrix}-\begin{pmatrix}6&0&-9\\3&21&-3\end{pmatrix}$$

$$=\begin{pmatrix}0&4&11\\5&-11&15\end{pmatrix}$$

EXERCISE 1C.1

1 Let $\mathbf{A}=\begin{pmatrix}1&2\\6&3\\5&4\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}-1&2\\-3&5\\0&2\end{pmatrix}$. Find:

a $4\mathbf{A}$
b $-2\mathbf{C}$
c $\mathbf{A}+2\mathbf{C}$
d $\mathbf{C}-\mathbf{A}$
e $-2\mathbf{A}+\frac{1}{2}\mathbf{C}$
f $\frac{1}{3}\mathbf{A}$

2 Consider two $m\times n$ matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$. Prove that:

a $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
b $k\mathbf{A}+k\mathbf{B}=k(\mathbf{A}+\mathbf{B})$
c $\mathbf{B}-\mathbf{A}=-(\mathbf{A}-\mathbf{B})$
d $(a+b)\mathbf{A}=a\mathbf{A}+b\mathbf{A}$ for all $a, b\in\mathbb{R}$
e $\underbrace{\mathbf{A}+\mathbf{A}+\mathbf{A}+....+\mathbf{A}}_{k \text{ of these}}=k\mathbf{A},\quad k\in\mathbb{Z}^+$

3 Find x and y such that:

a $\begin{pmatrix}x&x^2\\3&4\end{pmatrix}=\begin{pmatrix}y&9\\3&y+7\end{pmatrix}$

b $\begin{pmatrix}x&2y\\y&x\end{pmatrix}=\begin{pmatrix}-y&x\\x&y\end{pmatrix}$

4 Two teachers are comparing the grades their students have scored in recent exams.
Last year Keith's students obtained 9 As, 12 Bs, and 7 Cs.
This year his students obtained 8 As, 14 Bs, and 5 Cs.
Tatiana's students obtained 12 As, 6 Bs, and 13 Cs last year.
This year they obtained 9 As, 9 Bs, and 10 Cs.

- **a** Write the results of Keith's students in a 3×2 matrix $\mathbf{K}$.
- **b** Write the results of Tatiana's students in a 3×2 matrix $\mathbf{T}$.
- **c** Find $\mathbf{K} + \mathbf{T}$ and $\mathbf{K} - \mathbf{T}$.
- **d** Explain the significance of the matrices found in **c**.

8-A
14-B
5-C
9-A
9-B
10-C

5 **a** Let $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 0 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 0 & 5 \\ -2 & -3 \end{pmatrix}$.

Find $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$ and $\mathbf{A} + (\mathbf{B} + \mathbf{C})$.

b Prove that if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are any $m \times n$ matrices, then $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

6 Let $\mathbf{O}$ be the zero $m \times n$ matrix. Prove that for all $m \times n$ matrices $\mathbf{A}$:

- **a** $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$
- **b** $\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$

MATRIX ALGEBRA

We have made several discoveries about general $m \times n$ matrices. In the following table we compare these with algebraic facts for real numbers. Assume that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and the zero matrix $\mathbf{O}$ are matrices of the same order.

Ordinary algebra	Matrix algebra
• If a and b are real numbers then $a+b$ is also a real number.	• If $\mathbf{A}$ and $\mathbf{B}$ are matrices then $\mathbf{A} + \mathbf{B}$ is a matrix of the same order.
• $a+b=b+a$	• $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
• $(a+b)+c=a+(b+c)$	• $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
• $a+0=0+a=a$	• $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$
• $a+(-a)=(-a)+a=0$	• $\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$
• $k(a+b)=ka+kb$	• $k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$

Example 10

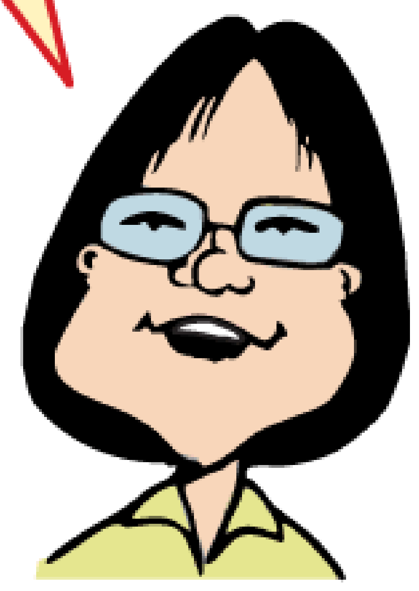
Write $\mathbf{X}$ in terms of $\mathbf{A}$ and $\mathbf{B}$, if:

- **a** $\mathbf{X} + 2\mathbf{A} = \mathbf{B}$
- **b** $5\mathbf{X} = \mathbf{A}$

a
$$\begin{aligned} \mathbf{X} + 2\mathbf{A} &= \mathbf{B} \\ \therefore\ \mathbf{X} + 2\mathbf{A} + (-2\mathbf{A}) &= \mathbf{B} + (-2\mathbf{A}) \\ \therefore\ \mathbf{X} + \mathbf{O} &= \mathbf{B} - 2\mathbf{A} \\ \therefore\ \mathbf{X} &= \mathbf{B} - 2\mathbf{A} \end{aligned}$$

b
$$\begin{aligned} 5\mathbf{X} &= \mathbf{A} \\ \therefore\ \tfrac{1}{5}(5\mathbf{X}) &= \tfrac{1}{5}\mathbf{A} \\ \therefore\ 1\mathbf{X} &= \tfrac{1}{5}\mathbf{A} \\ \therefore\ \mathbf{X} &= \tfrac{1}{5}\mathbf{A} \end{aligned}$$

We always write $\frac{1}{5}\mathbf{A}$ and not $\dfrac{\mathbf{A}}{5}$



EXERCISE 1C.2

1 Simplify:

a $3\mathbf{A} + 4\mathbf{A}$ **b** $\mathbf{C} - 5\mathbf{C}$ **c** $2\mathbf{M} - 2\mathbf{M}$

d $-\mathbf{X} + \mathbf{X}$ **e** $3(\mathbf{A} + \mathbf{B}) - \mathbf{B}$ **f** $2\mathbf{B} - (\mathbf{A} - \mathbf{B})$

g $\mathbf{A} - (2\mathbf{A} + \mathbf{C})$ **h** $2(\mathbf{A} + \mathbf{B}) - (\mathbf{A} - \mathbf{B})$ **i** $\mathbf{A} - 2\mathbf{D} - \frac{1}{2}(\mathbf{D} - \mathbf{A})$

2 Write $\mathbf{X}$ in terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$:

a $\mathbf{X} + \mathbf{B} = 2\mathbf{A}$ **b** $\mathbf{B} - \mathbf{X} = \mathbf{C}$ **c** $\mathbf{B} + 2\mathbf{X} = \mathbf{C}$

d $\frac{1}{2}\mathbf{X} + \mathbf{A} = 2\mathbf{C}$ **e** $3(\mathbf{X} - \mathbf{B}) = 2\mathbf{B} + \mathbf{C}$ **f** $\mathbf{C} - \frac{5}{2}\mathbf{X} = \mathbf{A} - \frac{1}{2}\mathbf{C}$

3 Suppose $\mathbf{A} = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 0 & 5 \end{pmatrix}$, and $3\mathbf{A} - 2\mathbf{X} = 3\mathbf{B}$.

a Make $\mathbf{X}$ the subject of the equation.

b Hence find matrix $\mathbf{X}$.

D MATRIX MULTIPLICATION

Stefan needs 2 new tyres, 18 ball bearings, and a bottle of oil for his go-kart. Stefan can buy these items at two shops.

At shop A, tyres cost €16.50 each, ball bearings cost €0.55 each, and a bottle of oil costs €5.95.

At shop B, tyres cost €14.75 each, ball bearings cost €0.70 each, and a bottle of oil costs €6.50.

18
BALL
BEARINGS
OIL

We can represent this information using a quantities matrix

$$\mathbf{Q} = \begin{array}{ccc} \big(2 & 18 & 1 \big) \\ \uparrow & \uparrow & \uparrow \\ \text{tyres} & \text{ball bearings} & \text{oil} \end{array}$$

and a cost matrix $\mathbf{C} = \begin{pmatrix} 16.50 & 14.75 \\ 0.55 & 0.70 \\ 5.95 & 6.50 \end{pmatrix}$.

To find the *total cost* of the items in each store, Stefan needs to multiply the number of items by their respective costs.

In shop A, the total cost is

$$2 \times €16.50 + 18 \times €0.55 + 1 \times €5.95 = €48.85\,.$$

In shop B, the total cost is

$$2 \times €14.75 + 18 \times €0.70 + 1 \times €6.50 = €48.60\,.$$

To calculate these values using matrices, we use **matrix multiplication**:

$$\mathbf{QC} = \begin{pmatrix} 2 & 18 & 1 \end{pmatrix} \times \begin{pmatrix} 16.50 & 14.75 \\ 0.55 & 0.70 \\ 5.95 & 6.50 \end{pmatrix} = \begin{pmatrix} \overset{\text{row } \mathbf{Q} \times \text{column 1}}{\overset{\downarrow}{48.85}} & \overset{\text{row } \mathbf{Q} \times \text{column 2}}{\overset{\downarrow}{48.60}} \end{pmatrix}$$

$$\underbrace{\boxed{1}\times\boxed{3} \longleftarrow \text{the same} \longrightarrow \boxed{3}\times\boxed{2}}_{\text{resultant matrix}} \qquad\qquad \boxed{1}\times\boxed{2}$$

Now suppose Stefan's friend Jason also needs supplies for his go-kart. Jason needs 4 new tyres, 12 ball bearings, and 1 bottle of oil.

The quantities matrix for both Stefan and Jason is

$$\begin{array}{cccl} \big(2 & 18 & 1 \big) & \longleftarrow \text{Stefan} \\ \big(4 & 12 & 1 \big) & \longleftarrow \text{Jason} \\ \uparrow & \uparrow & \uparrow & \\ \text{tyres} & \text{ball bearings} & \text{oil} & \end{array}$$

Stefan's *total cost* at shop A is €48.85 and at shop B is €48.60.

Jason's *total cost* at shop A is $4 \times €16.50 + 12 \times €0.55 + 1 \times €5.95 = €78.55$
and at shop B is $4 \times €14.75 + 12 \times €0.70 + 1 \times €6.50 = €73.90$.

So, using matrices we require that

$$\begin{pmatrix}2&18&1\\4&12&1\end{pmatrix}\times\begin{pmatrix}16.50&14.75\\0.55&0.70\\5.95&6.50\end{pmatrix}=\begin{pmatrix}\overset{\substack{\text{row 1}\times\text{column 1}\\\downarrow}}{48.85}&\overset{\substack{\text{row 1}\times\text{column 2}\\\downarrow}}{48.60}\\\underset{\substack{\uparrow\\\text{row 2}\times\text{column 1}}}{78.55}&\underset{\substack{\uparrow\\\text{row 2}\times\text{column 2}}}{73.90}\end{pmatrix}$$

$$\underbrace{\boxed{2}\times\boxed{3}\longleftarrow\text{the same}\longrightarrow\boxed{3}\times\boxed{2}}_{\text{resultant matrix}}\qquad\qquad\boxed{2}\times\boxed{2}$$

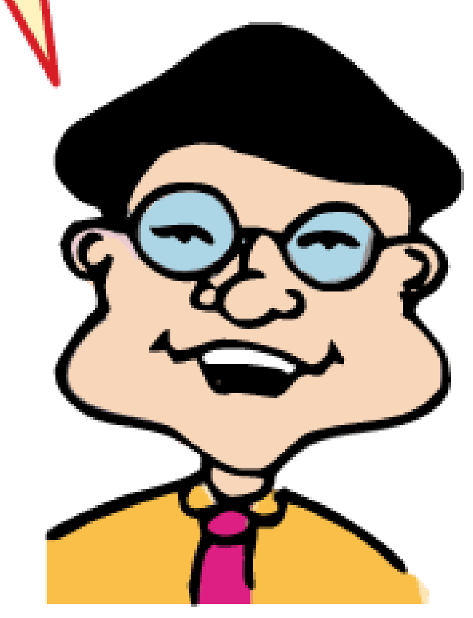
Having observed the usefulness of multiplying matrices in the contextual examples above, we now define matrix multiplication more formally.

The **product** of an $m\times n$ matrix **A** with an $n\times p$ matrix **B**, is the $m\times p$ matrix **AB** in which the element in the rth row and cth column is the sum of the products of the elements in the rth row of **A** with the corresponding elements in the cth column of **B**.

If $\mathbf{C}=\mathbf{AB}$ then $c_{ij}=\sum\limits_{r=1}^{n}a_{ir}b_{rj}=a_{i1}b_{1j}+a_{i2}b_{2j}+....+a_{in}b_{nj}$

for each pair i and j with $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant p$.

Note that the product **AB** exists *only* if the number of columns of **A** equals the number of rows of **B**.



$\sum\limits_{r=1}^{n}$ means the sum from $r=1$ to $r=n$.

For example:

If $\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}p&q\\r&s\end{pmatrix}$, then $\mathbf{AB}=\begin{pmatrix}ap+br&aq+bs\\cp+dr&cq+ds\end{pmatrix}$.

If $\mathbf{C}=\underset{2\times 3}{\begin{pmatrix}a&b&c\\d&e&f\end{pmatrix}}$ and $\mathbf{D}=\underset{3\times 1}{\begin{pmatrix}x\\y\\z\end{pmatrix}}$, then $\mathbf{CD}=\underset{2\times 1}{\begin{pmatrix}ax+by+cz\\dx+ey+fz\end{pmatrix}}$.

To get the matrix **AB** you multiply **rows by columns**. To get the element in the 5th row and 3rd column of **AB** (if it exists), multiply the 5th row of **A** by the 3rd column of **B**.

Example 11

Let $\mathbf{A}=\begin{pmatrix}2&3&1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2&3&1\\1&4&2\end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix}3&2\\2&4\\1&3\end{pmatrix}$.

Find: **a** **AC** **b** **BC**

a $\mathbf{A}$ is $1\times\boxed{3}$ and $\mathbf{C}$ is $\boxed{3}\times 2$ $\quad\therefore\ \mathbf{AC}$ is 1×2 ✓

$$\begin{aligned}\mathbf{AC}&=\begin{pmatrix}2&3&1\end{pmatrix}\begin{pmatrix}3&2\\2&4\\1&3\end{pmatrix}\\&=\begin{pmatrix}2\times3+3\times2+1\times1&2\times2+3\times4+1\times3\end{pmatrix}\\&=\begin{pmatrix}13&19\end{pmatrix}\end{aligned}$$

b **B** is $2\times\boxed{3}$ and **C** is $\boxed{3}\times 2$ $\therefore$ **BC** is 2×2

✓

$$\mathbf{BC}=\begin{pmatrix}2 & 3 & 1\\ 1 & 4 & 2\end{pmatrix}\begin{pmatrix}3 & 2\\ 2 & 4\\ 1 & 3\end{pmatrix}$$

$$=\begin{pmatrix}2\times 3+3\times 2+1\times 1 & 2\times 2+3\times 4+1\times 3\\ 1\times 3+4\times 2+2\times 1 & 1\times 2+4\times 4+2\times 3\end{pmatrix}$$

$$=\begin{pmatrix}13 & 19\\ 13 & 24\end{pmatrix}$$

EXERCISE 1D.1

1 Explain why **AB** cannot be found for $\mathbf{A}=\begin{pmatrix}-2 & 3 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}4 & 1 & -1\\ -2 & 0 & 3\end{pmatrix}$.

2 Suppose **A** is $3\times n$ and **B** is $m\times 2$.

- **a** When can we find **AB**?
- **b** If **AB** can be found, what is its order?
- **c** Explain why **BA** cannot be found.

3 Let $\mathbf{A}=\begin{pmatrix}3 & 2\\ 1 & 4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}6 & 5\end{pmatrix}$. Find: **a** **AB** **b** **BA**

4 Let $\mathbf{A}=\begin{pmatrix}3 & 1 & 4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2\\ 3\\ 1\end{pmatrix}$. Find: **a** **AB** **b** **BA**

5 Find:

- **a** $\begin{pmatrix}1 & 3 & 2\end{pmatrix}\begin{pmatrix}3 & 0 & 1\\ 2 & 2 & 3\\ 1 & 1 & 0\end{pmatrix}$
- **b** $\begin{pmatrix}-1 & 0 & 1\\ -2 & 2 & -1\\ 0 & 3 & 1\end{pmatrix}\begin{pmatrix}1\\ -3\\ 2\end{pmatrix}$
- **c** $\begin{pmatrix}1 & 1 & 0\\ -2 & 1 & 3\end{pmatrix}\begin{pmatrix}2 & 1 & 1 & -1\\ 0 & -2 & 3 & 1\\ 3 & 0 & 2 & 1\end{pmatrix}$
- **d** $\begin{pmatrix}2 & 1\\ 3 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}2 & 1 & 0 & -3\\ 0 & 4 & 1 & 0\end{pmatrix}$

6 At a new Chinese restaurant, the szechuan eggplant costs \$8.95, the roast duck costs \$12.95, and the crispy skin chicken costs \$9.95.

In the first month of operation, the restaurant sells 156 serves of eggplant, 193 serves of duck, and 218 serves of crispy skin chicken.

In the second month, the restaurant sells 183 serves of eggplant, 284 serves of duck, and 257 serves of crispy skin chicken.

- **a** Write the costs as a 3×1 matrix **C**, and the numbers as a 2×3 matrix **N**.
- **b** Find **NC** and interpret the resulting matrix.
- **c** Find the total income from these dishes over the two months.

USING TECHNOLOGY FOR MATRIX OPERATIONS

Click on the icon to obtain **graphics calculator** instructions for performing operations with matrices.

Alternatively, you can click on the **Matrix Operations** icon to obtain computer software for these tasks.

MATRIX OPERATIONS

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 1D.2

1 Use technology to find:

a $\begin{pmatrix} 2 & 6 & 0 & 7 \\ 3 & 2 & 8 & 6 \\ 1 & 4 & 0 & 2 \\ 3 & 0 & 1 & 8 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 6 \\ 11 \end{pmatrix}$

b $5.22\begin{pmatrix} 1 & 0 & 6 & 8 & 9 \\ 2 & 7 & 4 & 5 & 0 \\ 8 & 2 & 4 & 4 & 6 \end{pmatrix}$

c $2\begin{pmatrix} 13 & 12 & 4 \\ 11 & 12 & 8 \\ 7 & 9 & 7 \end{pmatrix} + 3\begin{pmatrix} 3 & 6 & 11 \\ 2 & 9 & 8 \\ 3 & 13 & 17 \end{pmatrix}$

d $0.4\begin{pmatrix} 13 & 12 & 4 \\ 11 & 12 & 8 \\ 7 & 9 & 7 \end{pmatrix} - 1.3\begin{pmatrix} 3 & 6 & 11 \\ 2 & 9 & 8 \\ 3 & 13 & 17 \end{pmatrix}$

2 A bus company runs four tours. Tour A costs \$125, Tour B costs \$315, Tour C costs \$405, and Tour D costs \$375. The numbers of clients they had over the summer period are shown in the table below.

$$\begin{array}{r c} & \begin{array}{cccc} \text{Tour A} & \text{Tour B} & \text{Tour C} & \text{Tour D} \end{array} \\ \begin{array}{r} \text{November} \\ \text{December} \\ \text{January} \\ \text{February} \end{array} & \begin{pmatrix} 50 & 42 & 18 & 65 \\ 65 & 37 & 25 & 82 \\ 120 & 29 & 23 & 75 \\ 42 & 36 & 19 & 72 \end{pmatrix} \end{array}$$

Use matrix methods to find the total income for the tour company.

PROPERTIES OF MATRIX MULTIPLICATION

INVESTIGATION 1 MATRIX MULTIPLICATION

In this Investigation we find the properties of matrix multiplication which are like those of ordinary number multiplication, and those which are not.

What to do:

1 For ordinary arithmetic $2 \times 3 = 3 \times 2$, and in algebra $ab = ba$.
For matrices, does **AB** always equal **BA**?

Hint: Try $\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$.

2 **a** If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, find **AO** and **OA**.

b What can be deduced from the results in **a**?

c Prove your result in **b** is true for all square matrices **A** where **O** is the zero square matrix of the same order.

3 **a** Find **AB** for:

i $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ **ii** $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$

b Explain the significance of this result.

4 For all real numbers a, b, and c, we have the **distributive law** $a(b+c) = ab + ac$.

a Use any three 2×2 matrices **A**, **B**, and **C** to verify that $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

b Now let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

Prove that in general, $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ for **A**, **B**, and **C** of appropriate order.

5 **a** If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $w = z = 1$ and $x = y = 0$ is a solution for any values of a, b, c, and d.

b For any real number a, we know that $a \times 1 = 1 \times a = a$.
Is there a matrix **I** such that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices **A**?

c Suppose $\mathbf{I}_n$ is an $n \times n$ matrix with 1s along the main diagonal and zeros everywhere else.

$$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

So, if $\mathbf{X} = \mathbf{I}_n$ then $(x_{ij}) = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j. \end{cases}$

Prove that $\mathbf{AI}_n = \mathbf{A}$ for all $n \times n$ matrices **A**.

$\mathbf{I}_n$ is called the $n \times n$ **identity matrix**.

6 Suppose $\mathbf{A}^k = \underbrace{\mathbf{AAAA}\ldots\mathbf{A}}_{k \text{ of these}}$

a If $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & -2 \end{pmatrix}$ find: **i** $\mathbf{A}^2$ **ii** $\mathbf{A}^3$

b If $\mathbf{B} = \begin{pmatrix} 2 & 0 & -3 \\ -1 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$ find: **i** $\mathbf{B}^2$ **ii** $\mathbf{B}^4$

c If $\mathbf{I}_n$ is the $n \times n$ identity matrix, deduce that $\mathbf{I}_n^k = \mathbf{I}$ for all $k \in \mathbb{Z}^+$.

d If $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \\ -2 & 1 \end{pmatrix}$, explain why $\mathbf{A}^2$ cannot be found.

7 For all real numbers a, b, and c we have the **associative law** $(ab)c = a(bc)$.

a Prove that for all 2×2 matrices **A**, **B**, and **C**, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

b Suppose $\mathbf{A} = (a_{ij})$ is 1×2, $\mathbf{B} = (b_{ij})$ is 2×3, and $\mathbf{C} = (c_{ij})$ is 3×4.
Show that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

In the **Investigation** you should have found that:

Ordinary algebra	**Matrix algebra**
If a and b are real numbers then so is ab. {closure}	If $\mathbf{A}$ and $\mathbf{B}$ are matrices that can be multiplied then $\mathbf{AB}$ is also a matrix. {closure}
$ab = ba$ for all a, b {commutative}	In general $\mathbf{AB} \neq \mathbf{BA}$. {non-commutative}
$a0 = 0a = 0$ for all a	If $\mathbf{O}$ is a zero matrix then $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ for all $\mathbf{A}$.
$ab = 0 \Leftrightarrow a = 0$ or $b = 0$ {Null Factor law}	$\mathbf{AB}$ may be $\mathbf{O}$ without requiring $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.
$a(b + c) = ab + ac$ {distributive law}	$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ {distributive law}
$a \times 1 = 1 \times a = a$ {identity law}	If $\mathbf{I}_n$ is the $n \times n$ **identity matrix** then $\mathbf{AI}_n = \mathbf{I}_n\mathbf{A} = \mathbf{A}$ for all $n \times n$ matrices $\mathbf{A}$. {identity law}
a^n exists for all $a \geqslant 0$ and $n \in \mathbb{R}$	$\mathbf{A}^n$ exists provided $\mathbf{A}$ is square and $n \in \mathbb{Z}^+$
$a(bc) = (ab)c$ for all $a, b, c \in \mathbb{R}$ {associative law}	$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ provided $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are of appropriate order. {associative law}

Note that in general, $\mathbf{A}(k\mathbf{B}) = k(\mathbf{AB}) \neq k\mathbf{BA}$. We can change the order in which we multiply by a scalar, but we cannot reverse the order in which we multiply matrices.

Example 12

Expand and simplify:

a $(2\mathbf{A} + \mathbf{I})^2$ **b** $(\mathbf{A} - 2\mathbf{B})^2$

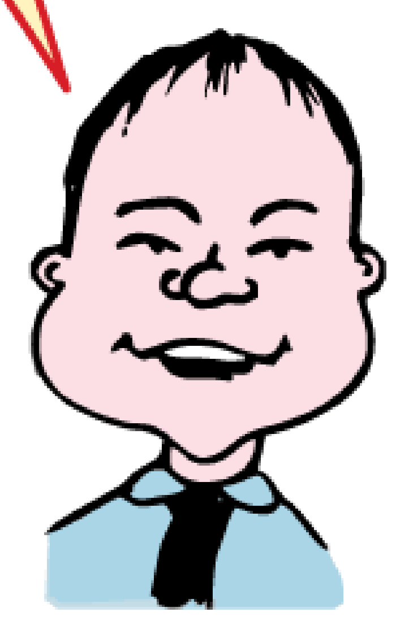
a

$$\begin{aligned}
&(2\mathbf{A} + \mathbf{I})^2 \\
&= (2\mathbf{A} + \mathbf{I})(2\mathbf{A} + \mathbf{I}) && \{\mathbf{X}^2 = \mathbf{XX} \text{ by definition}\} \\
&= (2\mathbf{A} + \mathbf{I})2\mathbf{A} + (2\mathbf{A} + \mathbf{I})\mathbf{I} && \{\text{distributive law}\} \\
&= 4\mathbf{A}^2 + 2\mathbf{IA} + 2\mathbf{AI} + \mathbf{I}^2 && \{\text{distributive law}\} \\
&= 4\mathbf{A}^2 + 2\mathbf{A} + 2\mathbf{A} + \mathbf{I} && \{\mathbf{AI} = \mathbf{IA} = \mathbf{A} \text{ and } \mathbf{I}^2 = \mathbf{I}\} \\
&= 4\mathbf{A}^2 + 4\mathbf{A} + \mathbf{I}
\end{aligned}$$

b

$$\begin{aligned}
&(\mathbf{A} - 2\mathbf{B})^2 \\
&= (\mathbf{A} - 2\mathbf{B})(\mathbf{A} - 2\mathbf{B}) && \{\mathbf{X}^2 = \mathbf{XX} \text{ by definition}\} \\
&= (\mathbf{A} - 2\mathbf{B})\mathbf{A} + (\mathbf{A} - 2\mathbf{B})(-2\mathbf{B}) && \{\text{distributive law}\} \\
&= \mathbf{A}^2 - 2\mathbf{BA} - 2\mathbf{AB} + 4\mathbf{B}^2 && \{\text{distributive law}\}
\end{aligned}$$

We cannot simplify **b** further, since in general $\mathbf{BA} \neq \mathbf{AB}$.



EXERCISE 1D.3

1 Expand and simplify where possible:

a $\mathbf{X}(2\mathbf{X} + \mathbf{I})$ **b** $(3\mathbf{I} + \mathbf{B})\mathbf{B}$ **c** $\mathbf{D}(\mathbf{D}^2 + 3\mathbf{D} + 2\mathbf{I})$ **d** $(\mathbf{A} + \mathbf{B})(\mathbf{C} - \mathbf{D})$

e $(\mathbf{B} - \mathbf{A})(\mathbf{B} + \mathbf{A})$ **f** $(\mathbf{A} - 2\mathbf{I})^2$ **g** $(5\mathbf{I} - 2\mathbf{B})^2$ **h** $(\mathbf{A} + \mathbf{B})^3$

Example 13

Suppose $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$. Write $\mathbf{A}^3$ and $\mathbf{A}^4$ in the form $k\mathbf{A} + l\mathbf{I}$ where $k, l \in \mathbb{Z}$.

$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A} \times \mathbf{A}^2 \\ &= \mathbf{A}(2\mathbf{A} - \mathbf{I}) \\ &= 2\mathbf{A}^2 - \mathbf{AI} \\ &= 2(2\mathbf{A} - \mathbf{I}) - \mathbf{A} \\ &= 4\mathbf{A} - 2\mathbf{I} - \mathbf{A} \\ &= 3\mathbf{A} - 2\mathbf{I} \end{aligned}$$

$$\begin{aligned} \mathbf{A}^4 &= \mathbf{A} \times \mathbf{A}^3 \\ &= \mathbf{A}(3\mathbf{A} - 2\mathbf{I}) \\ &= 3\mathbf{A}^2 - 2\mathbf{AI} \\ &= 3(2\mathbf{A} - \mathbf{I}) - 2\mathbf{A} \\ &= 6\mathbf{A} - 3\mathbf{I} - 2\mathbf{A} \\ &= 4\mathbf{A} - 3\mathbf{I} \end{aligned}$$

2 Suppose $\mathbf{A}^2 = 3\mathbf{A} + 2\mathbf{I}$. Write in the form $k\mathbf{A} + l\mathbf{I}$ where $k, l \in \mathbb{Z}$:

a $\mathbf{A}^3$ **b** $\mathbf{A}^4$ **c** $\mathbf{A}^8$

3 Suppose $\mathbf{A}$ is a matrix with the property $\mathbf{A}^2 = \mathbf{I}$. Simplify:

a $\mathbf{A}(2\mathbf{A} + 3\mathbf{I})$ **b** $(\mathbf{A} - \mathbf{I})^2$ **c** $\mathbf{A}(\mathbf{A} + 5\mathbf{I})^2$

4 Show using counter-examples that the following are not true in general:

a $\mathbf{A}^2 = \mathbf{O} \Rightarrow \mathbf{A} = \mathbf{O}$ **b** $\mathbf{A}^2 = \mathbf{A} \Rightarrow \mathbf{A} = \mathbf{O}$ or $\mathbf{I}$

5 Find all 2×2 matrices $\mathbf{A}$ for which $\mathbf{A}^2 = \mathbf{A}$.

6 Find the error in the argument:

$$\begin{aligned} &\mathbf{A}^2 = 2\mathbf{A} \\ \Rightarrow\ &\mathbf{A}^2 - 2\mathbf{A} = \mathbf{O} \\ \Rightarrow\ &\mathbf{A}(\mathbf{A} - 2\mathbf{I}) = \mathbf{O} \\ \Rightarrow\ &\mathbf{A} = \mathbf{O} \text{ or } 2\mathbf{I} \end{aligned}$$

7 Explain why the binomial expansion for real numbers can also be used to expand $(\mathbf{A} + k\mathbf{I})^n$, $n \in \mathbb{Z}^+$, but cannot be used to expand $(\mathbf{A} + \mathbf{B})^n$.

E MATRIX TRANSPOSE

The **transpose** $\mathbf{A}^{\mathrm{T}}$ of matrix $\mathbf{A}$ is the matrix obtained by writing the rows of $\mathbf{A}$ as the columns of $\mathbf{A}^{\mathrm{T}}$.

If $\mathbf{A} = (a_{ij})$ is an $m \times n$ matrix, then $\mathbf{A}^{\mathrm{T}} = (a_{ji})$ is an $n \times m$ matrix.

For example, if $\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 3 & 1 \end{pmatrix}$, then $\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \\ 4 & 1 \end{pmatrix}$.

The square matrix $\mathbf{A}$ is:

- **symmetric** if $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$
- **skew-symmetric** if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$.

For example:

- $\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$ is symmetric
- $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ is skew-symmetric.

PROPERTIES OF TRANSPOSE

Provided that the orders of the matrices are appropriate for the operations to be performed:

- $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$
- $(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$ and $(\mathbf{A} - \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} - \mathbf{B}^{\mathrm{T}}$
- $(s\mathbf{A})^{\mathrm{T}} = s\mathbf{A}^{\mathrm{T}}$ for any scalar $s \in \mathbb{R}$
- $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$

Example 14

Prove that $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ for matrices **A, B** of appropriate order so that multiplications may be performed.

If $\mathbf{C} = \mathbf{AB}$ then $c_{ij} = \sum_{n=1}^{k} a_{in}b_{nj}$

$\therefore$ $(\mathbf{AB})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}$ where $c_{ji} = \sum_{n=1}^{k} a_{jn}b_{ni}$

If we let $\mathbf{D} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, then

$$\begin{aligned} d_{ij} &= \sum_{n=1}^{k} b_{ni}a_{jn} \\ &= \sum_{n=1}^{k} a_{jn}b_{ni} \\ &= c_{ji} \end{aligned}$$

Since $c_{ji} = d_{ij}$ for all i, j, $\mathbf{C}^{\mathrm{T}} = \mathbf{D}$

$\therefore$ $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$

EXERCISE 1E

1 For $\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 & 3 \\ 1 & -2 & -1 \end{pmatrix}$, find:

a $(\mathbf{A}^T)^T$ **b** $(\mathbf{A}+\mathbf{B})^T$ **c** $\mathbf{A}^T+\mathbf{B}^T$ **d** $(3\mathbf{A})^T$ **e** $3\mathbf{A}^T$

f $(\mathbf{A}-\mathbf{B})^T$ **g** $\mathbf{A}^T-\mathbf{B}^T$ **h** $(-2\mathbf{B})^T$ **i** $-2\mathbf{B}^T$

2 For $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & 4 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 3 & 2 \\ 4 & 0 & 5 \end{pmatrix}$, find:

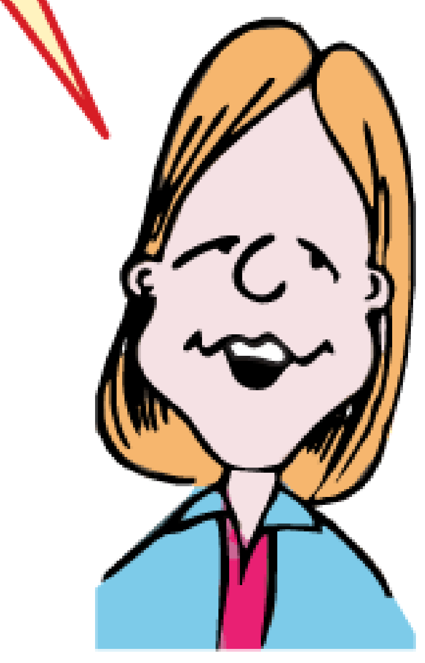
a $\mathbf{AB}$ **b** $(\mathbf{AB})^T$ **c** $\mathbf{A}^T\mathbf{B}^T$ **d** $\mathbf{B}^T\mathbf{A}^T$

3 Assuming the matrices are of suitable shape for the operations being performed, prove that:

a $(\mathbf{A}^T)^T = \mathbf{A}$ **b** $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T+\mathbf{B}^T$ **c** $(s\mathbf{A})^T = s\mathbf{A}^T$

4 **a** Prove that $(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3)^T = \mathbf{A}_3^T\mathbf{A}_2^T\mathbf{A}_1^T$.

b Generalise the result in **a** using $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, ..., \mathbf{A}_n$. Use mathematical induction to prove your generalisation is true for all $n \in \mathbb{Z}^+$.



For help with mathematical induction see the HL Core course.

5 Prove that:

a if $\mathbf{A}$ is symmetric, then $\mathbf{A}^T$ is symmetric

b if $\mathbf{A}$ and $\mathbf{B}$ are symmetric, then $\mathbf{A}+\mathbf{B}$ is symmetric

c if $\mathbf{A}$ and $\mathbf{B}$ are symmetric, $\mathbf{AB}$ is symmetric $\Leftrightarrow$ $\mathbf{AB} = \mathbf{BA}$.

6 Prove that $\mathbf{A} = (a_{ij})$ is skew symmetric $\Leftrightarrow$ $a_{ij} = -a_{ji}$.

7 Suppose $\mathbf{A}$ is a square matrix.

a Show that $\mathbf{AA}^T$, $\mathbf{A}^T\mathbf{A}$, and $\mathbf{A}+\mathbf{A}^T$ are symmetric.

b Is $\mathbf{A}-\mathbf{A}^T$ symmetric? Explain your answer.

8 Give examples of 3×3 matrices which are

a symmetric **b** skew-symmetric.

9 **a** Suppose $\mathbf{A}$ is symmetric. If $\mathbf{P}$ is a matrix of appropriate size, comment on whether $\mathbf{P}^T\mathbf{AP}$ is symmetric or skew-symmetric.

b Suppose $\mathbf{A}$ is skew-symmetric. If $\mathbf{P}$ is a matrix of appropriate size, comment on whether $\mathbf{P}^T\mathbf{AP}$ is symmetric or skew-symmetric.

F MATRIX DETERMINANT AND INVERSE

The real numbers 3 and $\frac{1}{3}$ are called **multiplicative inverses** because when they are multiplied together, the result is the multiplicative identity 1: $3 \times \frac{1}{3} = \frac{1}{3} \times 3 = 1$.

For the matrices $\begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$, we notice that $\begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ and $\begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$.

We say that $\begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$ are *multiplicative inverses* of each other.

The **multiplicative inverse** of $\mathbf{A}$, denoted $\mathbf{A}^{-1}$, satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

To find the multiplicative inverse of a matrix **A**, we need a matrix which, when multiplied by **A**, gives the identity matrix **I**.

FINDING THE INVERSE OF A 2×2 MATRIX

Suppose $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has inverse $\mathbf{A}^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$

$$\therefore\ \mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \mathbf{I}$$

$$\therefore\ \begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore\ \begin{cases} aw+by=1 & \text{.... (1)} \\ cw+dy=0 & \text{.... (2)} \end{cases} \quad \text{and} \quad \begin{cases} ax+bz=0 & \text{.... (3)} \\ cx+dz=1 & \text{.... (4)} \end{cases}$$

Solving (1) and (2) simultaneously for w and y gives: $w = \dfrac{d}{ad-bc}$ and $y = \dfrac{-c}{ad-bc}$.

Solving (3) and (4) simultaneously for x and z gives: $x = \dfrac{-b}{ad-bc}$ and $z = \dfrac{a}{ad-bc}$.

So, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad-bc \neq 0$, then $\mathbf{A}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

In this case
$$\begin{aligned} \mathbf{A}^{-1}\mathbf{A} &= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & bd-bd \\ ac-ac & -bc+ad \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathbf{I} \quad \text{also,} \end{aligned}$$

so $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

Just as the real number 0 does not have a multiplicative inverse, some matrices do not have a multiplicative inverse. This occurs when $\det \mathbf{A} = ad - bc = 0$.

For the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

- the value $ad - bc$ is called the **determinant** of matrix $\mathbf{A}$, denoted $\det \mathbf{A}$ or $|\mathbf{A}|$
- if $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is **invertible** or **non-singular**, and $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- if $\det \mathbf{A} = 0$, then $\mathbf{A}$ is **singular**, and $\mathbf{A}^{-1}$ does not exist.

Example 15

Find, if it exists, the inverse matrix of:

a $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$

b $\mathbf{B} = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}$

a $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$

$\therefore \det \mathbf{A} = 3(1) - (-2)(-2) = -1$

$$\therefore \mathbf{A}^{-1} = \tfrac{1}{-1} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -2 & -3 \end{pmatrix}$$

b $\mathbf{B} = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}$

$$\begin{aligned} \therefore \det \mathbf{B} &= 3(4) - (-6)(-2) \\ &= 12 - 12 \\ &= 0 \end{aligned}$$

$\therefore \mathbf{B}^{-1}$ does not exist.

EXERCISE 1F.1

1 Find $\begin{pmatrix} 2 & -4 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} -\frac{5}{2} & 2 \\ -\frac{3}{2} & 1 \end{pmatrix}$ and hence find the inverse of $\begin{pmatrix} 2 & -4 \\ 3 & -5 \end{pmatrix}$.

2 Find the determinant of each of the following matrices. Hence find the inverse, if it exists.

a $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} 5 & 0 \\ 3 & 0 \end{pmatrix}$ **e** $\begin{pmatrix} a & a \\ -a & 1 \end{pmatrix}$

3 For $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$, find:

a $\det \mathbf{A}$ **b** $\det(-\mathbf{A})$ **c** $\det(2\mathbf{A})$

4 Prove that if $\mathbf{A}$ is any 2×2 matrix and k is a constant, then $\det(k\mathbf{A}) = k^2 \det(\mathbf{A})$.

5 Prove that $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

6 Suppose $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ -4 & 6 \\ 1 & -1 \end{pmatrix}$.

a Find $\mathbf{AB}$.

b Use $\mathbf{BA}$ to explain why $\mathbf{A}$ and $\mathbf{B}$ are not inverses.

c Explain why only square matrices have inverses.

Example 16

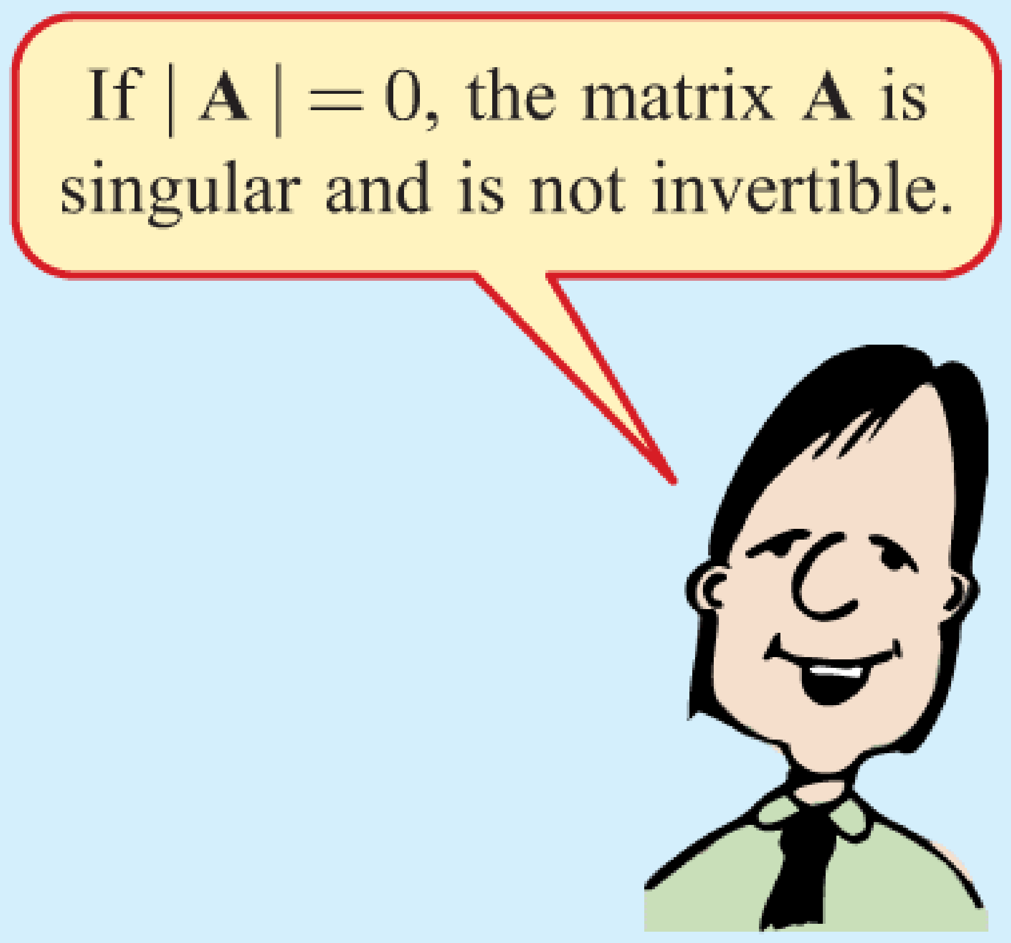
Suppose $\mathbf{A} = \begin{pmatrix} 4 & k \\ 2 & -1 \end{pmatrix}$. Find $\mathbf{A}^{-1}$ in terms of k, and state the values of k for which $\mathbf{A}^{-1}$ exists.

$$\begin{aligned}\det \mathbf{A} &= 4(-1) - k(2) \\ &= -4 - 2k \\ &= -(2k+4)\end{aligned}$$

$$\mathbf{A}^{-1} = \frac{1}{-(2k+4)} \begin{pmatrix} -1 & -k \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2k+4} & \dfrac{k}{2k+4} \\ \dfrac{2}{2k+4} & \dfrac{-4}{2k+4} \end{pmatrix}$$

$\therefore$ $\mathbf{A}^{-1}$ exists provided $2k + 4 \neq 0$

$\therefore$ $\mathbf{A}^{-1}$ exists for $k \in \mathbb{R}$, $k \neq -2$.


If $|\mathbf{A}| = 0$, the matrix $\mathbf{A}$ is singular and is not invertible.

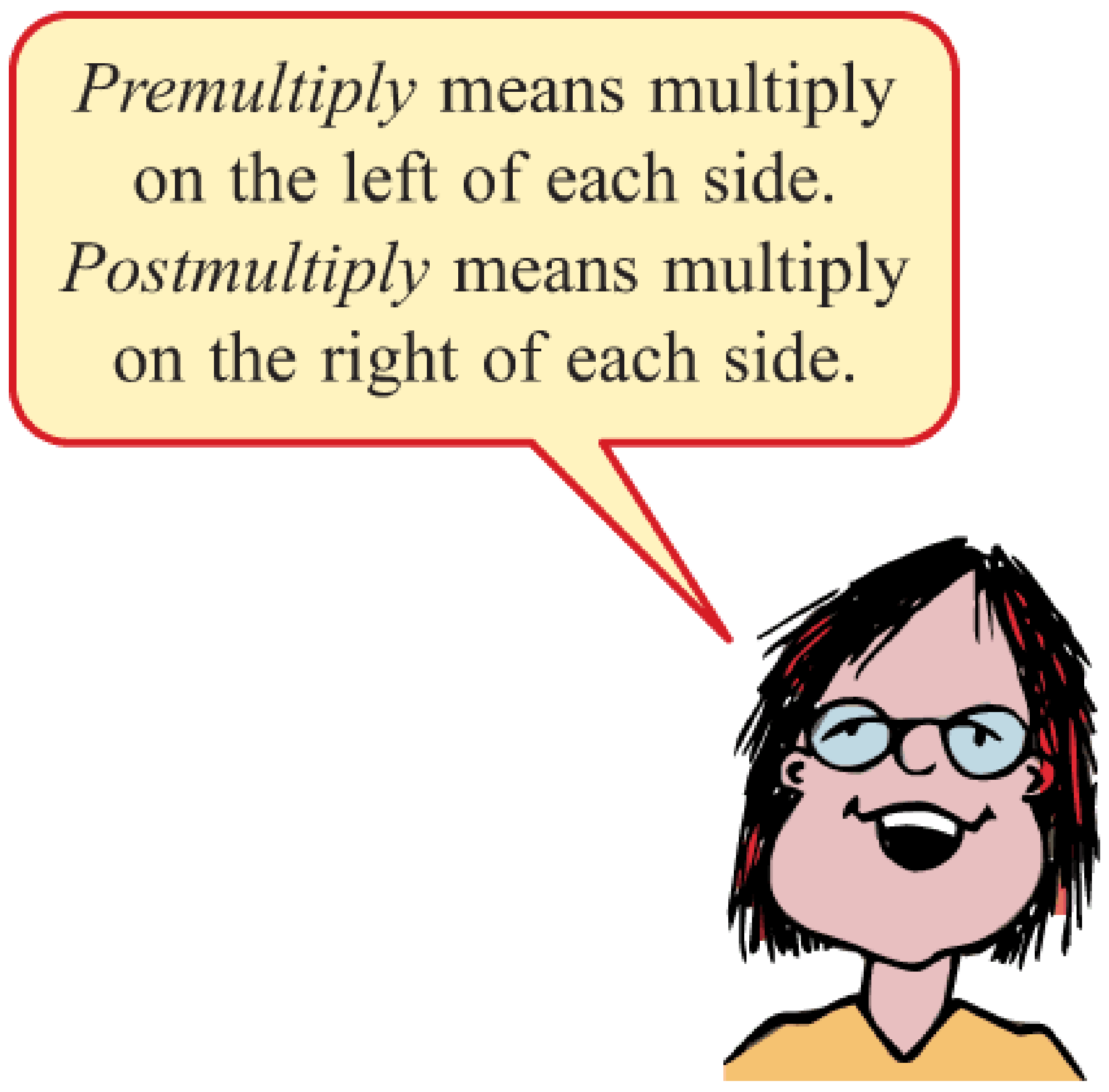
7 For each of the following matrices $\mathbf{A}$, find $\mathbf{A}^{-1}$ and state the values of k for which $\mathbf{A}^{-1}$ exists.

a $\mathbf{A} = \begin{pmatrix} 2 & k \\ 3 & -6 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} k-5 & -3 \\ 2 & k \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} k & 12 \\ k+1 & k+5 \end{pmatrix}$

FURTHER MATRIX ALGEBRA

In this section we consider matrix algebra with inverse matrices. Be careful that you use multiplication correctly. In particular, remember that:

- We can only perform matrix multiplication if the orders of the matrices allow it.
- If we *pre*multiply on one side then we must *pre*multiply on the other. This is important because, in general, $\mathbf{AB} \neq \mathbf{BA}$. The same applies if we *post*multiply.


Premultiply means multiply on the left of each side.
Postmultiply means multiply on the right of each side.

INVESTIGATION 2 PROPERTIES OF INVERSE MATRICES

In this Investigation, we consider some properties of invertible 2×2 matrices.

What to do:

1 A matrix $\mathbf{A}$ is **self-inverse** when $\mathbf{A} = \mathbf{A}^{-1}$.

For example, if $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ then $\mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{A}$.

a Show that if $\mathbf{A} = \mathbf{A}^{-1}$, then $\mathbf{A}^2 = \mathbf{I}$.

b Show that there are exactly 4 self-inverse matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

2 **a** Given $\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$, find $\mathbf{A}^{-1}$ and $(\mathbf{A}^{-1})^{-1}$.

b If $\mathbf{A}$ is any invertible matrix, simplify $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1})$ and $(\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1}$ by replacing $\mathbf{A}^{-1}$ by $\mathbf{B}$.

c What can be deduced from **b**?

3 Suppose k is a non-zero number and $\mathbf{A}$ is an invertible matrix.

a Simplify $(k\mathbf{A})(\frac{1}{k}\mathbf{A}^{-1})$ and $(\frac{1}{k}\mathbf{A}^{-1})(k\mathbf{A})$.

b What can you conclude from your results?

4 **a** Suppose $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -4 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & 2 \\ 2 & 1 \end{pmatrix}$. Find, in simplest form:

i $\mathbf{A}^{-1}$ **ii** $\mathbf{B}^{-1}$ **iii** $(\mathbf{AB})^{-1}$

iv $(\mathbf{BA})^{-1}$ **v** $\mathbf{A}^{-1}\mathbf{B}^{-1}$ **vi** $\mathbf{B}^{-1}\mathbf{A}^{-1}$

b Choose any two invertible matrices and repeat **a**.

c Simplify $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})$ and $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB})$ given that $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist. What can you conclude from your results?

5 **a** Given $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, find $(\mathbf{A}^{-1})^{\mathrm{T}}$ and $(\mathbf{A}^{\mathrm{T}})^{-1}$.

b Choose another invertible matrix $\mathbf{A}$ and repeat **a**.

c Prove that for all invertible 2×2 matrices $\mathbf{A}$, $(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1}$.

6 **a** Given $\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$, find:

i $\mathbf{A}^{-1}$

ii $(\mathbf{A}^2)^{-1}$ and $(\mathbf{A}^{-1})^2$

iii $(\mathbf{A}^3)^{-1}$ and $(\mathbf{A}^{-1})^3$

b Prove that if $\mathbf{A}$ is an invertible 2×2 matrix, then $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$.

c Use mathematical induction to prove that for any invertible 2×2 matrix $\mathbf{A}$, $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$ for all $n \in \mathbb{Z}^+$.

For help with mathematical induction see the HL Core course.

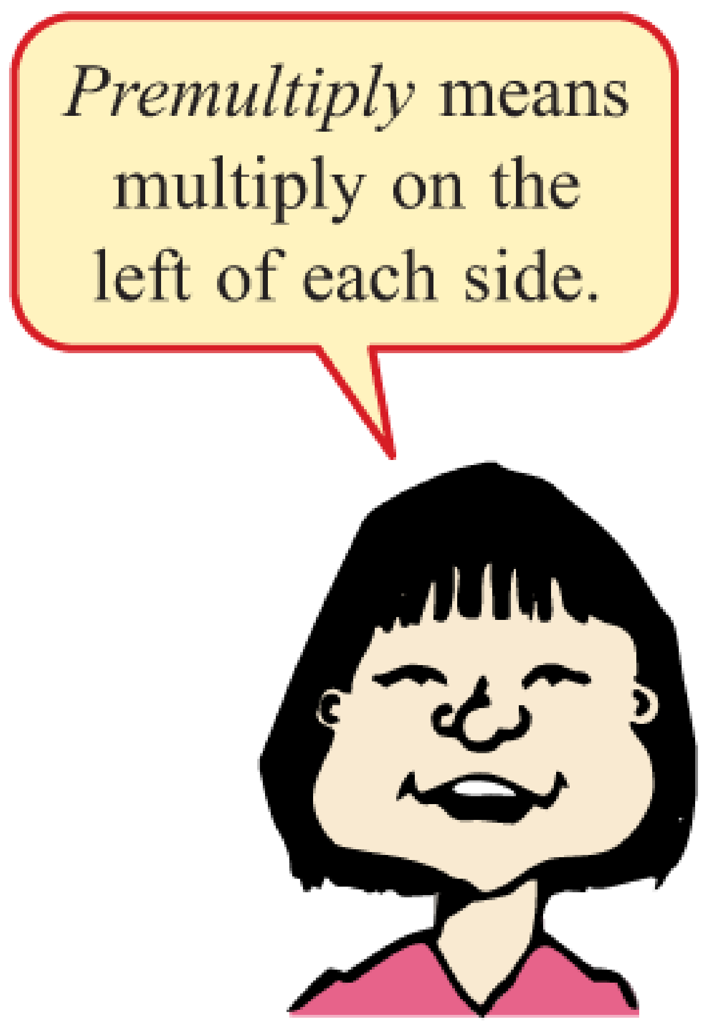
From the **Investigation** you should have found that if $\mathbf{A}$ and $\mathbf{B}$ are invertible, then:

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1}$
- $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$ for all $n \in \mathbb{Z}^+$

Example 17

Suppose $\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$. Find $\mathbf{A}^{-1}$ in the linear form $r\mathbf{A} + s\mathbf{I}$, where r and s are scalars.

$$\begin{aligned} \mathbf{A}^2 &= 3\mathbf{A} - \mathbf{I} \\ \therefore\ \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}(3\mathbf{A} - \mathbf{I}) \qquad \{\text{premultiplying both sides by } \mathbf{A}^{-1}\} \\ \therefore\ \mathbf{A}^{-1}\mathbf{A}\mathbf{A} &= 3\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\mathbf{I} \\ \therefore\ \mathbf{I}\mathbf{A} &= 3\mathbf{I} - \mathbf{A}^{-1} \\ \therefore\ \mathbf{A}^{-1} &= -\mathbf{A} + 3\mathbf{I} \end{aligned}$$


Premultiply means multiply on the left of each side.

EXERCISE 1F.2

1 Consider the matrix equation $\mathbf{AXB} = \mathbf{C}$.

a Show that if $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist, then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}$.

b Find $\mathbf{X}$ such that $\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \mathbf{X} \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$.

2 Suppose $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are 2×1 matrices, and $\mathbf{A}$ and $\mathbf{B}$ are invertible 2×2 matrices. If $\mathbf{X} = \mathbf{AY}$ and $\mathbf{Y} = \mathbf{BZ}$, write:

a $\mathbf{X}$ in terms of $\mathbf{Z}$ **b** $\mathbf{Z}$ in terms of $\mathbf{X}$.

3 If $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$, write $\mathbf{A}^2$ in the linear form $p\mathbf{A} + q\mathbf{I}$ where p and q are scalars.

Hence write $\mathbf{A}^{-1}$ in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

4 Write $\mathbf{A}^{-1}$ in linear form, given that:

a $\mathbf{A}^2 = 2\mathbf{A} + \mathbf{I}$ **b** $3\mathbf{A} = 2\mathbf{I} - \mathbf{A}^2$ **c** $2\mathbf{A}^2 - 3\mathbf{A} - \mathbf{I} = \mathbf{O}$

5 It is known that $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$ where the matrices $\mathbf{A}$ and $\mathbf{B}$ are not necessarily invertible. Prove that $\mathbf{A}^2 = \mathbf{A}$.

6 Under what condition is it true that "if $\mathbf{AB} = \mathbf{AC}$ then $\mathbf{B} = \mathbf{C}$"?

7 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$, prove that $\mathbf{X}^3 = \mathbf{I}$.

8 If $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{O}$ and $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$, prove that $a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} = \mathbf{O}$.

9 If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find constants s and t such that $\mathbf{A}^2 + s\mathbf{A} + t\mathbf{I} = \mathbf{O}$.

10 Prove that if $\mathbf{AB}^{-1} = \mathbf{B}^{-1}\mathbf{A}$, then $\mathbf{AB} = \mathbf{BA}$.

11 A non-singular matrix $\mathbf{A}$ is **orthogonal** if $\mathbf{A}^{-1} = \mathbf{A}^T$.

a Prove that $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.

b Prove that if $\mathbf{A}$ and $\mathbf{B}$ are orthogonal then $\mathbf{AB}$ is also orthogonal.

c Prove that if $\mathbf{A}$ is orthogonal then $\mathbf{A}^{-1}$ is also orthogonal.

THE DETERMINANT OF A 3×3 MATRIX

The **determinant** of $\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is defined as

$$|\mathbf{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

where the 2×2 determinants are called the **minors** of $|\mathbf{A}|$.

Just like 2×2 matrices, a 3×3 matrix will have an inverse if $|\mathbf{A}| \neq 0$.

MATRIX OPERATIONS

GRAPHICS CALCULATOR INSTRUCTIONS

Example 18

Given $\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{pmatrix}$, find $|\mathbf{A}|$ without using technology.

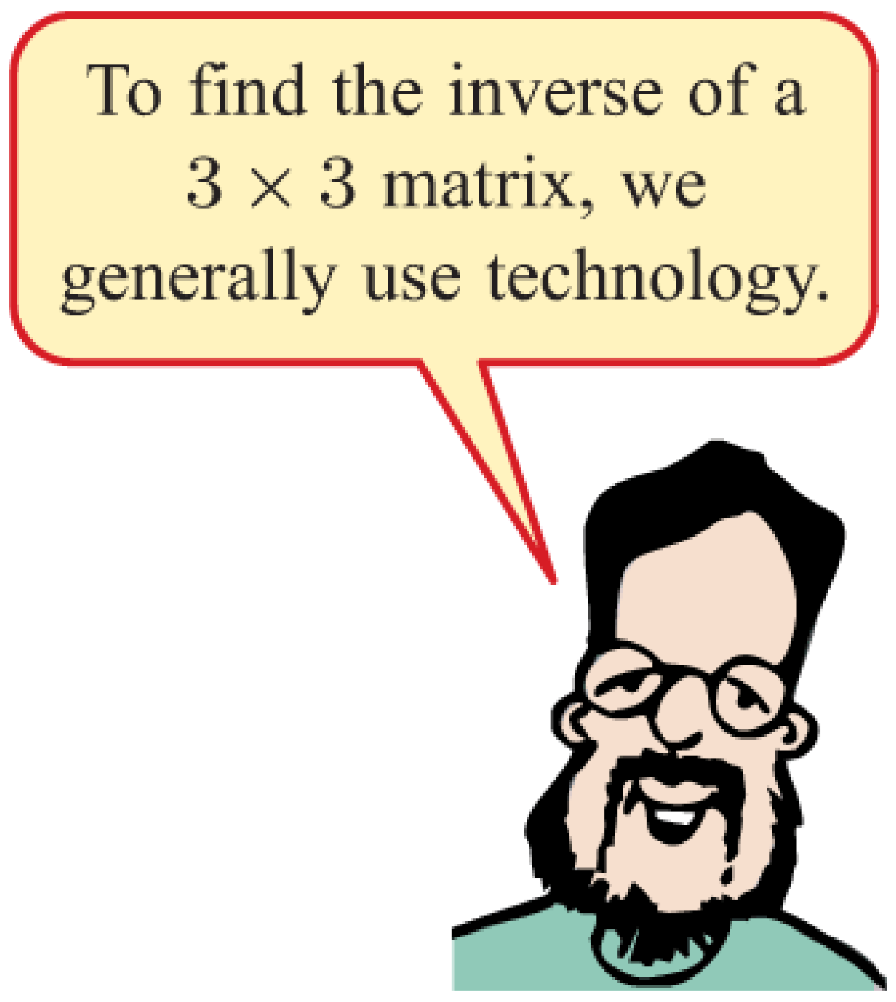
$$|\mathbf{A}| = 1 \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 4 \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}$$

same

same

same

$$\begin{aligned} &= 1(0 - -1) - 2(4 - 3) + 4(-2 - 0) \\ &= 1 - 2 - 8 \\ &= -9 \end{aligned}$$



To find the inverse of a 3×3 matrix, we generally use technology.

EXERCISE 1F.3

1 Evaluate without using technology:

a $\begin{vmatrix} 2 & 3 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 5 \end{vmatrix}$ **b** $\begin{vmatrix} -1 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{vmatrix}$ **c** $\begin{vmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$ **d** $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$

e $\begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix}$ **f** $\begin{vmatrix} 4 & 1 & 3 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{vmatrix}$ **g** $\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ -2 & 1 & 0 \end{vmatrix}$ **h** $\begin{vmatrix} 3 & -1 & 2 \\ 1 & -4 & 1 \\ -3 & 1 & -1 \end{vmatrix}$

Check your answers using technology.

2 Find the values of x for which the matrix $\begin{pmatrix} x & 2 & 9 \\ 3 & 1 & 2 \\ -1 & 0 & x \end{pmatrix}$ is singular.

Explain the significance of your answer.

3 Evaluate:

a $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$ **b** $\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$ **c** $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

4 For what values of a and b does $\begin{pmatrix} a^2 & 1 & 1 \\ 0 & a & b \\ 1 & 0 & 0 \end{pmatrix}$ have an inverse?

5 Find $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ as a product of 3 factors.

INVESTIGATION 3 ROW PROPERTIES OF 3×3 DETERMINANTS

By definition, $$|\mathbf{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad .... \ (1)$$

In this Investigation we explore how the determinant can be represented in other forms.

What to do:

1 By expanding (1), show that
$$|\mathbf{A}| = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1.$$

2 Show that we can also write:

a $$|\mathbf{A}| = -a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \quad .... \ (2)$$

b $$|\mathbf{A}| = a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad .... \ (3)$$

For the determinant of $\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, we say that (1) is the **expansion by the first row**, (2) is the **expansion by the second row**, and (3) is the **expansion by the third row**.

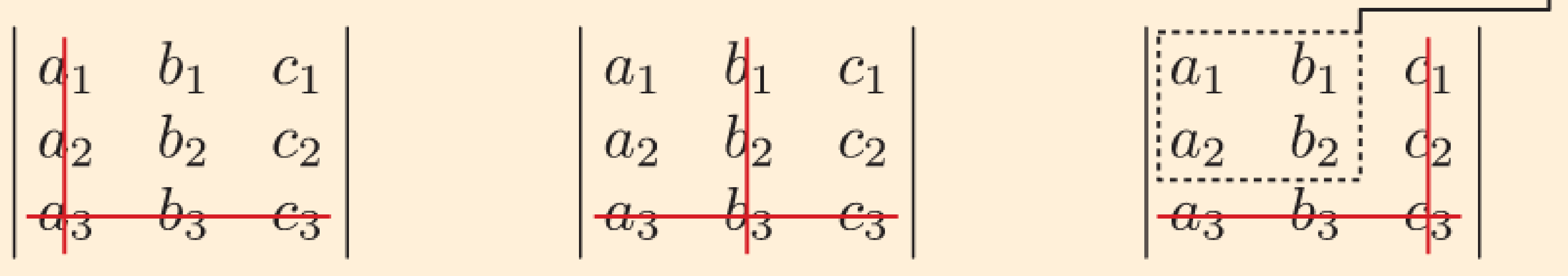
Notice that:

- the signs before each term are: $\begin{array}{ccc|l} + & - & + & \text{in (1)} \\ - & + & - & \text{in (2)} \\ + & - & + & \text{in (3)} \end{array}$
- to construct (3), $$|\mathbf{A}| = a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

cross out row and column of **A** containing a_3

cross out row and column of **A** containing b_3

cross out row and column of **A** containing c_3



DETERMINANT PROPERTIES FOR $n \times n$ MATRICES

- If matrix **B** is obtained from matrix **A** by interchanging two rows (or columns) then $|\mathbf{B}| = -|\mathbf{A}|$.
- If **A** has a row (or column) which is all zeros then $|\mathbf{A}| = 0$.
- If **A** has two identical rows (or columns) then $|\mathbf{A}| = 0$.
- If **B** is obtained from **A** by multiplying one row (or column) by $k \in \mathbb{R}$, then $|\mathbf{B}| = k|\mathbf{A}|$.
- If **B** is obtained from **A** by adding to one row (or column) a multiple of another row (or column), then $|\mathbf{B}| = |\mathbf{A}|$.
- $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$
- $|\mathbf{A}^T| = |\mathbf{A}|$

EXERCISE 1F.4

1 Check the first five properties using the general 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

2 Check the first five properties using the general 3×3 matrix $\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$.

3 If **A** is 3×3, explain why $|k\mathbf{A}| = k^3|\mathbf{A}|$.

4 Prove that $|\mathbf{A}^T| = |\mathbf{A}|$ for all 3×3 matrices **A**.

5 If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -1 & 1 \\ 4 & 1 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}$, verify that $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$.

6 If **A** is non-singular, prove that $|\mathbf{A}^{-1}| = \dfrac{1}{|\mathbf{A}|}$.

7 If **A** is an orthogonal matrix, prove that $|\mathbf{A}| = \pm 1$.

8 Consider $f(x) = \begin{vmatrix} 1 & x & x^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ and $\mathbf{A} = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$.

   a Prove that $f(x)$ has factors $(x-b)$ and $(x-c)$.

   b Hence prove that $|\mathbf{A}|$ has a factor $(a-b)(a-c)$.

   c Prove that $|\mathbf{A}|$ has a factor $(b-a)(b-c)$.

9 Find **A** given that $(\mathbf{A}^T - 3\mathbf{I})^{-1} = \begin{pmatrix} 2 & 4 & 1 \\ 1 & 0 & 1 \\ 3 & 3 & 2 \end{pmatrix}$.

G SOLVING SYSTEMS OF LINEAR EQUATIONS USING MATRICES

The systems of linear equations $\begin{cases} 3x - 2y = 10 \\ -x + 3y = -1 \end{cases}$

can be written in the form $\begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$.

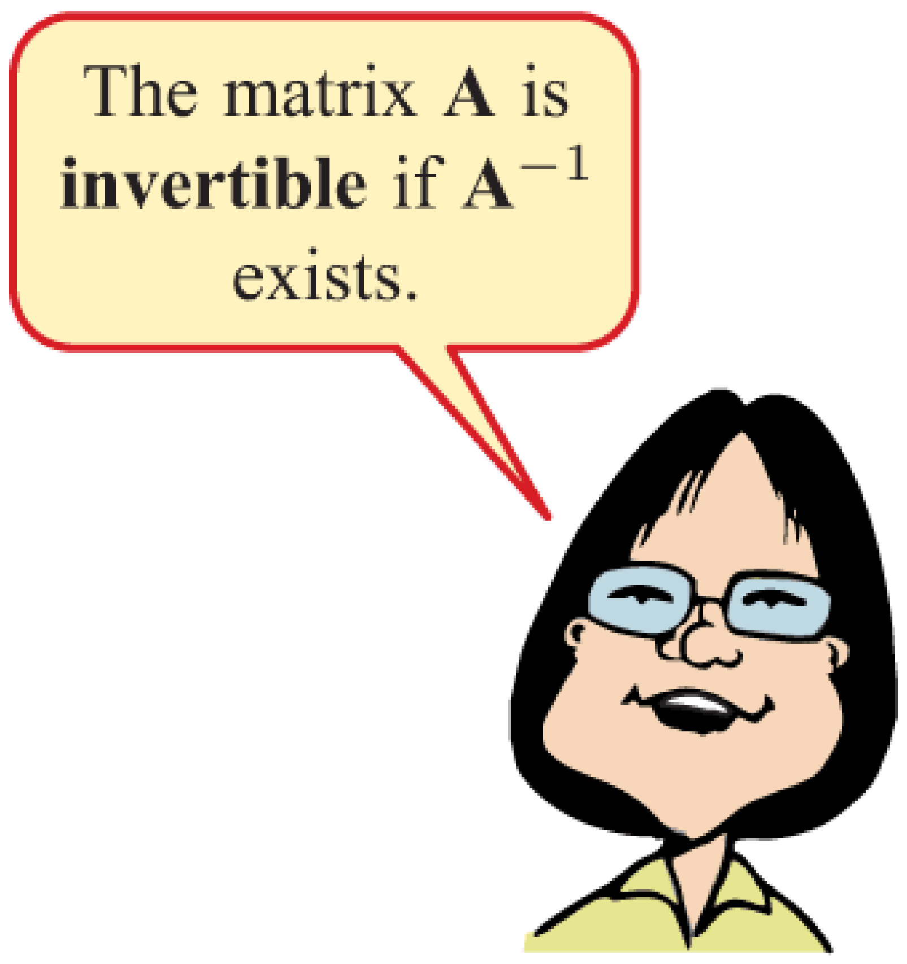
The solution $x = 4$, $y = 1$ can be checked using the matrix multiplication $\begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$.

An $n \times n$ system of linear equations can be written in the form $\mathbf{AX} = \mathbf{B}$ where $\mathbf{A}$ is an $n \times n$ square matrix of coefficients, $\mathbf{X}$ is an $n \times 1$ column matrix of variables, and $\mathbf{B}$ is an $n \times 1$ column matrix of constants.

Consider such a system of the form $\mathbf{AX} = \mathbf{B}$.

If the square matrix $\mathbf{A}$ is invertible, then the system has a unique solution which can be found as follows:

$$\begin{aligned} \mathbf{AX} &= \mathbf{B} \\ \therefore\ \mathbf{A}^{-1}(\mathbf{AX}) &= \mathbf{A}^{-1}\mathbf{B} \qquad \{\text{premultiplying by } \mathbf{A}^{-1}\} \\ \therefore\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$


The matrix **A** is **invertible** if $\mathbf{A}^{-1}$ exists.

If the square matrix is *not* invertible, then the system does not have a unique solution.

Example 19

Solve using matrices: $\begin{cases} 3x - 2y = 10 \\ -x + 3y = -1 \end{cases}$

In matrix form, the system is $\begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$.

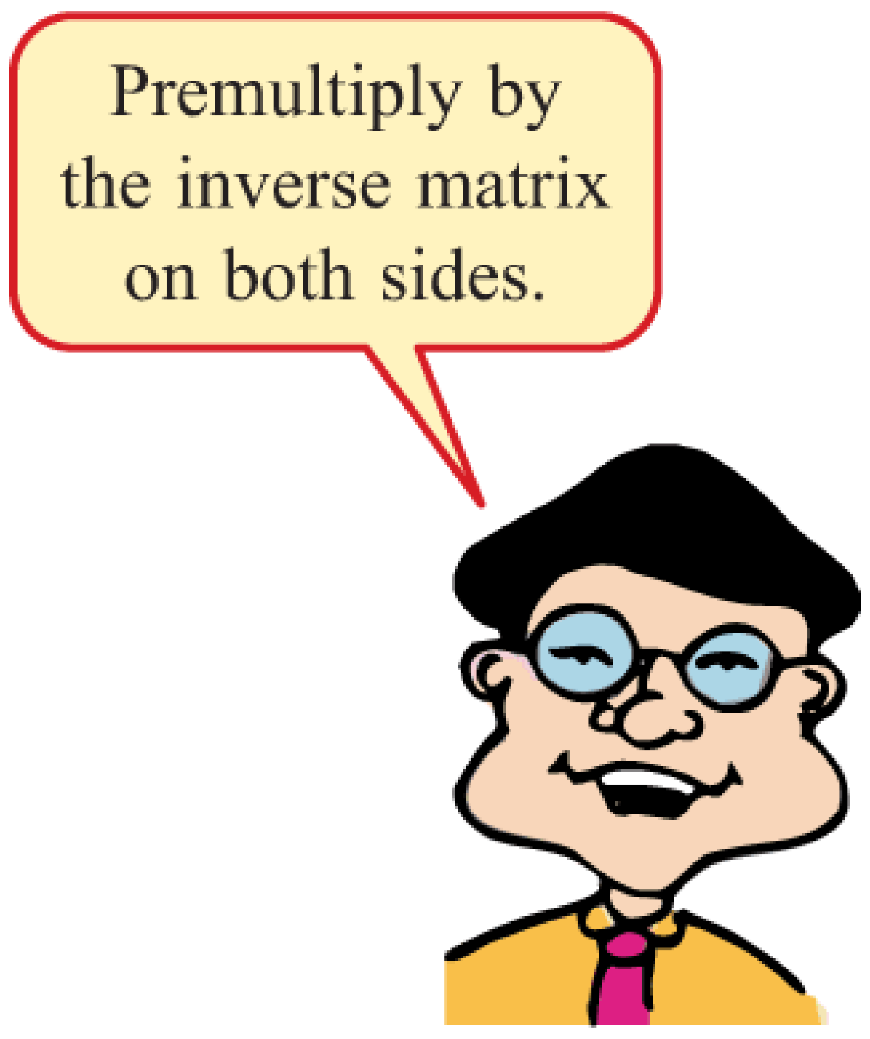
If $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix}$ then $|\mathbf{A}| = (3)(3) - (-2)(-1) = 7$

$$\therefore\ \mathbf{A}^{-1} = \tfrac{1}{7}\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 10 \\ -1 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{7}\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 10 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\therefore\ x = 4,\ y = 1$$


Premultiply by the inverse matrix on both sides.

EXERCISE 1G

1 Solve using matrices:

a $\begin{cases} 2x+4y=-6 \\ 5x-y=7 \end{cases}$ **b** $\begin{cases} x-3y=13 \\ -3x-2y=5 \end{cases}$ **c** $\begin{cases} 5x+2y=3 \\ -x-3y=15 \end{cases}$ **d** $\begin{cases} -2x+5y=4 \\ 3x-2y=20 \end{cases}$

2 Consider the system $\begin{cases} -3x+y=7 \\ 6x-2y=-7. \end{cases}$

a Write the system in the form $\mathbf{AX}=\mathbf{B}$ where $\mathbf{X}=\begin{pmatrix} x \\ y \end{pmatrix}$.

b Find $|\mathbf{A}|$. What does this tell us about the system?

3 Consider the system $\begin{cases} 2x-ky=4 \\ -2x+3y=-4. \end{cases}$

a Write the system in the form $\mathbf{AX}=\mathbf{B}$ where $\mathbf{X}=\begin{pmatrix} x \\ y \end{pmatrix}$.

b Find $|\mathbf{A}|$. What does this tell us about the system?

4 Consider the matrix equation $\begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}\mathbf{X}=\begin{pmatrix} 5 & 0 \\ -3 & 14 \end{pmatrix}$.

a Does this equation represent a system of linear equations? Explain your answer.

b Find $\mathbf{X}$ using an inverse matrix.

5 Show that if $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions of $\mathbf{AX}=\mathbf{B}$ then $\mathbf{X}_3=t\mathbf{X}_1+(1-t)\mathbf{X}_2$ is also a solution of $\mathbf{AX}=\mathbf{B}$, for all $t\in\mathbb{R}$.
Explain the significance of this result.

6 For what values of k does the system have a unique solution?

a $\begin{cases} x+2y-3z=5 \\ 2x-y-z=8 \\ kx+y+2z=14 \end{cases}$ **b** $\begin{cases} 2x-y-4z=8 \\ 3x-ky+z=1 \\ 5x-y+kz=-2 \end{cases}$

Example 20

Solve the system $\begin{cases} x-y-z=2 \\ x+y+3z=7 \\ 9x-y-3z=-1 \end{cases}$ using matrix methods and a graphics calculator.

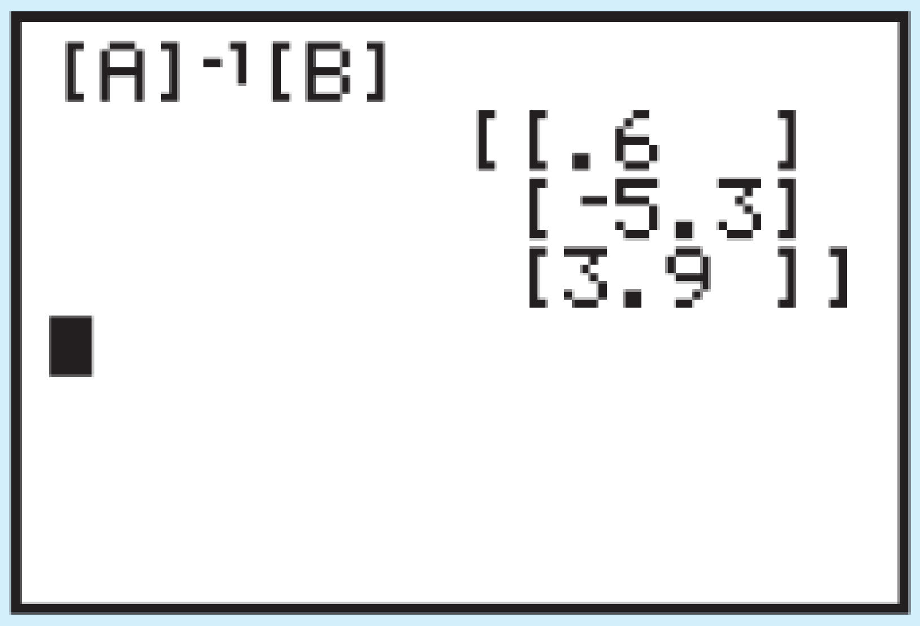
GRAPHICS CALCULATOR INSTRUCTIONS

In matrix form, the system is: $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}$

The system has the form $\mathbf{A}\quad\mathbf{X}\quad=\quad\mathbf{B}$, so $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$.

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}=\begin{pmatrix} 0.6 \\ -5.3 \\ 3.9 \end{pmatrix}$$

$\therefore\ x=0.6,\ y=-5.3,\ z=3.9$



```
[A]⁻¹[B]
        [[.6   ]
         [-5.3]
         [3.9 ]]
█
```

7 Use matrix methods and technology to solve:

a $$\begin{cases} 3x+2y-z=14 \\ x-y+2z=-8 \\ 2x+3y-z=13 \end{cases}$$

b $$\begin{cases} x-y-2z=4 \\ 5x+y+2z=-6 \\ 3x-4y-z=17 \end{cases}$$

c $$\begin{cases} x+3y-z=15 \\ 2x+y+z=7 \\ x-y-2z=0 \end{cases}$$

d $$\begin{cases} x+2y-z=23 \\ x-y+3z=-23 \\ 7x+y-4z=62 \end{cases}$$

e $$\begin{cases} 10x-y+4z=-9 \\ 7x+3y-5z=89 \\ 13x-17y+23z=-309 \end{cases}$$

f $$\begin{cases} 1.3x+2.7y-3.1z=8.2 \\ 2.8x-0.9y+5.6z=17.3 \\ 6.1x+1.4y-3.2z=-0.6 \end{cases}$$

8 Describe the limitations of using matrix methods for solving systems of linear equations.

Example 21

A rental company has three different makes of car for hire: P, Q, and R. These cars are located at yards A and B on either side of a city, or else are being rented. In total they have 150 cars. At yard A they have 20% of P, 40% of Q, and 30% of R, which is 46 cars in total. At yard B they have 40% of P, 20% of Q, and 50% of R, which is 54 cars in total. How many of each car type does the company have?

BOB'S CAR RENTAL

Suppose the company has x of P, y of Q, and z of R.

It has 150 cars in total, so $x+y+z=150$ .... (1)

Yard A has 20% of P + 40% of Q + 30% of R, and this is 46 cars.

$$\therefore \ \tfrac{2}{10}x+\tfrac{4}{10}y+\tfrac{3}{10}z=46$$

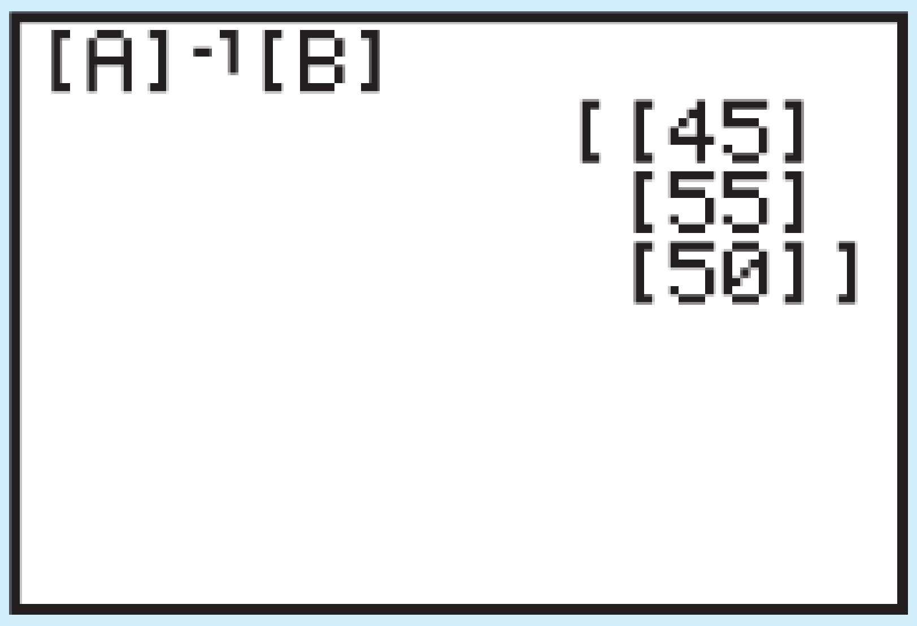
$$\therefore \ 2x+4y+3z=460 \quad .... \ (2)$$

Yard B has 40% of P + 20% of Q + 50% of R, and this is 54 cars.

$$\therefore \ \tfrac{4}{10}x+\tfrac{2}{10}y+\tfrac{5}{10}z=54$$

$$\therefore \ 4x+2y+5z=540 \quad .... \ (3)$$

We need to solve the system: $$\begin{cases} x+y+z=150 \\ 2x+4y+3z=460 \\ 4x+2y+5z=540 \end{cases}$$



```
[A]⁻¹[B]
          [[45]
           [55]
           [50]]
```

In matrix form, we write:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 4 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 150 \\ 460 \\ 540 \end{pmatrix}$$

$$\therefore \ \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 4 & 2 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 150 \\ 460 \\ 540 \end{pmatrix}=\begin{pmatrix} 45 \\ 55 \\ 50 \end{pmatrix} \quad \{\text{using technology}\}$$

$\therefore$ the company has 45 of P, 55 of Q, and 50 of R.

9 Managers, clerks, and labourers are paid according to an industry award.
Xenon employs 2 managers, 3 clerks, and 8 labourers with a total salary bill of €352 000.
Xanda employs 1 manager, 5 clerks, and 4 labourers with a total salary bill of €274 000.
Xylon employs 1 manager, 2 clerks, and 11 labourers with a total salary bill of €351 000.
Let x, y, and z represent the salaries (in thousands of euros) for managers, clerks, and labourers respectively.

- **a** Write the above information as a system of three equations.
- **b** Solve the system of equations.
- **c** Determine the total salary bill for the company Xulu which employs 3 managers, 8 clerks, and 37 labourers according to the industry award.

10 A mixed nut company uses cashews, macadamias, and Brazil nuts to make three gourmet mixes. The table alongside indicates the weight in hundreds of grams of each kind of nut required to make a kilogram of mix.

	Mix A	Mix B	Mix C
Cashews	5	2	6
Macadamias	3	4	1
Brazil nuts	2	4	3

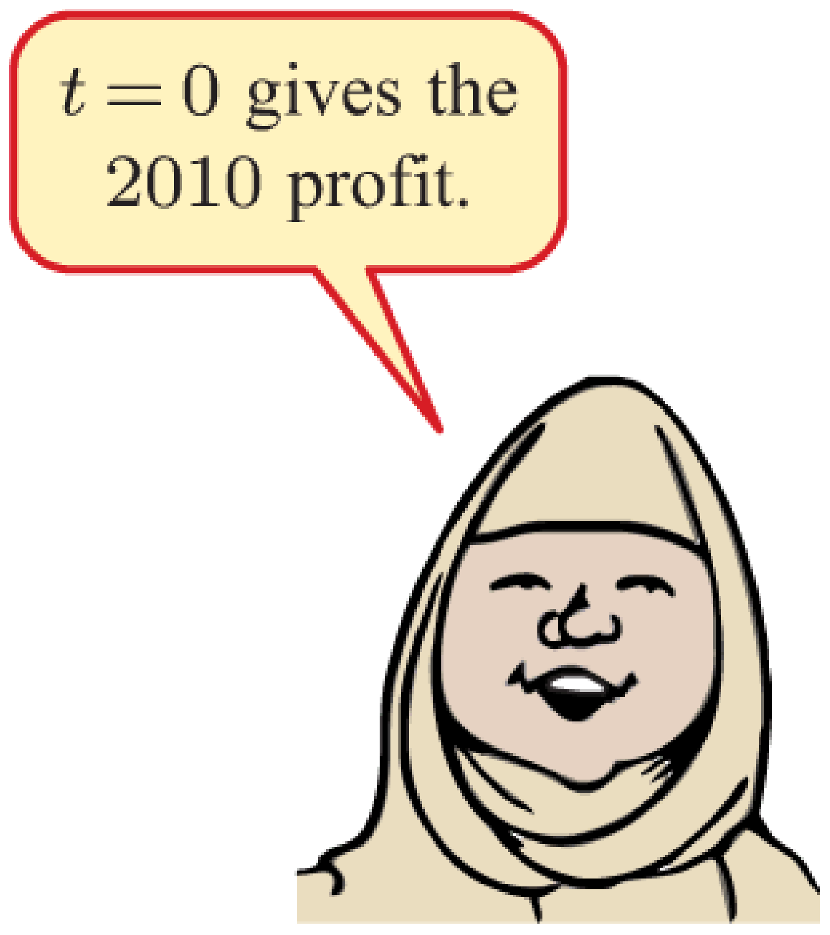
1 kg of mix A cost \$12.50 to produce, 1 kg of mix B costs \$12.40, and 1 kg of mix C costs \$11.70.

- **a** Determine the cost per kilogram of each of the different kinds of nuts.
- **b** Hence, find the cost per kilogram to produce a mix containing 400 grams of cashews, 200 grams of macadamias, and 400 grams of Brazil nuts.

11 Susan and Elki opened a new business in 2007. Their annual profit was £160 000 in 2010, £198 000 in 2011, and £240 000 in 2012. Based on this information, they believe that their annual profit can be predicted by the model

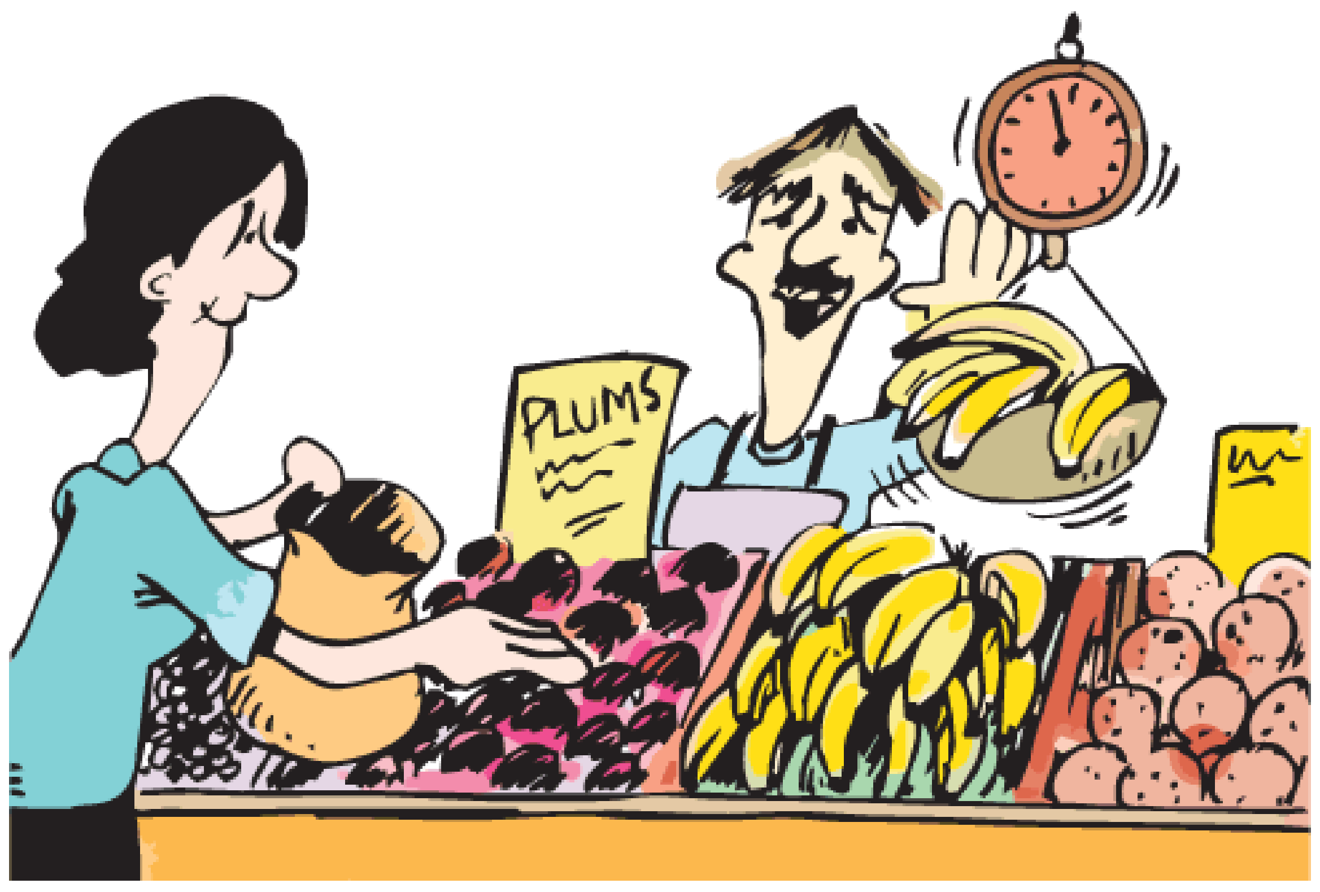
$$P(t) = at + b + \frac{c}{t+4} \text{ pounds}$$

where t is the number of years after 2010.


$t = 0$ gives the 2010 profit.

- **a** Find the values of a, b, and c which fit the profits for 2010, 2011, and 2012.
- **b** The profit in 2009 was £130 000. Does this profit fit the model in **a**?
- **c** Susan and Elki believe their profit will continue to grow according to this model. Predict their profit in 2013 and 2015.

12 If Jan bought one orange, two apples, a pear, a cabbage, and a lettuce, the total cost would be \$6.30. Two oranges, one apple, two pears, one cabbage, and one lettuce would cost a total of \$6.70. One orange, two apples, three pears, one cabbage, and one lettuce would cost a total of \$7.70. Two oranges, two apples, one pear, one cabbage, and three lettuces would cost a total of \$9.80. Three oranges, three apples, five pears, two cabbages, and two lettuces would cost a total of \$10.90.



- **a** Write this information in the form $\mathbf{AX} = \mathbf{B}$ where $\mathbf{A}$ is a quantities matrix, $\mathbf{X}$ is the cost per item column matrix, and $\mathbf{B}$ is the total costs column matrix.
- **b** Explain why $\mathbf{X}$ cannot be found from the given information.
- **c** If the last line of information was replaced with "three oranges, one apple, two pears, two cabbages, and one lettuce cost a total of \$9.20", can the system be solved now? If so, what is the solution?

ACTIVITY CRYPTOGRAPHY

Cryptography is the study of encoding and decoding messages. Cryptography was first developed for the military to send secret messages. Today it is also used to maintain privacy when information is transmitted on public communication services such as the internet.

To send a coded message, it must first be **encrypted** into code called **ciphertext**. When the recipient wishes to read the message, the ciphertext must be **deciphered**.

A simple method for encrypting messages is to use matrix addition or multiplication. The messages are then deciphered using either matrix subtraction or an inverse matrix.

Suppose the letters of the alphabet are assigned integer values, with Z assigned 0 as shown below:

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	0

The word SEND can be written as the string of numbers 19 5 14 4 which we can write in 2×2 matrix form $\begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix}$.

Now suppose we encrypt the message by adding the matrix $\begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix}$.

$$\begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 21 & 12 \\ 27 & 9 \end{pmatrix}$$

Before this matrix can be transmitted all of its numbers must be written in **modulo** 26, or mod 26. This means that any number not in the range 0 to 25 is adjusted to be in it by adding or subtracting multiples of 26.

The matrix to be sent is therefore $\begin{pmatrix} 21 & 12 \\ 1 & 9 \end{pmatrix}$, and this is sent as the string 21 12 1 9. By itself, the string says ULAI, which has no apparent meaning.

The message SEND MONEY PLEASE could be broken into groups of four letters, and each group is then encoded.

SEND|MONE|YPLE|ASEE ← repeat the last letter to make group of 4.
This is a dummy letter.

For MONE the matrix required is $\begin{pmatrix} 13 & 15 \\ 14 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 15 & 22 \\ 27 & 10 \end{pmatrix} \equiv \begin{pmatrix} 15 & 22 \\ 1 & 10 \end{pmatrix} \pmod{26}$

For YPLE the matrix required is $\begin{pmatrix} 25 & 16 \\ 12 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 27 & 23 \\ 25 & 10 \end{pmatrix} \equiv \begin{pmatrix} 1 & 23 \\ 25 & 10 \end{pmatrix} \pmod{26}$

For ASEE the matrix required is $\begin{pmatrix} 1 & 19 \\ 5 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 26 \\ 18 & 10 \end{pmatrix} \equiv \begin{pmatrix} 3 & 0 \\ 18 & 10 \end{pmatrix} \pmod{26}$

So, the whole message is 21 12 1 9 15 22 1 10 1 23 25 10 3 0 18 10

The person decoding the message needs to know in advance to subtract $\begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix}$ from each matrix of numbers, in order to decipher the message.

What to do:

1 Perform the matrix subtractions to check that the original message is obtained.

2 Use the code given to decode the message:

21 12 1 9 22 15 18 25 20 22 2 21 21 1 2
25 10 12 0 20 23 1 21 20 8 1 21 10 15 2
5 23 3 6 12 4

3 Create your own matrix addition code. Encrypt a short message. Supply the decoding matrix to a friend so that he or she can decode it.

4 A code is **broken** when someone discovers how the messages can be decoded. Breaking codes involving matrix addition is relatively easy, but breaking codes involving matrix multiplication is more difficult.

Consider the encryption matrix $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$.

The word SEND is encoded as

$$\begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 43 & 67 \\ 32 & 50 \end{pmatrix} \equiv \begin{pmatrix} 17 & 15 \\ 6 & 24 \end{pmatrix} (\text{mod}\, 26).$$

a What is the coded form of SEND MONEY PLEASE?

b What matrix needs to be supplied to the receiver so that the message can be deciphered? Check your answer by decoding the message.

c Create your own matrix multiplication code using a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc = 1$.

d What are the problems in using a 2×2 matrix when $ad - bc \neq 1$? How can these problems be overcome?

5 Research **Hill ciphers** and explain how they differ from the methods given previously.

H ELEMENTARY MATRICES

When we solved systems of linear equations using Gaussian elimination, we saw that the process required systematic application of elementary row operations. If we need to solve a very large system of equations, it is too time consuming to perform the operations by hand. There are also no easy rules for a computer to follow in order to obtain the inverse of the matrix of coefficients.

Instead, we can program a computer to perform Gaussian elimination for us. To do this we premultiply the matrix of coefficients by a series of **elementary matrices**, each of which achieves an elementary row operation. We will see that if we also add a **check matrix**, the procedure will also (eventually) provide us with the inverse of the matrix of coefficients, if such a matrix exists.

An **elementary matrix E**, is a square matrix which, when postmultiplied by matrix **A**, achieves an elementary row operation.

Consider the 3×4 matrix $\mathbf{A}=\begin{pmatrix}1&2&3&4\\-2&0&1&6\\4&1&-1&2\end{pmatrix}$.

Since **A** has 3 rows, **E** must be 3×3.

Notice that:

- If $\mathbf{E}=\begin{pmatrix}1&0&0\\0&5&0\\0&0&1\end{pmatrix}$ then $\mathbf{EA}=\begin{pmatrix}1&0&0\\0&5&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\-2&0&1&6\\4&1&-1&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\-10&0&5&30\\4&1&-1&2\end{pmatrix}$.

  E has multiplied row 2 of **A** by 5, so **E** is equivalent to $5R_2\to R_2$.

- If $\mathbf{E}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$ then $\mathbf{EA}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&2&3&4\\-2&0&1&6\\4&1&-1&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&1&-1&2\\-2&0&1&6\end{pmatrix}$.

  E has interchanged rows 2 and 3 of **A**, so **E** is equivalent to $R_2\leftrightarrow R_3$.

- If $\mathbf{E}=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}$ then $\mathbf{EA}=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\-2&0&1&6\\4&1&-1&2\end{pmatrix}=\begin{pmatrix}1&2&3&4\\0&4&7&14\\4&1&-1&2\end{pmatrix}$.

  E has replaced row 2 with 2 row 1 + row 2, so **E** is equivalent to $2R_1+R_2\to R_2$.

From examples like those above we observe that:

- If a matrix **A** has n rows, then any elementary matrix **E** must be $n\times n$.
- To find an elementary matrix **E**, we start with the identity matrix $\mathbf{I}_n$. Then:
  - to swap R_i with R_j, we swap row i and row j in $\mathbf{I}_n$
  - to replace R_i by kR_i, we replace the 1 in row i by k
  - to replace R_i by R_i+aR_j, we replace the 0 in row i, column j with a.

GAUSSIAN ELIMINATION WITH CHECK MATRIX

Suppose an augmented matrix $(\mathbf{A}\,|\,\mathbf{B})$ can be converted into reduced row echelon form using a series of elementary matrices $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, ..., $\mathbf{E}_k$.

The final matrix in this procedure will be $\mathbf{E}_k\mathbf{E}_{k-1}\,....\,\mathbf{E}_2\mathbf{E}_1(\mathbf{A}\,|\,\mathbf{B})$.

The matrix $\mathbf{W} = \mathbf{E}_k\mathbf{E}_{k-1}\ldots\mathbf{E}_2\mathbf{E}_1$ is called the **check matrix**. It can be found by inserting $\mathbf{I}_n$ between $\mathbf{A}$ and $\mathbf{B}$ in the augmented matrix $(\mathbf{A}\,|\,\mathbf{B})$.

Under Gaussian elimination, $(\mathbf{A}\,|\,\mathbf{B})$ becomes $\mathbf{W}(\mathbf{A}\,|\,\mathbf{B}) = (\mathbf{WA}\,|\,\mathbf{WB})$,

and so $(\mathbf{A}\,|\,\mathbf{I}\,|\,\mathbf{B})$ becomes $\mathbf{W}(\mathbf{A}\,|\,\mathbf{I}\,|\,\mathbf{B}) = (\mathbf{WA}\,|\,\mathbf{W}\,|\,\mathbf{WB})$

$(\mathbf{WA}\,|\,\mathbf{W}\,|\,\mathbf{WB})$ is the **reduced row echelon form with check matrix**.

Example 22

For the system $\begin{cases} x_1 + x_2 = 3 \\ x_1 + 2x_2 = 5 \\ 2x_1 - x_2 = 4\,, \end{cases}$ find the reduced row echelon form with check matrix.

Write down the corresponding elementary matrix at each step.

For the given system,

$$(\mathbf{A}\,|\,\mathbf{I}\,|\,\mathbf{B})$$

$$= \left(\begin{array}{cc|ccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ 1 & 2 & 0 & 1 & 0 & 5 \\ 2 & -1 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|ccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 2 & -1 & 0 & 0 & 1 & 4 \end{array}\right) \qquad R_2 - R_1 \to R_2 \qquad \mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \left(\begin{array}{cc|ccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & -3 & -2 & 0 & 1 & -2 \end{array}\right) \qquad R_3 - 2R_1 \to R_3 \qquad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\sim \left(\begin{array}{cc|ccc|c} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & -3 & -2 & 0 & 1 & -2 \end{array}\right) \qquad R_1 - R_2 \to R_1 \qquad \mathbf{E}_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}}_{\mathbf{WA}}\;\middle|\;\underbrace{\begin{matrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -5 & 3 & 1 \end{matrix}}_{\mathbf{W}}\;\middle|\;\underbrace{\begin{matrix} 1 \\ 2 \\ 4 \end{matrix}}_{\mathbf{WB}}\right) \qquad R_3 + 3R_2 \to R_3 \qquad \mathbf{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

Check: $$\mathbf{W}(\mathbf{A}\,|\,\mathbf{B}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -5 & 3 & 1 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 2 & -1 & 4 \end{array}\right)$$

$$= \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}}_{\mathbf{WA}}\;\middle|\;\underbrace{\begin{matrix} 1 \\ 2 \\ 4 \end{matrix}}_{\mathbf{WB}}\right)$$

USING GAUSSIAN ELIMINATION TO FIND MATRIX INVERSES

If the square matrix $\mathbf{A}$ is invertible, then its inverse matrix $\mathbf{A}^{-1}$ can be found using Gaussian elimination to reduce $(\mathbf{A}\,|\,\mathbf{I})$ to the form $(\mathbf{I}\,|\,\mathbf{A}^{-1})$.

Proof: Under Gaussian elimination, $(\mathbf{A}\,|\,\mathbf{I})$ becomes $(\mathbf{WA}\,|\,\mathbf{W})$.
If $\mathbf{WA}=\mathbf{I}$ then $\mathbf{W}=\mathbf{A}^{-1}$, and $(\mathbf{A}\,|\,\mathbf{I})$ becomes $(\mathbf{I}\,|\,\mathbf{A}^{-1})$.

Note that if $\mathbf{A}^{-1}$ exists then

$$\begin{aligned}\mathbf{A}&=\mathbf{W}^{-1}\\&=(\mathbf{E}_k\mathbf{E}_{k-1}\,....\,\mathbf{E}_2\mathbf{E}_1)^{-1}\\&=\mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\,....\,\mathbf{E}_{k-1}^{-1}\mathbf{E}_k^{-1}\end{aligned}$$

Example 23

Use elementary row operations to find $\mathbf{A}^{-1}$ for $\mathbf{A}=\begin{pmatrix}0&4&2\\1&1&3\\3&-1&1\end{pmatrix}$.

$$(\mathbf{A}\,|\,\mathbf{I})=\left(\begin{array}{ccc|ccc}0&4&2&1&0&0\\1&1&3&0&1&0\\3&-1&1&0&0&1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}1&1&3&0&1&0\\0&4&2&1&0&0\\3&-1&1&0&0&1\end{array}\right)\qquad R_2\leftrightarrow R_1$$

$$\sim\left(\begin{array}{ccc|ccc}1&1&3&0&1&0\\0&4&2&1&0&0\\0&-4&-8&0&-3&1\end{array}\right)\qquad R_3-3R_1\to R_3$$

$$\sim\left(\begin{array}{ccc|ccc}1&1&3&0&1&0\\0&4&2&1&0&0\\0&0&-6&1&-3&1\end{array}\right)\qquad R_3+R_2\to R_3$$

$$\sim\left(\begin{array}{ccc|ccc}1&1&3&0&1&0\\0&1&\frac{1}{2}&\frac{1}{4}&0&0\\0&0&1&-\frac{1}{6}&\frac{1}{2}&-\frac{1}{6}\end{array}\right)\qquad \begin{array}{l}\frac{1}{4}R_2\to R_2\\-\frac{1}{6}R_3\to R_3\end{array}$$

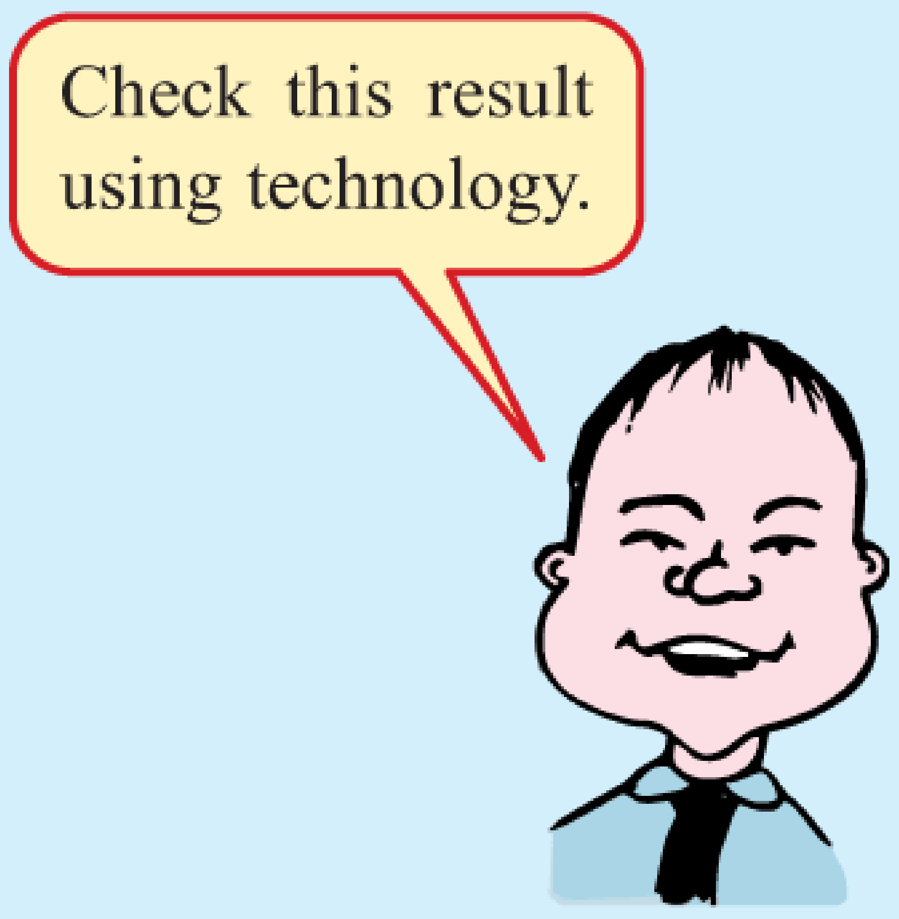
$$\sim\left(\begin{array}{ccc|ccc}1&1&3&0&1&0\\0&1&0&\frac{1}{3}&-\frac{1}{4}&\frac{1}{12}\\0&0&1&-\frac{1}{6}&\frac{1}{2}&-\frac{1}{6}\end{array}\right)\qquad R_2-\tfrac{1}{2}R_3\to R_2$$

$$\sim\left(\begin{array}{ccc|ccc}1&0&3&-\frac{1}{3}&\frac{5}{4}&-\frac{1}{12}\\0&1&0&\frac{1}{3}&-\frac{1}{4}&\frac{1}{12}\\0&0&1&-\frac{1}{6}&\frac{1}{2}&-\frac{1}{6}\end{array}\right)\qquad R_1-R_2\to R_1$$

$$\sim\left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{6}&-\frac{1}{4}&\frac{5}{12}\\0&1&0&\frac{1}{3}&-\frac{1}{4}&\frac{1}{12}\\0&0&1&-\frac{1}{6}&\frac{1}{2}&-\frac{1}{6}\end{array}\right)\qquad R_1-3R_3\to R_1$$

$$=(\mathbf{I}\,|\,\mathbf{A}^{-1})$$

$$\therefore\quad \mathbf{A}^{-1}=\begin{pmatrix}\frac{1}{6}&-\frac{1}{4}&\frac{5}{12}\\\frac{1}{3}&-\frac{1}{4}&\frac{1}{12}\\-\frac{1}{6}&\frac{1}{2}&-\frac{1}{6}\end{pmatrix}$$


Check this result using technology.

EXERCISE 1H

1 State the elementary 3×3 matrix which corresponds to the row operation:

a $R_2 \leftrightarrow R_3$ **b** $-4R_1 \to R_1$ **c** $R_2 - 2R_1 \to R_2$ **d** $R_1 \leftrightarrow R_3$

e $\frac{1}{3}R_3 \to R_3$ **f** $R_3 + \frac{1}{2}R_1 \to R_3$ **g** $R_3 + 5R_2 \to R_3$ **h** $R_2 + 6R_1 \to R_2$

2 State the corresponding elementary row operation for the elementary matrix:

a $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

d $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **e** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

3 Matrix $\mathbf{A} = \begin{pmatrix} -1 & 1 & 4 \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{pmatrix}$ is subjected to the following elementary row operations in succession:

$$R_1 \leftrightarrow R_2, \quad R_2 + R_1 \to R_2, \quad R_3 - 2R_1 \to R_3, \quad \tfrac{1}{3}R_2 \to R_2, \quad R_3 + 5R_2 \to R_3.$$

a What does $\mathbf{A}$ reduce to under these operations?

b Use elementary matrices to show that they produce the same result.

4 For each of the following systems:

i use Gaussian elimination with check matrix to convert the system to reduced row echelon form

ii verify your answer using the check matrix.

a $\begin{cases} x_1 + 3x_2 = 4 \\ 2x_1 - x_2 = 3 \\ 3x_1 - 5x_2 = 2 \end{cases}$ **b** $\begin{cases} x_1 + x_2 + 2x_3 = 1 \\ 2x_1 \qquad - 3x_3 = 4 \end{cases}$ **c** $\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 4 \\ 2x_1 + x_2 + 5x_3 + x_4 = 7 \\ x_2 + x_3 + 6x_4 = 4 \end{cases}$

5 Without using technology, use Gaussian elimination to find the inverse of:

a $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 2 & 3 \\ 2 & -1 & 3 \end{pmatrix}$

d $\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{pmatrix}$

6 There are three types of elementary $n \times n$ matrices:

- $\mathbf{E}_s$ for swapping two rows
- $\mathbf{E}_k$ for multiplying a row by a non-zero constant k
- $\mathbf{E}_a$ for adding a multiple of one row to another row.

a Find the determinants of these elementary matrices:

i $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **ii** $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ **iii** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$

b Prove that:

i $|\mathbf{E}_s| = -1$ **ii** $|\mathbf{E}_k| = k$ **iii** $|\mathbf{E}_a| = 1$

c Hence, prove that every check matrix is invertible.

d Find the inverses of these elementary matrices:

i $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **ii** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **iii** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ **iv** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

v $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ **vi** $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **vii** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ **viii** $\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

e Hence, describe an inverse for each of the $n \times n$ elementary matrix types.

7 Consider $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

a Find elementary matrices $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ such that $\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}_3$.

b Write $\mathbf{A}$ as a product of elementary matrices.

8 Given $\mathbf{A} = \begin{pmatrix} a & b & a \\ 0 & c & a \\ 0 & 0 & d \end{pmatrix}$ where $a, c, d \neq 0$, use Gaussian elimination to find $\mathbf{A}^{-1}$.

9 **a** Given $\mathbf{A} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ where $a, d, f \neq 0$, use Gaussian elimination to find $\mathbf{A}^{-1}$.

b Suppose matrix $\mathbf{B}$ is obtained from matrix $\mathbf{A}$ by an elementary row operation. Show that if $\mathbf{C}$ is of appropriate size then $\mathbf{BC}$ is obtained from $\mathbf{AC}$ under the same elementary operation.

I VECTOR SPACES

An $n \times 1$ column matrix is commonly referred to as a **vector**.

We commonly use vectors to describe the coordinates of points in space.

For example:

- in 2-dimensions, the point $(3,\ 4)$ is given the position vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$
- in 3-dimensions, the point $(2,\ -1,\ 5)$ is given the position vector $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.

$\mathbb{R}^n$ is called the **real Cartesian space** in n-dimensions.

For example:

- $\mathbb{R} = \mathbb{R}^1$ is the set of real numbers on the number line
- $\mathbb{R}^2$ is the 2-dimensional or Cartesian plane
- $\mathbb{R}^3$ is the 3-dimensional Cartesian space.

EQUALITY AND OPERATIONS IN $\mathbb{R}^n$

Using our knowledge of matrices, we can define vector equality and operations between vectors as follows:

If $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$, and $c \in \mathbb{R}$:

- **equality** $\mathbf{u} = \mathbf{v} \Leftrightarrow u_i = v_i$ for all $i = 1,\ 2,\ 3,\ ....,\ n$.
- **addition** $\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_n + v_n \end{pmatrix}$
- **scalar multiplication** $c\mathbf{u} = \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \\ \vdots \\ cu_n \end{pmatrix}$
- **zero vector** $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
- **negative vector** $-\mathbf{v} = -1\mathbf{v} = \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \\ \vdots \\ -v_n \end{pmatrix}$

VECTOR PROPERTIES IN $\mathbb{R}^n$

For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$, and constants $c_1, c_2, k \in \mathbb{R}$, the following properties apply:

- $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ {commutative}
- $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ {associative}
- $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ {additive identity}
- $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$ {inverse}
- $c_1(c_2\mathbf{u})=c_1c_2\mathbf{u}$
- $(c_1+c_2)\mathbf{u}=c_1\mathbf{u}+c_2\mathbf{u}$
- $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

These properties are provable using the definitions of operations above.

Example 24

Prove that for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $k \in \mathbb{R}$, $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$.

$\mathbf{u}$, $\mathbf{v}$ are $n \times 1$ matrices, so we let $\mathbf{u}=(u_{i1})$ and $\mathbf{v}=(v_{i1})$, $i=1, 2, 3, ..., n$.

Now $k\mathbf{u}+k\mathbf{v}$

$$\begin{aligned}
&= k(u_{i1})+k(v_{i1}) \\
&= k[(u_{i1})+(v_{i1})] \\
&= k(u_{i1}+v_{i1}) \qquad \{\text{addition}\} \\
&= k(\mathbf{u}+\mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n, \ k \in \mathbb{R}.
\end{aligned}$$

ORTHOGONALITY

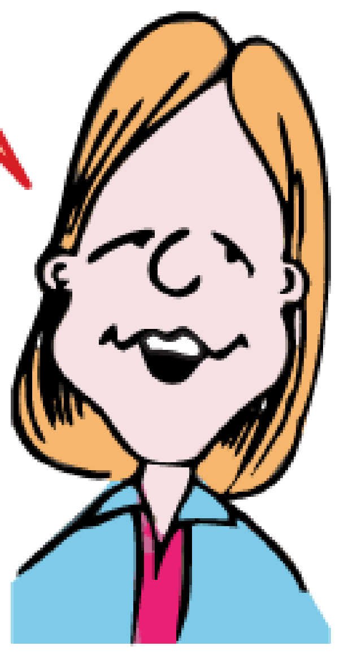
If $\mathbf{v}_1=(a_i)$ and $\mathbf{v}_2=(b_i)$ are $n \times 1$ matrices then the **vector dot product** of $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$\mathbf{v}_1 \bullet \mathbf{v}_2 = \mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2 = \sum_{i=1}^{n} a_ib_i = a_1b_1+a_2b_2+....+a_nb_n.$$

For example, if $\mathbf{v}_1=\begin{pmatrix}3\\-2\\1\end{pmatrix}$ and $\mathbf{v}_2=\begin{pmatrix}-1\\1\\4\end{pmatrix}$

then $\mathbf{v}_1 \bullet \mathbf{v}_2 = \mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2 = \begin{pmatrix}3 & -2 & 1\end{pmatrix}\begin{pmatrix}-1\\1\\4\end{pmatrix}$

$$\begin{aligned}
&= (3)(-1)+(-2)(1)+(1)(4) \\
&= -1
\end{aligned}$$



You will study vector dot product more thoroughly in the HL Core course.

Two non-zero $n \times 1$ vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are **orthogonal** if $\mathbf{v}_1 \bullet \mathbf{v}_2 = \mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2 = 0$.

Geometrically, if two vectors are orthogonal then they are perpendicular. This result is proven in the HL Core course.

Three or more non-zero $n \times 1$ vectors are **mutually orthogonal** if each is orthogonal to every other.

For example, if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are mutually orthogonal then $\mathbf{v}_1 \bullet \mathbf{v}_2 = \mathbf{v}_1 \bullet \mathbf{v}_3 = \mathbf{v}_2 \bullet \mathbf{v}_3 = 0$.

Example 25

Determine whether the vectors $\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\begin{pmatrix}-1\\0\\2\end{pmatrix}$, and $\begin{pmatrix}0\\1\\1\end{pmatrix}$ are mutually orthogonal.

$$\begin{pmatrix}2\\-1\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\0\\2\end{pmatrix} = -2+0+2=0 \qquad \begin{pmatrix}2\\-1\\1\end{pmatrix}\bullet\begin{pmatrix}0\\1\\1\end{pmatrix} = 0-1+1=0$$

$$\begin{pmatrix}-1\\0\\2\end{pmatrix}\bullet\begin{pmatrix}0\\1\\1\end{pmatrix} = 0+0+2=2$$

$\therefore$ the vectors are *not* mutually orthogonal.

LINEAR COMBINATIONS OF VECTORS

Vector $\mathbf{w}$ is a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_k$ if it can be written in the form $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + ... + c_k\mathbf{v}_k$ for some constants $c_i \in \mathbb{R}$.

Example 26

Show that $\begin{pmatrix}-1\\0\\0\end{pmatrix}$ is a linear combination of $\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\begin{pmatrix}2\\-1\\1\end{pmatrix}$, and $\begin{pmatrix}3\\0\\1\end{pmatrix}$.

Suppose $\begin{pmatrix}-1\\0\\0\end{pmatrix} = c_1\begin{pmatrix}1\\2\\0\end{pmatrix} + c_2\begin{pmatrix}2\\-1\\1\end{pmatrix} + c_3\begin{pmatrix}3\\0\\1\end{pmatrix}$ for some $c_i \in \mathbb{R}$.

Equating elements, we have the system of linear equations $\left\{\begin{array}{rl} c_1+2c_2+3c_3 &= -1\\ 2c_1 - c_2 &= 0\\ c_2 + c_3 &= 0.\end{array}\right.$

Using technology, we obtain the unique solution $c_1 = 1$, $c_2 = 2$, $c_3 = -2$.

Thus $\begin{pmatrix}-1\\0\\0\end{pmatrix} = 1\begin{pmatrix}1\\2\\0\end{pmatrix} + 2\begin{pmatrix}2\\-1\\1\end{pmatrix} - 2\begin{pmatrix}3\\0\\1\end{pmatrix}$, which is a linear combination of the vectors.

From the **Example**, we write the linear combination as

$$\begin{aligned}\begin{pmatrix}-1\\0\\0\end{pmatrix} &= 1\begin{pmatrix}1\\2\\0\end{pmatrix} + 2\begin{pmatrix}2\\-1\\1\end{pmatrix} - 2\begin{pmatrix}3\\0\\1\end{pmatrix}\\ &= \begin{pmatrix}1&2&3\\2&-1&0\\0&1&1\end{pmatrix}\begin{pmatrix}1\\2\\-2\end{pmatrix}\\ &= (\mathbf{v}_1\,|\,\mathbf{v}_2\,|\,\mathbf{v}_3)\begin{pmatrix}1\\2\\-2\end{pmatrix}\end{aligned}$$

In general,

> If $\mathbf{w}$ is a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_k$, then
>
> $$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + ... + c_k\mathbf{v}_k = (\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \mathbf{v}_3 \,|\, ... \,|\, \mathbf{v}_k)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_k \end{pmatrix}$$
>
> and so $\mathbf{w}$ can be written in the form $\mathbf{Ac}$.

Example 27

a Show that $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$ can be written as a linear combination of $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$.

b Hence write $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$ as a matrix product.

a Suppose $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} = c_1\mathbf{u} + c_2\mathbf{v}$ for some $c_1, c_2 \in \mathbb{R}$

$$\therefore \quad c_1\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$$

$$\therefore \quad \left\{ \begin{aligned} 2c_1 + c_2 &= 4 \\ 3c_1 + 4c_2 &= 1 \\ c_1 - c_2 &= 5 \end{aligned} \right. \quad \text{which has augmented matrix} \quad \left(\begin{array}{cc|c} 2 & 1 & 4 \\ 3 & 4 & 1 \\ 1 & -1 & 5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

$$\therefore \quad c_1 = 3 \text{ and } c_2 = -2$$

Thus $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$, which is a linear combination of $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$.

b $\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} = (\mathbf{v}_1 \,|\, \mathbf{v}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

BASIC UNIT VECTORS

In $\mathbb{R}^2$, any vector is a linear combination of the unit vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

In $\mathbb{R}^3$, any vector is a linear combination of the unit vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

In $\mathbb{R}^n$, any vector is a linear combination of the **basic unit vectors** $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, ..., \mathbf{e}_n$ where $\mathbf{e}_i$ consists of 1 in the ith position and zeros everywhere else.

EXERCISE 1I.1

1 Prove that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$:

a $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ **b** $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ **c** $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

2 Prove that for all $\mathbf{u} \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$:

a $c_1(c_2\mathbf{u}) = c_1c_2\mathbf{u}$ **b** $(c_1+c_2)\mathbf{u} = c_1\mathbf{u}+c_2\mathbf{u}$

3 Determine whether the following sets of vectors are mutually orthogonal:

a $\begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$

c $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

4 Explain geometrically why any set of 3 vectors in $\mathbb{R}^2$ cannot be mutually orthogonal.

5 Find k such that:

a $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ k \end{pmatrix}$ are orthogonal

b $\begin{pmatrix} 3 \\ 1 \\ k \end{pmatrix}, \begin{pmatrix} 1 \\ k-1 \\ 1 \end{pmatrix}, \begin{pmatrix} k \\ -4 \\ -7 \end{pmatrix}$ are mutually orthogonal.

6 Write $\begin{pmatrix} 8 \\ -3 \\ -15 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$.

7 **a** Write $\begin{pmatrix} -3 \\ -2 \\ 9 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$.

b Hence write $\begin{pmatrix} -3 \\ -2 \\ 9 \end{pmatrix}$ as a matrix product.

8 **a** Write $\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

b Hence write $\begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$ as a matrix product.

9 Show that $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \Leftrightarrow a = 2b$.

10 **a** Show that $\begin{pmatrix}3\\-2\\0\end{pmatrix}$ cannot be written as a linear combination of $\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\-8\end{pmatrix}$.

b Give a geometrical interpretation of the result in **a**.

11 Can every vector in $\mathbb{R}^3$ be written as a linear combination of $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$, and $\begin{pmatrix}0\\0\\2\end{pmatrix}$?

Explain your answer.

12 Prove that in $\mathbb{R}^n$, the set of basic unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots., \mathbf{e}_n$ where $\mathbf{e}_i$ consists of 1 in the ith position and zeros everywhere else, is mutually orthogonal.

SUBSPACES

A **subspace of** $\mathbb{R}^n$ is a *non-empty* subset W of $\mathbb{R}^n$ such that W is closed under vector addition and scalar multiplication.

By "closed under vector addition and scalar multiplication", we mean that the sum of any two vectors in W is also in W, and any scalar multiple of a vector in W is also in W.

To show that W is a subspace of $\mathbb{R}^n$ we need to establish that:

(1) W is non-empty

(2) for every $\mathbf{u}, \mathbf{v} \in W$ and $c \in \mathbb{R}$, $\mathbf{u} + \mathbf{v} \in W$ and $c\mathbf{u} \in W$.

Every subspace of $\mathbb{R}^n$ contains the zero vector $\mathbf{0}$.

Example 28

Show that $W = \left\{ \begin{pmatrix}x\\y\\x+y\end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

(1) If $x = y = 0$, $\mathbf{0} = \begin{pmatrix}0\\0\\0\end{pmatrix} \in \mathbb{R}^3 \Rightarrow W$ is non-empty.

(2) Let $\mathbf{u} = \begin{pmatrix}x_1\\y_1\\x_1+y_1\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}x_2\\y_2\\x_2+y_2\end{pmatrix}$ be in W, and let $c \in \mathbb{R}$

$$\therefore \quad \mathbf{u}+\mathbf{v} = \begin{pmatrix}x_1\\y_1\\x_1+y_1\end{pmatrix} + \begin{pmatrix}x_2\\y_2\\x_2+y_2\end{pmatrix} = \begin{pmatrix}(x_1+x_2)\\(y_1+y_2)\\(x_1+x_2)+(y_1+y_2)\end{pmatrix} \text{ which } \in W$$

$$\text{and} \quad c\mathbf{u} = c\begin{pmatrix}x_1\\y_1\\x_1+y_1\end{pmatrix} = \begin{pmatrix}cx_1\\cy_1\\cx_1+cy_1\end{pmatrix} \text{ which is also } \in W.$$

Thus, W is non-empty, and is closed under vector addition and scalar multiplication.

$\Rightarrow$ W is a subspace of $\mathbb{R}^3$.

EXERCISE 11.2

1 **a** Prove that every subspace of $\mathbb{R}^n$ contains $\mathbf{0}$.

b Show that $W = \{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

2 Determine whether $W = \left\{ \begin{pmatrix} x \\ x-3 \\ z \end{pmatrix} \middle| \; x,\, z \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

3 Explain why $\left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \; a=2,\; b=0,\; c+d=2 \right\}$ is not a subspace of $\mathbb{R}^4$ whereas $\left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \; a=0,\; c+d=0 \right\}$ is a subspace of $\mathbb{R}^4$.

4 Prove that $\mathbb{R}^n$ is a subspace of itself.

5 Show that $W = \left\{ \begin{pmatrix} x \\ 2x+1 \\ 0 \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$ is not a subspace of $\mathbb{R}^3$.

6 Consider the homogeneous set of linear equations $\mathbf{A}\mathbf{x} = \mathbf{0}$ where $\mathbf{A}$ is an $m \times n$ matrix.

Show that the solution set $W = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \middle| \; \mathbf{A}\mathbf{x} = \mathbf{0} \right\}$ is a subspace of $\mathbb{R}^n$.

7 Prove that every subspace of:

a $\mathbb{R}^2$ is either $\{\mathbf{0}\}$, a line through O, or $\mathbb{R}^2$

b $\mathbb{R}^3$ is either $\{\mathbf{0}\}$, a line through O, a plane through O, or $\mathbb{R}^3$.

SPANNING SET

W is a **spanning set** of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ if W is the set of *all linear combinations* of these vectors.

$$W = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \ldots + c_k\mathbf{v}_k \mid c_i \in \mathbb{R}\}$$

We write $W = \text{lin}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k\}$

We also say W is the **span** of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k$, or that W is the **linear space spanned** by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k$.

Example 29

Explain what these spanning sets represent:

a $W = \text{lin}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ **b** $W = \text{lin}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ **c** $W = \text{lin}\left\{\begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-1\end{pmatrix}\right\}$

a
$$\begin{aligned} W &= \text{lin}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \\ &= \left\{c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R}\right\} \\ &= \left\{\begin{pmatrix}c_1\\c_2\end{pmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R}\right\} \\ &= \mathbb{R}^2 \end{aligned}$$

b
$$\begin{aligned} W &= \text{lin}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\} \\ &= \left\{c_1\begin{pmatrix}0\\1\\0\end{pmatrix} \,\middle|\, c_1 \in \mathbb{R}\right\} \\ &= \left\{\begin{pmatrix}0\\c_1\\0\end{pmatrix} \,\middle|\, c_1 \in \mathbb{R}\right\} \end{aligned}$$

which is the y-axis in 3-dimensional Cartesian space.

c
$$\begin{aligned} W &= \text{lin}\left\{\begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-1\end{pmatrix}\right\} \\ &= \left\{c_1\begin{pmatrix}3\\1\\0\end{pmatrix} + c_2\begin{pmatrix}2\\0\\-1\end{pmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R}\right\} \\ &= \left\{\begin{pmatrix}3c_1+2c_2\\c_1\\-c_2\end{pmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R}\right\} \end{aligned}$$

Now if $x = 3c_1 + 2c_2$, $y = c_1$, and $z = -c_2$ then

$x = 3y - 2z$

$\therefore\ x - 3y + 2z = 0$

Thus $W = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, x - 3y + 2z = 0\right\}$ which is a plane in $\mathbb{R}^3$ passing through the origin O.

Example 30

Determine whether the vectors $\mathbf{v}_1 = \begin{pmatrix}1\\-2\\1\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}3\\1\\2\end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix}1\\5\\0\end{pmatrix}$ span $\mathbb{R}^3$.

Suppose $\begin{pmatrix}x\\y\\z\end{pmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\begin{pmatrix}1\\-2\\1\end{pmatrix} + c_2\begin{pmatrix}3\\1\\2\end{pmatrix} + c_3\begin{pmatrix}1\\5\\0\end{pmatrix}$

for some $c_1, c_2, c_3 \in \mathbb{R}$

$$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 1 & 5 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

This is a linear system of the form $\mathbf{x} = \mathbf{Ac}$ where
$$\begin{aligned} |\mathbf{A}| &= 1\begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} - 3\begin{vmatrix} -2 & 5 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} \\ &= 1(-10) - 3(-5) + 1(-5) \\ &= 0 \end{aligned}$$

Since $|\mathbf{A}| = 0$, $\mathbf{A}^{-1}$ does not exist.

$\therefore$ $\mathbf{Ac} = \mathbf{x}$ does not have a unique solution $\mathbf{c} = \mathbf{A}^{-1}\mathbf{x}$ for each vector $\mathbf{x}$.

Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ do not span $\mathbb{R}^3$.

Example 31

a Show that $W = \text{lin}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ spans $\mathbb{R}^3$.

b Hence write $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ as a linear combination of the spanning vectors.

a Suppose $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ for some $c_1, c_2, c_3 \in \mathbb{R}$

$$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

This is a linear system of the form $\mathbf{x} = \mathbf{Ac}$ where

$$|\mathbf{A}| = 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0 + 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 + 1 = 2$$

Since $|\mathbf{A}| \neq 0$, $\mathbf{A}^{-1}$ exists, and so a non-trivial solution exists for $\mathbf{c}$

$\Rightarrow$ W spans $\mathbb{R}^3$.

b If $\mathbf{x} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ then $\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \quad \{\text{using technology}\}$$

$$\therefore \quad \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

From these examples, we observe the following theorem on spanning vectors:

If $\mathbf{A} = (\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \mathbf{v}_3 \,|\, ... \,|\, \mathbf{v}_n)$, then the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n$:

- **span** $\mathbb{R}^n$ if $|\,\mathbf{A}\,| \neq 0$
- **do not span** $\mathbb{R}^n$ if $|\,\mathbf{A}\,| = 0$.

EXERCISE 11.3

1 Explain what these spanning sets represent:

a $W = \text{lin}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ **b** $W = \text{lin}\left\{\begin{pmatrix}0\\0\\2\end{pmatrix}\right\}$ **c** $W = \text{lin}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$

d $W = \text{lin}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\3\end{pmatrix}\right\}$ **e** $W = \text{lin}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$

2 **a** Show that $W = \text{lin}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\}$ spans $\mathbb{R}^2$.

b Hence write $\begin{pmatrix}7\\8\end{pmatrix}$ as a linear combination of the spanning vectors.

3 Determine whether $W = \text{lin}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}-2\\3\\1\end{pmatrix}\right\}$ spans $\mathbb{R}^3$.

4 **a** Show that $W = \text{lin}\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right\}$ spans $\mathbb{R}^3$.

b Hence write $\begin{pmatrix}6\\7\\8\end{pmatrix}$ as a linear combination of the spanning vectors.

5 Find the equation of the plane in $\mathbb{R}^3$ which is spanned by $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\2\\3\end{pmatrix}$.

6 Consider $S = \left\{\begin{pmatrix}x\\y\\2x\end{pmatrix} \middle|\, x, y \in \mathbb{R}\right\}$.

a Is S a subspace of $\mathbb{R}^3$?

b Does $W_1 = \text{lin}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\3\end{pmatrix}\right\}$ span S? Explain your answer.

c Does $W_2 = \text{lin}\left\{\begin{pmatrix}-1\\3\\-2\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}\right\}$ span S? Explain your answer.

7 Prove that $W = \text{lin}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r\}$ is the smallest subspace of $\mathbb{R}^n$ containing the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r$.

LINEAR INDEPENDENCE OF VECTORS

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r$ are **linearly independent** if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + ... + x_r\mathbf{v}_r = \mathbf{0}$ has only the trivial solution $x_1 = x_2 = x_3 = ... = x_r = 0$.

If there is a non-zero solution to the equation, the vectors are **linearly dependent**.

Example 32

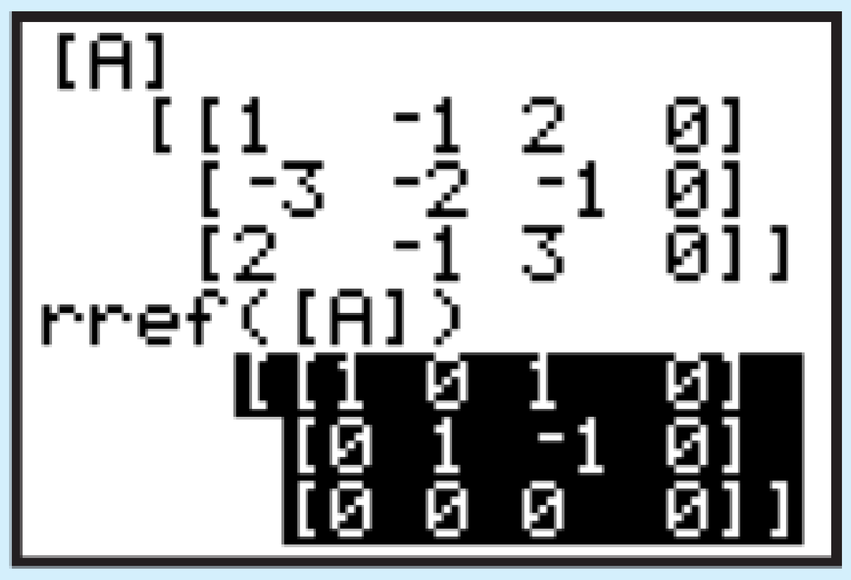
Determine whether the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ are linearly independent.

If $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ then $(\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \mathbf{v}_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} 1 & -1 & 2 \\ -3 & -2 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The system has AM $\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -3 & -2 & -1 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$



```
[A]
   [[1   -1 2  0]
    [-3 -2 -1 0]
    [2   -1 3  0]]
rref([A])
       [[1  0 1  0]
        [0 1 -1 0]
        [0 0 0  0]]
```

If $x_3 = t$ then $x_2 = t$ and $x_1 = -t$

$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$, which is a non-trivial solution.

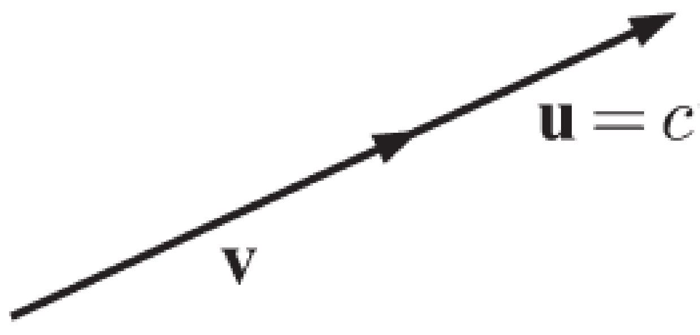
$\therefore$ $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

LINEAR DEPENDENCE IN $\mathbb{R}^2$

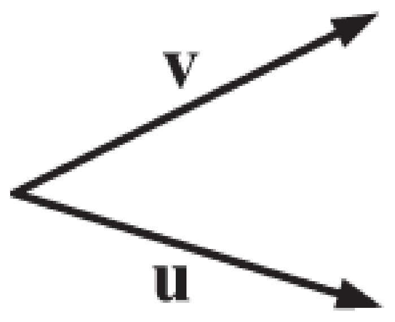
The following are **useful facts** about linear dependence in $\mathbb{R}^2$.

(1) Any list of vectors in $\mathbb{R}^2$ which includes $\mathbf{0}$ is linearly dependent.

(2) Any two vectors in $\mathbb{R}^2$ are linearly dependent $\Leftrightarrow$ one is a multiple of the other *or* one is $\mathbf{0}$.



are linearly dependent



are linearly independent

(3) Any list of 3 or more vectors in $\mathbb{R}^2$ is linearly dependent.

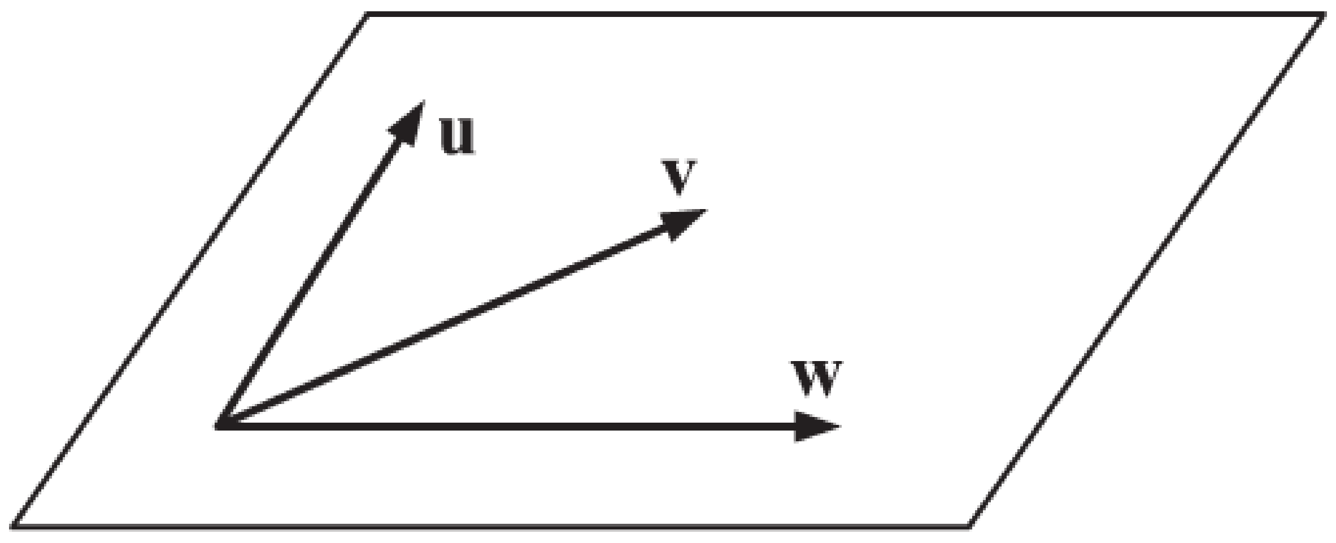
LINEAR DEPENDENCE IN $\mathbb{R}^3$

The following are **useful facts** about linear dependence in $\mathbb{R}^3$.

(1) Any list of vectors in $\mathbb{R}^3$ which includes $\mathbf{0}$ is linearly dependent.

(2) Any three vectors in $\mathbb{R}^3$ are linearly dependent $\Leftrightarrow$ one is a linear combination of the other two *or* one is $\mathbf{0}$.

(3) Any three vectors **u**, **v**, and **w** in $\mathbb{R}^3$ are linearly dependent if they lie in the same plane.



(4) Any list of four or more vectors in $\mathbb{R}^3$ is linearly dependent.

THEOREM ON LINEAR INDEPENDENCE

The list of vectors $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r\}$ is linearly dependent $\Leftrightarrow$ one of the vectors is a linear combination of the other vectors.

Proof:

$(\Rightarrow)$ The vectors of V are linearly dependent

$\Rightarrow$ $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + ... + c_r\mathbf{v}_r = \mathbf{0}$ for some scalars c_i which are not all zero.

$\Rightarrow$ $c_i\mathbf{v}_i = -c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - .... - c_{i-1}\mathbf{v}_{i-1} - c_{i+1}\mathbf{v}_{i+1} - .... - c_r\mathbf{v}_r$ where $c_i \neq 0$

$\Rightarrow$ $\mathbf{v}_i = -\frac{c_1}{c_i}\mathbf{v}_1 - \frac{c_2}{c_i}\mathbf{v}_2 - .... - \frac{c_{i-1}}{c_i}\mathbf{v}_{i-1} - \frac{c_{i+1}}{c_i}\mathbf{v}_{i+1} - .... - \frac{c_r}{c_i}\mathbf{v}_r$

$\Rightarrow$ $\mathbf{v}_i$ is a linear combination of the other vectors.

$(\Leftarrow)$ $\mathbf{v}_i$ is a linear combination of the other vectors

$\Rightarrow$ $\mathbf{v}_i = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_{i-1}\mathbf{v}_{i-1} + x_{i+1}\mathbf{v}_{i+1} + ... + x_r\mathbf{v}_r$

$\Rightarrow$ $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + ... + x_{i-1}\mathbf{v}_{i-1} - 1\mathbf{v}_i + x_{i+1}\mathbf{v}_{i+1} + ... + x_r\mathbf{v}_r = \mathbf{0}$

$\Rightarrow$ there exists a non-trivial solution for the x_is

$\Rightarrow$ the vectors of V are linearly dependent.

EXERCISE 11.4

1 Determine whether each set of vectors in $\mathbb{R}^3$ is linearly independent.

a $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ **b** $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ **c** $\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\-2\end{pmatrix}, \begin{pmatrix}2\\3\\4\end{pmatrix}\right\}$

2 For what values of $t \in \mathbb{R}$ are $\begin{pmatrix}t\\1\\1\end{pmatrix}, \begin{pmatrix}1\\t\\1\end{pmatrix}, \begin{pmatrix}1\\1\\t\end{pmatrix}$ linearly dependent?

3 If **u** and **v** are any two vectors in $\mathbb{R}^n$, show that the following vectors are linearly dependent:

a **u**, **v**, **v** **b** **u**, **v**, **u** − **v** **c** **u**, **u** + **v**, **u** − **v**

4 Prove that any two vectors in $\mathbb{R}^2$ are linearly dependent $\Leftrightarrow$ one vector is a scalar multiple of the other *or* one of them is $\mathbf{0}$.

5 Prove that any set of vectors in $\mathbb{R}^3$ which includes $\mathbf{0}$ are linearly dependent.

6 Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^3$, then so is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

7 The **vector cross-product** of any two non-zero vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ in $\mathbb{R}^3$ is defined as the vector $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$.

a If $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$, verify that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ are linearly independent.

b Is the result in **a** true in general?

8 Prove that any set of three non-zero vectors in $\mathbb{R}^3$ are linearly dependent $\Leftrightarrow$ one of them is a linear combination of the other two.

BASIS AND DIMENSION FOR A VECTOR SPACE

Suppose W is a subspace of $\mathbb{R}^n$. The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r\}$ form a **basis** for W $\Leftrightarrow$ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r$ are linearly independent *and* these vectors span W.

From this definition we establish the following **basic basis property**:

If $\mathbf{w}$ is any vector in W, then $\mathbf{w}$ can be written as a *unique* linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r$.

Proof:

Let $\mathbf{w} \in W = \text{lin}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r\}$ and suppose that the linear combination is not unique.

$\Rightarrow$ $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + ... + a_r\mathbf{v}_r$ and $\mathbf{w} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 + ... + b_r\mathbf{v}_r$ for some $a_i, b_i \in \mathbb{R}$ where not all $a_i = b_i$, $i = 1, 2, 3, ..., r$.

$$\Rightarrow \sum_{i=1}^{r} a_i\mathbf{v}_i = \sum_{i=1}^{r} b_i\mathbf{v}_i$$

$$\Rightarrow \sum_{i=1}^{r} (a_i - b_i)\mathbf{v}_i = \mathbf{0}$$

$$\Rightarrow a_i - b_i = 0 \quad \text{for } i = 1, 2, 3, ..., r \quad \{\text{since } \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r \text{ are linearly independent}\}$$

$$\Rightarrow a_i = b_i \quad \text{for } i = 1, 2, 3, ..., r$$

This is a contradiction, so the linear combination must be unique.

The **dimension** of a subspace W is the number of vectors in a basis of W.

For example, $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ forms the **standard basis** for $\mathbb{R}^n$, and $\mathbb{R}^n$ has dimension n.

Example 33

Let $\mathbf{v}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}2\\0\\1\end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix}3\\2\\1\end{pmatrix}$. Show that $W = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$.

Suppose there exist x_1, x_2, x_3 in $\mathbb{R}$ such that $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$

$$\Rightarrow \quad x_1\begin{pmatrix}1\\1\\0\end{pmatrix} + x_2\begin{pmatrix}2\\0\\1\end{pmatrix} + x_3\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}1&2&3\\1&0&2\\0&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

which has augmented matrix $\left(\begin{array}{ccc|c}1&2&3&0\\1&0&2&0\\0&1&1&0\end{array}\right) \sim \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)$ {using technology}

The system has the trivial solution $x_1 = x_2 = x_3 = 0$, so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

Now $\mathbf{A} = (\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \mathbf{v}_3) = \begin{pmatrix}1&2&3\\1&0&2\\0&1&1\end{pmatrix}$ has $|\,\mathbf{A}\,| = -1 \neq 0$

$\therefore$ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$.

Since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent and span $\mathbb{R}^3$, they form a basis of $\mathbb{R}^3$.

Example 34

Consider $S = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \middle|\, x + 3y = 0,\ x, y, z \in \mathbb{R} \right\}$ and $W = \left\{ \begin{pmatrix}-3\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$.

a Show that:

  i S is a subspace of $\mathbb{R}^3$ **ii** W spans S **iii** W is a basis of S.

b State the value of $\dim(S)$.

a **i** (1) $\begin{pmatrix}0\\0\\0\end{pmatrix} = \mathbf{0} \in S \Rightarrow S$ is non-empty.

(2) Let $\mathbf{u} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ be in S,

so $u_1 + 3u_2 = 0$, $v_1 + 3v_2 = 0$ .... $(*)$

Now $\mathbf{u} + \mathbf{v} = \begin{pmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{pmatrix}$ where

$$\begin{aligned}&(u_1+v_1) + 3(u_2+v_2)\\ &= (u_1+3u_2) + (v_1+3v_2)\\ &= 0 \qquad \{\text{from } (*)\}\end{aligned}$$

and $c\mathbf{u} = \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \end{pmatrix}$ where

$$\begin{aligned} &(cu_1) + 3(cu_2) \\ &= c(u_1 + 3u_2) \\ &= 0 \qquad \{\text{from } (\,*\,)\} \end{aligned}$$

$\therefore$ $\mathbf{u}, \mathbf{v} \in S \;\Rightarrow\; \mathbf{u} + \mathbf{v} \in S$ and $c\mathbf{u} \in S$

Thus S is closed under vector addition and scalar multiplication.

$\therefore$ S is a subspace of $\mathbb{R}^3$ {using (1) and (2)}

ii $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are both in S, since they both satisfy $x + 3y = 0$.

If $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ then $c_1 = y = -\dfrac{x}{3}$ and $c_2 = z$ is a non-trivial solution for which $x + 3y = -3c_1 + 3c_1 = 0$.

Thus W spans S.

iii (1) If $x_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

then $-3x_1 = 0$, $x_1 = 0$, $x_2 = 0$

$\therefore$ $x_1 = x_2 = 0$ {the trivial solution}

$\therefore$ $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent.

(2) From **ii**, W spans S.

Thus W is a basis of S {using (1) and (2)}

b $\dim(S) = 2$ {since 2 linearly independent vectors span S}

Example 35

Find a basis for and state the dimension of the solution space of

$$\begin{cases} x_1 - x_2 - 2x_3 + 3x_4 + x_5 = 0 \\ x_1 - x_2 - 2x_3 \phantom{{}+3x_4} + x_5 = 0 \\ \phantom{x_1 -{}} x_2 + x_3 + x_4 \phantom{{}+x_5} = 0 \\ 2x_1 + x_2 - x_3 \phantom{{}+3x_4} + 2x_5 = 0. \end{cases}$$

The system has AM $\left(\begin{array}{ccccc|c} 1 & -1 & -2 & 3 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} \boxed{1} & 0 & -1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ {using technology}

The free variables are x_3 and x_5.

Letting $x_3 = s$, $x_5 = t$, we find $x_4 = 0$, $x_2 + s = 0$, and $x_1 - x_3 + x_5 = 0$

$\therefore$ $x_2 = -s$ and $x_1 = s - t$

Thus $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = s\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ where $s, t \in \mathbb{R}$

$\therefore$ $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ span the solution space and are linearly independent.

$\therefore$ $S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the solution space, and $\dim(S) = 2$.

Theorem 1 on bases:

If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n$ is a basis of vector space V, then every set of m vectors in V where $m > n$ is linearly dependent.

Proof:

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, ..., \mathbf{w}_m$ be a set of m vectors in V, where $m > n$.

If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n$ is a basis of V, then each of the $\mathbf{w}_i$ is expressible as a linear combination of these vectors. We can write the system of linear equations:

$$\begin{cases} \mathbf{w}_1 = c_{11}\mathbf{v}_1 + c_{12}\mathbf{v}_2 + c_{13}\mathbf{v}_3 + ... + c_{1n}\mathbf{v}_n \\ \mathbf{w}_2 = c_{21}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + c_{23}\mathbf{v}_3 + ... + c_{2n}\mathbf{v}_n \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \\ \mathbf{w}_m = c_{m1}\mathbf{v}_1 + c_{m2}\mathbf{v}_2 + c_{m3}\mathbf{v}_3 + ... + c_{mn}\mathbf{v}_n. \end{cases}$$

Suppose $k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + k_3\mathbf{w}_3 + ... + k_m\mathbf{w}_n = \mathbf{0}$.

$\therefore$ $(k_1c_{11} + k_2c_{21} + ... + k_mc_{m1})\mathbf{v}_1 + (k_1c_{12} + k_2c_{22} + ... + k_mc_{m2})\mathbf{v}_2 + ...$
$+ (k_1c_{1n} + k_2c_{2n} + ... + k_mc_{mn})\mathbf{v}_n = \mathbf{0}$

But $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n$ are a basis of V, so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n$ are linearly independent.

$\therefore$ we require each coefficient of $\mathbf{v}_i$ to be zero, and we are left with the system

$$\begin{cases} k_1c_{11} + k_2c_{21} + ... + k_mc_{m1} = 0 \\ k_1c_{12} + k_2c_{22} + ... + k_mc_{m2} = 0 \\ \quad \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\ k_1c_{1n} + k_2c_{2n} + ... + k_mc_{mn} = 0 \end{cases}$$

This is a system of equations with more unknowns than equations, so there must exist non-trivial solutions for $k_1, k_2, k_3, ..., k_n$.

$\therefore$ $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, ..., \mathbf{w}_m$ are linearly dependent.

Theorem 2 on bases:

Two distinct bases of a vector space contain the same number of vectors.

Proof:

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n$ and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, ..., \mathbf{w}_m$ be the two distinct bases.

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n$ is a basis and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, ..., \mathbf{w}_m$ are linearly independent, by **Theorem 1 on bases**, $m \leqslant n$.

Likewise, since $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, ..., \mathbf{w}_m$ is a basis and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_n$ are linearly independent, $n \leqslant m$.

Consequently $m = n$.

For example, the standard basis of $\mathbb{R}^n$ $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, ..., \mathbf{e}_n$ consists of n vectors.

$\therefore$ by **Theorem 2 on bases**, every basis of $\mathbb{R}^n$ contains n vectors.

EXERCISE 11.5

1 Show that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ form a basis for $\mathbb{R}^3$.

2 Which of these sets of vectors form a basis for $\mathbb{R}^3$?

a $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

b $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix} \right\}$

c $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 10 \end{pmatrix} \right\}$

3 Find bases for these subspaces of $\mathbb{R}^3$:

a the plane $x + 2y - 3z = 0$

b the plane with equation $x + z = 0$

c the line $x = t$, $y = -5t$, $z = 2t$, $t \in \mathbb{R}$

d all vectors of the form $\begin{pmatrix} a \\ b \\ a-b \end{pmatrix}$ where $a, b \in \mathbb{R}$.

4 Find a basis, and the dimension of the solution space, of:

a $\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 - x_2 + 7x_3 = 0 \end{cases}$

b $\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + 2x_3 = 0 \\ 2x_1 + 4x_2 - 5x_3 = 0 \end{cases}$

c $\begin{cases} x_1 + x_2 - x_3 + x_4 = 0 \\ 2x_1 - x_2 + 2x_3 - x_4 = 0 \end{cases}$

d $\begin{cases} x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 0 \\ x_1 + x_2 + x_4 + 3x_5 = 0 \\ 2x_1 + 3x_2 + x_4 + 3x_5 = 0 \\ 2x_1 + 3x_2 + 2x_3 + 2x_4 + 5x_5 = 0 \end{cases}$

e $\begin{cases} x_1 + x_2 + 2x_3 - x_4 + 3x_5 = 0 \\ 2x_1 - x_2 + x_4 + 2x_5 = 0 \\ 3x_1 + 2x_3 + 5x_5 = 0 \\ x_1 - 2x_2 - 2x_3 + 2x_4 - x_5 = 0 \\ 4x_1 + x_2 + x_3 - x_4 + 3x_5 = 0 \end{cases}$

5 Determine the dimension of these subspaces of $\mathbb{R}^4$:

a all vectors of the form $\begin{pmatrix} a \\ b \\ 0 \\ c \end{pmatrix}$ **b** all vectors of the form $\begin{pmatrix} a \\ b \\ 2a \\ a+b \end{pmatrix}$.

6 If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the basis of a vector space V in $\mathbb{R}^3$, show that $\{v_1 + v_2 + v_3,\ v_1 + v_2,\ v_1\}$ is also a basis of V.

NULL SPACE

The **null space** of the $m \times n$ matrix $\mathbf{A}$ is the set of all solutions to the homogeneous equation $\mathbf{Ax} = \mathbf{0}$.

The null space of $\mathbf{A}$ is denoted Null $\mathbf{A}$, and Null $\mathbf{A} = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{0},\ \mathbf{x} \in \mathbb{R}^n\}$.

The null space of an $m \times n$ matrix $\mathbf{A}$ has the following properties:

- Null $\mathbf{A}$ are vectors in $\mathbb{R}^n$.
- Null $\mathbf{A}$ always contains $\mathbf{0}$, the trivial solution.
- Null $\mathbf{A}$ depends on the number of columns of $\mathbf{A}$.
- If $\mathbf{A}$ is invertible, Null $\mathbf{A} = \{\mathbf{0}\}$.

The **nullity** of matrix $\mathbf{A}$ is the dimension of the null space of $\mathbf{A}$.

Example 36

Find Null $\mathbf{A}$ and nullity $(\mathbf{A})$, if $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & 4 & 0 & -1 & 2 \\ 3 & 6 & 1 & 0 & 5 \end{pmatrix}$.

$\mathbf{Ax} = \mathbf{0}$ has AM $\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 3 & 0 \\ 2 & 4 & 0 & -1 & 2 & 0 \\ 3 & 6 & 1 & 0 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \boxed{1} & \frac{3}{2} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ {using technology}

Letting $x_2 = r$, $x_4 = s$, $x_5 = t$, we find $x_3 = -\frac{3}{2}s - 2t$ and $x_1 = -2r + \frac{1}{2}s - t$

$$\therefore \text{ Null } \mathbf{A} \text{ is } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = r\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \quad \text{where } r, s, t \in \mathbb{R}$$

$$\text{which is the subspace spanned by } \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$\therefore$ nullity $(\mathbf{A}) = 3$

COLUMN AND ROW SPACE

For the $m \times n$ matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, we say that:

- $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$ are its **column vectors**
- $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix}, \begin{pmatrix} a_{21} & a_{22} & \dots & a_{2n} \end{pmatrix}, \dots, \begin{pmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ are its **row vectors**.

The **column space** of matrix **A** is the subspace of $\mathbb{R}^n$ spanned by the column vectors of **A**.

The **row space** of matrix **A** is the subspace of $\mathbb{R}^n$ spanned by the row vectors of **A**.

The following theorems are important for finding bases for the row and column spaces of a given matrix. They are given without proof.

Theorem 1: The row space of a matrix is not changed by elementary row operations.

This theorem does not hold for the column space of a matrix.

Theorem 2: Suppose matrix **B** is obtained from matrix **A** by elementary row operations.

If the column vectors of **B** form a basis for the column space of **B**, then the corresponding column vectors of **A** form a basis for the column space of **A**.

Theorem 3: If matrix **A** is converted to matrix **R** in **reduced row echelon form**, then:

- its row vectors with leading 1s form a basis for the row space of **R**
- its column vectors with leading 1s form a basis for the column space of **R**.

The **row rank** of matrix **A** is the dimension of the row space of **A**.

The **column rank** of matrix **A** is the dimension of the column space of **A**.

Since row rank and column rank are both determined by the number of leading 1s in the reduced row-echelon form matrix, the row rank of **A** is equal to the column rank of **A**.

Hence we define rank (**A**) by

$$\text{rank}\,(\mathbf{A}) = \text{row rank of } \mathbf{A} = \text{column rank of } \mathbf{A}.$$

Example 37

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & 4 & 0 & -1 & 2 \\ 3 & 6 & 1 & 0 & 5 \end{pmatrix}$.

a Find a basis for the row space of **A**.

b Find a basis for the column space of **A**.

c Find rank (**A**).

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & 4 & 0 & -1 & 2 \\ 3 & 6 & 1 & 0 & 5 \end{pmatrix} \sim \mathbf{R} = \begin{pmatrix} \boxed{1} & 2 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & \boxed{1} & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \{\text{using technology}\}$$

a By **Theorem 3**, a basis for the row space of **R** is
$\{(1 \;\; 2 \;\; 0 \;\; -\frac{1}{2} \;\; 1), (0 \;\; 0 \;\; 1 \;\; \frac{3}{2} \;\; 2)\}$
$\therefore$ by **Theorem 1**, a basis for the row space of **A** is
$\{(1 \;\; 2 \;\; 0 \;\; -\frac{1}{2} \;\; 1), (0 \;\; 0 \;\; 1 \;\; \frac{3}{2} \;\; 2)\}$.

b By **Theorem 3**, a basis for the column space of **R** is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

$\therefore$ by **Theorem 2**, a basis for the column space of **A** is $\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

c rank (**A**) = row rank of **A** = column rank of **A** = 2.

FINDING A BASIS FOR THE ROW SPACE USING $\mathbf{A}^T$

An alternative method for finding a basis for the row space of **A** is:

Step 1: Find $\mathbf{A}^T$ {since $\mathbf{A}^T$ converts the row space of **A** into the column space of $\mathbf{A}^T$}.

Step 2: Convert $\mathbf{A}^T$ into reduced row echelon form **R**.

Step 3: From the leading 1s in **R**, choose the column vectors of $\mathbf{A}^T$ which form a basis for the column space of $\mathbf{A}^T$.

Step 4: Write down a basis for the row space of **A**.

This method is sometimes referred to as "selecting a basis from the rows of **A**".

Example 38

Using the transpose method, find a basis for the row space of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & 4 & 0 & -1 & 2 \\ 3 & 6 & 1 & 0 & 5 \end{pmatrix}$.

For $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & 4 & 0 & -1 & 2 \\ 3 & 6 & 1 & 0 & 5 \end{pmatrix}$, $\mathbf{A}^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \end{pmatrix} \sim \mathbf{R} = \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ {using technology}

The first two columns of **R** form a basis for the column space of **R**

$\Rightarrow$ the first two columns of $\mathbf{A}^T$ from a basis for the column space of $\mathbf{A}^T$

$\Rightarrow$ the first two rows of **A** form a basis for the row space of **A**

$\Rightarrow$ $\{(1 \;\; 2 \;\; 1 \;\; 1 \;\; 3), (2 \;\; 4 \;\; 0 \;\; -1 \;\; 2)\}$ forms a basis for the row space of **A**.

Theorem on system consistency:

The system $\mathbf{Ax} = \mathbf{b}$ is consistent $\Leftrightarrow$ $\mathbf{b}$ is in the column space of $\mathbf{A}$.

Proof:

Let $\mathbf{A} = (\mathbf{c}_1 \,|\, \mathbf{c}_2 \,|\, \mathbf{c}_3 \,|\, ... \,|\, \mathbf{c}_n)$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$.

$$\begin{aligned} \mathbf{Ax} = \mathbf{b} \quad &\Leftrightarrow \quad (\mathbf{c}_1 \,|\, \mathbf{c}_2 \,|\, \mathbf{c}_3 \,|\, ... \,|\, \mathbf{c}_n) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{b} \\ &\Leftrightarrow \quad x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 + ... + x_n\mathbf{c}_n = \mathbf{b} \\ &\Leftrightarrow \quad \mathbf{b} \text{ is in the column space of } \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, ..., \mathbf{c}_n\}. \end{aligned}$$

Example 39

Determine whether the system $\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 3 \\ 2x_1 + 4x_2 \qquad - x_4 = 2 \\ 3x_1 + 6x_2 + x_3 \qquad = 5 \end{cases}$ is consistent.

The system has the form $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 0 & -1 \\ 3 & 6 & 1 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

$\mathbf{A} \sim \begin{pmatrix} \boxed{1} & 2 & 0 & -\frac{1}{2} \\ 0 & 0 & \boxed{1} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{R}$ {using technology}

$\therefore$ a basis for the column space of $\mathbf{R}$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$\therefore$ a basis for the column space of $\mathbf{A}$ is $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Now if $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ then $\begin{cases} a + b = 3 \\ 2a = 2 \\ 3a + b = 5 \end{cases}$ which has solution $a = 1$, $b = 2$

$\therefore$ $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ is in the column space of $\mathbf{A}$

$\therefore$ $\mathbf{Ax} = \mathbf{b}$ is consistent {**Theorem on system consistency**}

DIMENSION

We have defined:

- the dimension of the row space of $\mathbf{A}$ is the rank of $\mathbf{A}$
- the dimension of the null space of $\mathbf{A}$ is nullity $(\mathbf{A})$.

Theorem on rank and nullity:

If n is the number of columns of $\mathbf{A}$ then rank $(\mathbf{A})$ + nullity $(\mathbf{A}) = n$.

Example 40

Verify the rank-nullity theorem for $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & 4 & 0 & -1 & 2 \\ 3 & 6 & 1 & 0 & 1 \end{pmatrix}$.

For $\mathbf{Ax} = \mathbf{0}$, the AM is $\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 3 & 0 \\ 2 & 4 & 0 & -1 & 2 & 0 \\ 3 & 6 & 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \boxed{1} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right)$

{using technology}

Letting $x_2 = s$, $x_4 = t$, we find $x_5 = 0$, $x_3 = -\frac{3}{2}t$, $x_1 = -2s + \frac{1}{2}t$

$\therefore$ Null $\mathbf{A}$ is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix}$, where $s, t \in \mathbb{R}$

which is the subspace spanned by $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} \right\}$.

$\therefore$ nullity $(\mathbf{A}) = 2$.

We will use the transpose method to find a basis for the row space of $\mathbf{A}$.

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \sim \mathbf{R} = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ the three rows of $\mathbf{A}$ form a basis for the row space of $\mathbf{A}$.

Thus $\{(1\ \ 2\ \ 1\ \ 1\ \ 3), (2\ \ 4\ \ 0\ \ -1\ \ 2), (3\ \ 6\ \ 1\ \ 0\ \ 1)\}$ forms a basis for the row space of $\mathbf{A}$.

$\therefore$ rank $(\mathbf{A}) = 3$.

$\therefore$ rank $(\mathbf{A})$ + nullity $(\mathbf{A}) = 3 + 2 = 5$, which is the number of columns of $\mathbf{A}$ ✓

EXERCISE 11.6

1 For the matrix $\mathbf{A}=\begin{pmatrix}1 & 0 & 1 & -2\\ 2 & -1 & 1 & 3\\ 1 & 3 & 2 & 1\end{pmatrix}$ list:

a the row vectors **b** the column vectors.

2 Find Null $\mathbf{A}$ and nullity $(\mathbf{A})$ for:

a $\mathbf{A}=\begin{pmatrix}1 & 2 & 1 & 0\\ 2 & -1 & 1 & 3\\ 3 & -4 & 1 & 1\end{pmatrix}$ **b** $\mathbf{A}=\begin{pmatrix}2 & 1 & 1 & 3 & 1\\ 3 & 3 & 1 & 5 & 6\\ 1 & -1 & 1 & 1 & -2\end{pmatrix}$

3 For each of the following matrices $\mathbf{A}$, find:

i a basis for the row space **ii** a basis for the column space **iii** rank $(\mathbf{A})$.

a $\mathbf{A}=\begin{pmatrix}1 & -2\\ -3 & 6\end{pmatrix}$ **b** $\mathbf{A}=\begin{pmatrix}1 & 0 & 1 & 2\\ 1 & 2 & -1 & 3\\ 3 & 2 & 1 & 5\end{pmatrix}$ **c** $\mathbf{A}=\begin{pmatrix}1 & 1 & 1 & 2 & 4\\ 2 & 0 & -1 & 1 & 0\\ 3 & -1 & -3 & 0 & -8\\ 2 & 2 & 2 & 4 & 1\\ 5 & 1 & -1 & 4 & -7\end{pmatrix}$

4 Use the transpose method to find a basis for the row space of:

a $\mathbf{A}=\begin{pmatrix}1 & 3 & 2\\ 2 & 6 & 4\end{pmatrix}$ **b** $\mathbf{A}=\begin{pmatrix}1 & 0 & 1 & 2\\ 1 & 2 & -1 & 3\\ 3 & 2 & 1 & 7\end{pmatrix}$ **c** $\mathbf{A}=\begin{pmatrix}1 & 1 & 1 & 2 & 4\\ 2 & 0 & -1 & 1 & 0\\ 3 & -1 & -3 & 0 & -8\\ 2 & 2 & 2 & 4 & 1\\ 5 & 1 & -1 & 4 & -7\end{pmatrix}$

5 Determine whether the system $\begin{cases} x_1+x_2-x_3=4\\ x_1-2x_2+x_3=6\\ 2x_1-x_2=11\end{cases}$ is consistent.

6 Verify the rank-nullity theorem for $\mathbf{A}=\begin{pmatrix}1 & 3 & 1 & -2 & 1\\ 2 & 6 & 4 & -8 & 3\\ -1 & -3 & 1 & -2 & 5\end{pmatrix}$.

7 Find a basis for the subspace of $\mathbb{R}^3$ spanned by:

a $\left\{\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$ **b** $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$

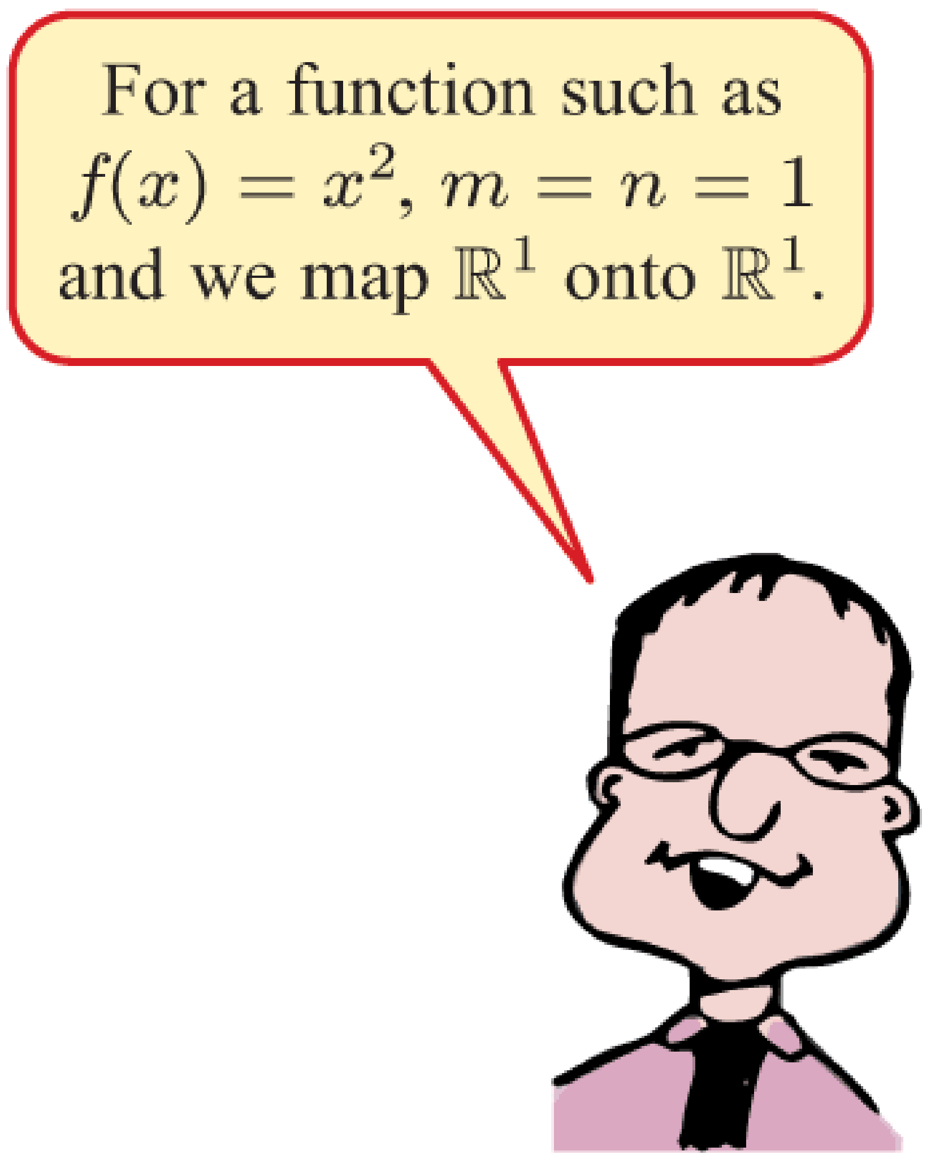
8 Find a basis for the subspace of $\mathbb{R}^4$ spanned by:

a $\left\{\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}\right\}$ **b** $\left\{\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}-1\\0\\1\\1\end{pmatrix},\begin{pmatrix}2\\1.5\\0\\-1\end{pmatrix}\right\}$

J LINEAR TRANSFORMATIONS

From the HL Core course you should be familiar with the idea of a **function** which maps one variable onto another. We defined a function as a relation in which no two different ordered pairs have the same first component. This means that each element in the **domain** of the function is mapped to a unique element in the **range** of the function.

We now expand our idea of a function to consider it as a mapping from one vector space to another. We consider a function T which maps a vector $\mathbf{v} \in \mathbb{R}^n$ to another vector $\mathbf{w} \in \mathbb{R}^m$. We say that $\mathbf{w}$ is the **image** of $\mathbf{v}$ under the **transformation** T, and write $\mathbf{w} = \mathrm{T}(\mathbf{v})$.


For a function such as $f(x) = x^2$, $m = n = 1$ and we map $\mathbb{R}^1$ onto $\mathbb{R}^1$.

For example, if $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{w} = \mathrm{T}(\mathbf{v}) = \begin{pmatrix} y \\ x+y \\ x-y \end{pmatrix}$ then T maps $\mathbb{R}^2$ onto $\mathbb{R}^3$.

When $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, we find $\mathbf{w} = \mathrm{T}\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$.

A vector-related function $\mathrm{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a **linear transformation** if:

(1) $\mathrm{T}(\mathbf{u} + \mathbf{v}) = \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ {*addition* property} and

(2) $\mathrm{T}(k\mathbf{u}) = k\mathrm{T}(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^n$, $k \in \mathbb{R}$ {*scalar multiplication* property}

Example 41

Show that $\mathrm{T} : \mathbb{R}^2 \mapsto \mathbb{R}^3$ where $\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ x+y \\ x-y \end{pmatrix}$ is a linear transformation.

Let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

(1) $\mathrm{T}(\mathbf{u} + \mathbf{v})$

$$= \mathrm{T}\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} y_1 + y_2 \\ x_1 + x_2 + y_1 + y_2 \\ x_1 + x_2 - (y_1 + y_2) \end{pmatrix}$$

$$= \begin{pmatrix} y_1 \\ x_1 + y_1 \\ x_1 - y_1 \end{pmatrix} + \begin{pmatrix} y_2 \\ x_2 + y_2 \\ x_2 - y_2 \end{pmatrix}$$

$$= \mathrm{T}\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + \mathrm{T}\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$$

$$= \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})$$

(2) $\mathrm{T}(k\mathbf{u})$

$$= \mathrm{T}\left(\begin{pmatrix} kx_1 \\ ky_1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} ky_1 \\ kx_1 + ky_1 \\ kx_1 - ky_1 \end{pmatrix}$$

$$= k\begin{pmatrix} y_1 \\ x_1 + y_1 \\ x_1 - y_1 \end{pmatrix}$$

$$= k\mathrm{T}\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right)$$

$$= k\mathrm{T}(\mathbf{u})$$

Since the addition and scalar multiplication properties are satisfied, T is a linear transformation.

In general, if a transformation involves higher powers of the variables, it will not be linear.

Example 42

Show by counter-example that $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ where $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y^2 \end{pmatrix}$ is not a linear transformation.

Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Now $T(\mathbf{u}+\mathbf{v}) = T\left(\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 16 \end{pmatrix}$

and $T(\mathbf{u}) + T(\mathbf{v}) = T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + T\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \end{pmatrix}$

For this $\mathbf{u}$ and $\mathbf{v}$, $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$

$\therefore$ T is not a linear transformation.

PROPERTIES OF LINEAR TRANSFORMATIONS

If $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation, then:

- $T(\mathbf{0}) = \mathbf{0}$
- $T(-\mathbf{u}) = -T(\mathbf{u})$
- $T(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + ... + k_r\mathbf{u}_r) = k_1T(\mathbf{u}_1) + k_2T(\mathbf{u}_2) + ... + k_rT(\mathbf{u}_r)$

Proof (of the first two properties):

- $$\begin{aligned} T(\mathbf{0}) &= T(\mathbf{0}+\mathbf{0}) \\ &= T(\mathbf{0}) + T(\mathbf{0}) \quad \{\text{addition property}\} \\ &= 2T(\mathbf{0}) \end{aligned}$$
  $\therefore$ $T(\mathbf{0}) = \mathbf{0}$
- $$\begin{aligned} T(-\mathbf{u}) &= T(-1\mathbf{u}) \\ &= -T(\mathbf{u}) \quad \{\text{scalar multiplication property}\} \end{aligned}$$

In $T(\mathbf{0}) = \mathbf{0}$, the $\mathbf{0}$s are different as the first $\mathbf{0}$ is in $\mathbb{R}^n$ and the other is in $\mathbb{R}^m$.

If a basis for a vector space V is known, the image of any vector $\mathbf{v} \in V$ can be found under the linear transformation $T : V \mapsto W$.

Proof:

If V has basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r\}$ and $T : V \mapsto W$, then $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, 2, 3, ..., r$.

Now as $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r\}$ is a basis of V, for any $\mathbf{v} \in V$ there exists scalars $a_1, a_2, a_3, ..., a_r \in \mathbb{R}$ such that $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + ... + a_r\mathbf{v}_r$.

$$\begin{aligned} \text{Hence} \quad T(\mathbf{v}) &= T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + ... + a_r\mathbf{v}_r) \\ &= a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + a_3T(\mathbf{v}_3) + ... + a_rT(\mathbf{v}_r) \\ &= a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 + ... + a_r\mathbf{w}_r \end{aligned}$$

$\therefore$ since the $\mathbf{w}_i$ are known, $T(\mathbf{v})$ can be determined.

Example 43

Suppose $\mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)=\begin{pmatrix}1\\2\end{pmatrix}$ and $\mathrm{T}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)=\begin{pmatrix}4\\1\end{pmatrix}$ for a linear transformation $\mathrm{T}:\mathbb{R}^2\mapsto\mathbb{R}^2$. Find $\mathrm{T}\left(\begin{pmatrix}3\\5\end{pmatrix}\right)$.

$\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ is the standard basis for $\mathbb{R}^2$.

$$\begin{aligned}\therefore\ \mathrm{T}\left(\begin{pmatrix}3\\5\end{pmatrix}\right)&=\mathrm{T}\left(3\begin{pmatrix}1\\0\end{pmatrix}+5\begin{pmatrix}0\\1\end{pmatrix}\right)\\&=3\mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)+5\mathrm{T}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)\\&=3\begin{pmatrix}1\\2\end{pmatrix}+5\begin{pmatrix}4\\1\end{pmatrix}\\&=\begin{pmatrix}23\\11\end{pmatrix}\end{aligned}$$

EXERCISE 1J.1

1 Show that $\mathrm{T}:\mathbb{R}^2\mapsto\mathbb{R}^3$ where $\mathrm{T}\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}x\\y\\x+2y\end{pmatrix}$ is a linear transformation.

2 Determine whether $\mathrm{T}:\mathbb{R}^3\mapsto\mathbb{R}^2$ where $\mathrm{T}\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=\begin{pmatrix}x+y\\z-y\end{pmatrix}$ is a linear transformation.

3 Show by counter-example that $\mathrm{T}:\mathbb{R}^3\mapsto\mathbb{R}^2$ where $\mathrm{T}\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=\begin{pmatrix}xz\\yz\end{pmatrix}$ is not a linear transformation.

4 Prove that if $\mathrm{T}:\mathbb{R}^n\mapsto\mathbb{R}^m$ is a linear transformation, then

$$\mathrm{T}(k_1\mathbf{u}_1+k_2\mathbf{u}_2+....+k_r\mathbf{u}_r)=k_1\mathrm{T}(\mathbf{u}_1)+k_2\mathrm{T}(\mathbf{u}_2)+....+k_r\mathrm{T}(\mathbf{u}_r).$$

5 $\mathrm{T}:\mathbb{R}^2\mapsto\mathbb{R}^2$ is a linear transformation where $\mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix}$ and $\mathrm{T}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix}$.

Find $\mathrm{T}\left(\begin{pmatrix}5\\-3\end{pmatrix}\right)$.

6 $\mathrm{T}:\mathbb{R}^2\mapsto\mathbb{R}^2$ is a linear transformation where $\mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix}$ and $\mathrm{T}\left(\begin{pmatrix}1\\2\end{pmatrix}\right)=\begin{pmatrix}-1\\-3\end{pmatrix}$.

Find $\mathrm{T}\left(\begin{pmatrix}4\\6\end{pmatrix}\right)$.

7 $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ is a linear transformation where $T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}2\\1\\7\end{pmatrix}$ and $T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}-3\\0\\-4\end{pmatrix}$.

Find $T\left(\begin{pmatrix}3\\7\end{pmatrix}\right)$.

8 $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a linear transformation where $T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\-1\\2\end{pmatrix}$, $T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}2\\1\\3\end{pmatrix}$,

and $T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}5\\1\\4\end{pmatrix}$.

Find: **a** $T\left(\begin{pmatrix}1\\3\\0\end{pmatrix}\right)$ **b** $T\left(\begin{pmatrix}a\\b\\c\end{pmatrix}\right)$

9 Determine which of these transformations are linear:

a $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ where $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}x\\y-x\\x^2\end{pmatrix}$

b $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ where $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}x-y\\2\end{pmatrix}$

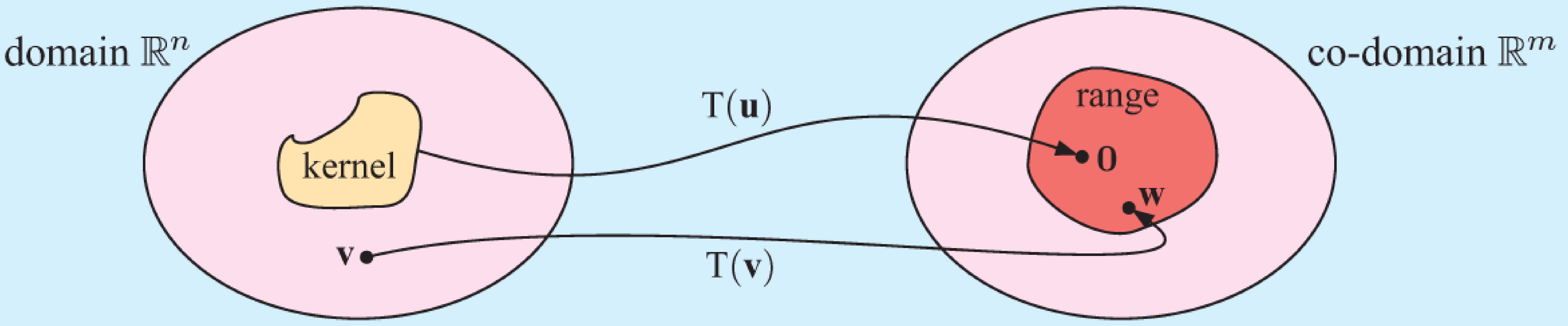
c $T : \mathbb{R}^3 \mapsto \mathbb{R}$ where $T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = \begin{pmatrix}x+y-2z\end{pmatrix}$

d $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ where $T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right) = \begin{pmatrix}x+y\\x-4z\end{pmatrix}$

KERNEL AND RANGE

For the linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$:

- the **domain** of T is $\mathbb{R}^n$
- the **co-domain** of T is $\mathbb{R}^m$
- the **kernel** (or **null space**) of T, denoted $\ker(T)$, is the set of all vectors **u** in the domain of T such that $T(\mathbf{u}) = \mathbf{0}$
- the **range** of T, denoted $\mathcal{R}(T)$, is the set of all vectors **w** in the co-domain of T such that $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in \mathbb{R}^n$.



Theorem on kernel and range:

If $\mathrm{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation, then:

- $\ker(\mathrm{T})$ is a subspace of the domain $\mathbb{R}^n$ and
- $\mathcal{R}(\mathrm{T})$ is a subspace of the co-domain $\mathbb{R}^m$.

Proof: For $\ker(\mathrm{T})$

(1) $\ker(\mathrm{T})$ is non-empty as $\mathrm{T}(\mathbf{0}) = \mathbf{0} \;\Rightarrow\; \mathbf{0} \in \ker(\mathrm{T})$.

(2) For all $\mathbf{u}_1, \mathbf{u}_2 \in \ker(\mathrm{T})$,

$$\begin{aligned} \mathrm{T}(\mathbf{u}_1 + \mathbf{u}_2) &= \mathrm{T}(\mathbf{u}_1) + \mathrm{T}(\mathbf{u}_2) && \{\text{addition property}\} \\ &= \mathbf{0} + \mathbf{0} && \{\mathbf{u}_1, \mathbf{u}_2 \in \ker(\mathrm{T})\} \\ &= \mathbf{0} \end{aligned}$$

$$\Rightarrow \quad \mathbf{u}_1 + \mathbf{u}_2 \in \ker(\mathrm{T})$$

Thus $\ker(\mathrm{T})$ is closed under vector addition.

(3) For $\mathbf{u} \in \ker(\mathrm{T})$, $k \in \mathbb{R}$,

$$\begin{aligned} \mathrm{T}(k\mathbf{u}) &= k\mathrm{T}(\mathbf{u}) && \{\text{scalar multiplication property}\} \\ &= k\mathbf{0} && \{\mathbf{u} \in \ker(\mathrm{T})\} \\ &= \mathbf{0} \end{aligned}$$

$$\Rightarrow \quad k\mathbf{u} \in \ker(\mathrm{T})$$

Thus $\ker(\mathrm{T})$ is closed under scalar multiplication.

From (1), (2), and (3), $\ker(\mathrm{T})$ is a subspace of the domain $\mathbb{R}^n$.

For $\mathcal{R}(\mathrm{T})$

(1) $\mathcal{R}(\mathrm{T})$ is non-empty as $\mathrm{T}(\mathbf{0}) = \mathbf{0} \;\Rightarrow\; \mathbf{0} \in \mathcal{R}(\mathrm{T})$.

(2) For all $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{R}(\mathrm{T})$,

$$\begin{aligned} \mathbf{w}_1 + \mathbf{w}_2 &= \mathrm{T}(\mathbf{v}_1) + \mathrm{T}(\mathbf{v}_2) && \text{for some } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n \\ &= \mathrm{T}(\mathbf{v}_1 + \mathbf{v}_2) && \{\text{addition property}\} \end{aligned}$$

$$\Rightarrow \quad \mathbf{w}_1 + \mathbf{w}_2 \in \mathcal{R}(\mathrm{T})$$

Thus $\mathcal{R}(\mathrm{T})$ is closed under vector addition.

(3) For all $\mathbf{w} \in \mathcal{R}(\mathrm{T})$, $k \in \mathbb{R}$,

$$\begin{aligned} k\mathbf{w} &= k\mathrm{T}(\mathbf{v}) && \text{for some } \mathbf{v} \in \mathbb{R}^n \\ &= \mathrm{T}(k\mathbf{v}) && \{\text{scalar multiplication property}\} \end{aligned}$$

$$\Rightarrow \quad k\mathbf{w} \in \mathcal{R}(\mathrm{T})$$

Thus $\mathcal{R}(\mathrm{T})$ is closed under scalar multiplication.

From (1), (2), and (3), $\mathcal{R}(\mathrm{T})$ is a subspace of the co-domain $\mathbb{R}^m$.

If $\mathrm{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation, then:

- **nullity** (T) is the dimension of $\ker(\mathrm{T})$, and
- **rank** (T) is the dimension of $\mathcal{R}(\mathrm{T})$.

Example 44

Consider $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ where $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ x+y \\ x-y \end{pmatrix}$. Find:

a $\ker(T)$ **b** $\mathcal{R}(T)$ **c** nullity (T) **d** rank (T)

a Let $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} y \\ x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore x = y = 0$$

$$\therefore \ker(T) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$$

b Consider $\mathbf{w} = T(\mathbf{v})$

$$= \begin{pmatrix} y \\ x+y \\ x-y \end{pmatrix}$$

$$= x\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \mathcal{R}(T) = \text{lin}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$$

c nullity $(T) = 0$

d rank $(T) = 2$

Example 45

Consider $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ where $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y \\ x+z \end{pmatrix}$. Find:

a $\ker(T)$ **b** $\mathcal{R}(T)$ **c** nullity (T) **d** rank (T)

a Let $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} y \\ x+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore y = 0 \text{ and } x+z = 0$$

$$\therefore y = 0 \text{ and } z = -x$$

$$\therefore \ker(T) = \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix}, \; x \in \mathbb{R}$$

$$= x\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \; x \in \mathbb{R}$$

$$= \text{lin}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$$

b Consider $\mathbf{w} = T(\mathbf{v})$

$$= \begin{pmatrix} y \\ x+z \end{pmatrix}$$

$$= y\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (x+z)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore \mathcal{R}(T) = \text{lin}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

$$= \mathbb{R}^2$$

c nullity $(T) = 1$

d rank $(T) = 2$

RANK-NULLITY THEOREM

From examples like these, we observe the following property of nullity, rank, and dimension:

For every linear transformation $\mathrm{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$, nullity (T) + rank $(\mathrm{T}) = n$.

Example 46

For the linear transformation $\mathrm{T} : \mathbb{R}^4 \mapsto \mathbb{R}^2$ where $\mathrm{T}\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} w+y \\ x+z \end{pmatrix}$, find:

a the kernel of T **b** rank (T) **c** the range of T.

a Let $\mathrm{T}\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} w+y \\ x+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \ w = -y \ \text{ and } \ z = -x$$

$$\therefore \ \ker(\mathrm{T}) = \begin{pmatrix} -y \\ x \\ y \\ -x \end{pmatrix}, \quad x,\, y \in \mathbb{R}$$

$$= x\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad x,\, y \in \mathbb{R}$$

$$= \text{lin}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

b From **a**, nullity $(\mathrm{T}) = 2$

Now nullity (T) + rank $(\mathrm{T}) = n = 4$

$\therefore$ rank $(\mathrm{T}) = 2$

c Consider $\mathbf{w} = \mathrm{T}(\mathbf{v})$

$$= \begin{pmatrix} w+y \\ x+z \end{pmatrix}$$

$$= (w+y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (x+z)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \ \mathcal{R}(\mathrm{T}) = \text{lin}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

$$= \mathbb{R}^2$$

THE STANDARD MATRIX OF A LINEAR TRANSFORMATION

Consider the linear transformation $\mathrm{T} : \mathbb{R}^2 \mapsto \mathbb{R}^3$ where

$$\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ x+y \\ x-y \end{pmatrix} = x\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

If we let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, we notice that $\mathrm{T}(\mathbf{u}) = \mathbf{A}\mathbf{u}$ where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$.

We also notice that $\mathrm{T}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \underset{\substack{\uparrow \\ \text{column 1} \\ \text{of } \mathbf{A}}}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$ and $\mathrm{T}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \underset{\substack{\uparrow \\ \text{column 2} \\ \text{of } \mathbf{A}}}{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}$

So, the effect of T on the standard vectors of $\mathbb{R}^2$ is to produce the column vectors of the matrix of T.

A is therefore called the **standard matrix** of T.

If $\mathrm{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation, the **standard matrix** for T is $\mathbf{A} = (\mathrm{T}(\mathbf{e}_1) \,|\, \mathrm{T}(\mathbf{e}_2) \,|\, \mathrm{T}(\mathbf{e}_3) \,|\, ... \,|\, \mathrm{T}(\mathbf{e}_n))$.

Example 47

Find the standard matrix of the linear transformation $\mathrm{T} : \mathbb{R}^2 \mapsto \mathbb{R}^3$ where

$$\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2y \\ x \\ x-y \end{pmatrix}.$$

Method 1: $\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, so $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}$.

Method 2: $\mathrm{T}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\mathrm{T}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, so $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}$.

If $\mathrm{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation with standard matrix $\mathbf{A} = (\mathrm{T}(\mathbf{e}_1) \,|\, \mathrm{T}(\mathbf{e}_2) \,|\, \mathrm{T}(\mathbf{e}_3) \,|\, ... \,|\, \mathrm{T}(\mathbf{e}_n))$, then $\mathrm{T}(\mathbf{v}) = \mathbf{A}\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

Proof:

Consider $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 + ... + v_n\mathbf{e}_n$

$$\begin{aligned} \therefore \quad \mathrm{T}(\mathbf{v}) &= \mathrm{T}(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 + ... + v_n\mathbf{e}_n) \\ &= v_1\mathrm{T}(\mathbf{e}_1) + v_2\mathrm{T}(\mathbf{e}_2) + v_3\mathrm{T}(\mathbf{e}_3) + ... + v_n\mathrm{T}(\mathbf{e}_n) \\ &= (\mathrm{T}(\mathbf{e}_1) \,|\, \mathrm{T}(\mathbf{e}_2) \,|\, \mathrm{T}(\mathbf{e}_3) \,|\, ... \,|\, \mathrm{T}(\mathbf{e}_n))\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} \\ &= \mathbf{A}\mathbf{v} \end{aligned}$$

The **converse** of this theorem is also true:

If $\mathrm{T}:\mathbb{R}^n \mapsto \mathbb{R}^m$ is defined by $\mathrm{T}(\mathbf{v})=\mathbf{Av}$, then T is a linear transformation.

Proof:

Since $\mathrm{T}(\mathbf{v})=\mathbf{Av}$,

$$\begin{aligned}\mathrm{T}(\mathbf{u}+\mathbf{v}) &= \mathbf{A}(\mathbf{u}+\mathbf{v})\\ &= \mathbf{Au}+\mathbf{Av}\\ &= \mathrm{T}(\mathbf{u})+\mathrm{T}(\mathbf{v})\end{aligned} \qquad \text{and} \qquad \begin{aligned}\mathrm{T}(k\mathbf{u}) &= \mathbf{A}k\mathbf{u}\\ &= k\mathbf{Au}\\ &= k\mathrm{T}(\mathbf{u})\end{aligned}$$

Since the addition and scalar multiplication properties hold, T is a linear transformation.

To find the range, we observe that $\mathcal{R}(\mathrm{T})$ is spanned by the column vectors of the transformation matrix **A**. We therefore use the property that the column space of **A** is the row space of $\mathbf{A}^{\mathrm{T}}$, the transpose of **A**.

Example 48

Consider the linear transformation $\mathrm{T}:\mathbb{R}^4 \mapsto \mathbb{R}^3$ where $\mathrm{T}(\mathbf{v})=\begin{pmatrix}1&1&2&3\\1&-2&1&4\\2&-1&1&-1\end{pmatrix}\mathbf{v}$. Find:

a ker(T) **b** $\mathcal{R}(\mathrm{T})$ **c** nullity (T) **d** rank (T).

a $\mathbf{Av}=\mathbf{0}$ has augmented matrix

$$\left(\begin{array}{cccc|c}1&1&2&3&0\\1&-2&1&4&0\\2&-1&1&-1&0\end{array}\right)$$

$$\sim\left(\begin{array}{cccc|c}1&0&0&-\frac{10}{3}&0\\0&1&0&-\frac{5}{3}&0\\0&0&1&4&0\end{array}\right)$$

{using technology}

Letting $x_4=t$, $t\in\mathbb{R}$, we find that $x_3=-4t$, $x_2=\frac{5}{3}t$, and $x_1=\frac{10}{3}t$.

$$\therefore \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=t\begin{pmatrix}\frac{10}{3}\\\frac{5}{3}\\-4\\1\end{pmatrix}, \quad t\in\mathbb{R}$$

$$\therefore \ \ker(\mathrm{T})=\operatorname{lin}\left\{\begin{pmatrix}\frac{10}{3}\\\frac{5}{3}\\-4\\1\end{pmatrix}\right\}$$

b $\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}1&1&2\\1&-2&-1\\2&1&1\\3&4&-1\end{pmatrix}$

$$\sim\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix} \quad \text{\{using technology\}}$$

$\therefore$ a basis for the row space of $\mathbf{A}^{\mathrm{T}}$ is $\{(1\ 0\ 0),\ (0\ 1\ 0),\ (0\ 0\ 1)\}$.

$\therefore$ a basis for the column space of **A** is

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$\begin{aligned}\therefore\ \mathcal{R}(\mathrm{T}) &= \operatorname{lin}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\\ &= \mathbb{R}^3\end{aligned}$$

c nullity $(\mathrm{T})=1$

d rank $(\mathrm{T})=3$

EXERCISE 1J.2

1 Consider the linear transformation $\text{T} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ where $\text{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix}$. Find:

a $\ker(\text{T})$ **b** $\mathcal{R}(\text{T})$ **c** nullity (T) **d** rank (T).

2 Consider the linear transformation $\text{T} : \mathbb{R}^3 \mapsto \mathbb{R}^4$ where $\text{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y \\ z - x \\ 2x \\ x + y + z \end{pmatrix}$. Find:

a $\ker(\text{T})$ **b** $\mathcal{R}(\text{T})$ **c** nullity (T) **d** rank (T).

3 Consider the linear transformation $\text{T} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ where $\text{T}(\mathbf{v}) = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix}\mathbf{v}$.

a Show that $\begin{pmatrix} 9 \\ -3 \end{pmatrix} \in \ker(\text{T})$, but $\begin{pmatrix} -3 \\ 0 \end{pmatrix} \notin \ker(\text{T})$.

b Determine whether the following vectors are in $\mathcal{R}(\text{T})$: **i** $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ **ii** $\begin{pmatrix} -4 \\ 8 \end{pmatrix}$

4 Find the standard matrix for:

a $\text{T} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ where $\text{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -y \\ x \end{pmatrix}$

b $\text{T} : \mathbb{R}^3 \mapsto \mathbb{R}^2$ where $\text{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + z \\ y - z \end{pmatrix}$

c $\text{T} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ where $\text{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y \\ -z \\ x + y \end{pmatrix}$

d $\text{T} : \mathbb{R}^2 \mapsto \mathbb{R}^4$ where $\text{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -x \\ x + y \\ y \\ y - x \end{pmatrix}$

5 $\text{T} : \mathbb{R}^4 \mapsto \mathbb{R}^2$ is a linear transformation with $\text{T}(\mathbf{v}) = \begin{pmatrix} 3 & 1 & 2 & -1 \\ 1 & 2 & 0 & 4 \end{pmatrix}\mathbf{v}$.

a Find $\ker(\text{T})$. **b** Find $\mathcal{R}(\text{T})$.

c Verify that nullity (T) + rank (T) = dimension of the domain.

6 Consider $\text{T} : \mathbb{R}^4 \mapsto \mathbb{R}^3$ where $\text{T}(\mathbf{v}) = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 9 & -4 \\ -2 & -1 & -3 & -1 \end{pmatrix}\mathbf{v}$.

a Find $\ker(\text{T})$. **b** Find $\mathcal{R}(\text{T})$.

c Show that $\begin{pmatrix} 0 \\ -7 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ -21 \\ 3 \\ -2 \end{pmatrix}$ are in $\ker(\text{T})$, but $\begin{pmatrix} -3 \\ 3 \\ 1 \\ 2 \end{pmatrix} \notin \ker(\text{T})$.

7 Consider $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ where $T\left(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right) = \begin{pmatrix} x+y-z \\ y+w-x \end{pmatrix}$. Find:

a $\ker(T)$ **b** $\mathcal{R}(T)$ **c** nullity (T) **d** rank (T)

8 If $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a reflection in the line $y = -x$, then $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -y \\ -x \end{pmatrix}$.

Determine the kernel and range for this linear transformation.

9 Consider $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ where $T\left(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right) = \begin{pmatrix} x-y \\ w+z \end{pmatrix}$.

a Find the standard matrix for T. **b** Find $\ker(T)$.

c Find $\mathcal{R}(T)$. **d** Verify the rank-nullity theorem.

COMPOSITION OF LINEAR TRANSFORMATIONS

Suppose $S : \mathbb{R}^k \mapsto \mathbb{R}^m$ is a linear transformation with standard matrix $\mathbf{A}(m \times k)$,
and $T : \mathbb{R}^n \mapsto \mathbb{R}^k$ is a linear transformation with standard matrix $\mathbf{B}(k \times n)$.

The **composition** of S and T is the linear transformation $S \circ T : \mathbb{R}^n \mapsto \mathbb{R}^m$ defined by $(S \circ T)(\mathbf{v}) = S(T(\mathbf{v}))$ for all $\mathbf{v} \in \mathbb{R}^n$, and this has standard matrix $\mathbf{AB}(m \times n)$.

Proof: For every $\mathbf{v} \in \mathbb{R}^n$,

$$\begin{aligned} &(S \circ T)(\mathbf{v}) \\ &= S(T(\mathbf{v})) \\ &= S(\mathbf{Bv}) \\ &= \mathbf{A}(\mathbf{Bv}) \\ &= \mathbf{AB}(\mathbf{v}) \end{aligned}$$

$\therefore$ $S \circ T$ has standard matrix **AB**.

Note that $T \circ S$ can only be defined if $m = n$, since the domain of T must lie in the range of S.

Example 49

Consider $S : \mathbb{R}^3 \mapsto \mathbb{R}^2$ where $S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+z \\ z-y \end{pmatrix}$ and $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ where $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x-y \\ x+y \\ 2x \end{pmatrix}$.

Find $S \circ T$ using:

a the definition of composition **b** standard matrices.

a $(\mathrm{S} \circ \mathrm{T})(\mathbf{v}) = \mathrm{S}(\mathrm{T}(\mathbf{v}))$

$$\begin{aligned} &= \mathrm{S}\left(\begin{pmatrix} x-y \\ x+y \\ 2x \end{pmatrix}\right) \\ &= \begin{pmatrix} (x-y)+2x \\ 2x-(x+y) \end{pmatrix} \\ &= \begin{pmatrix} 3x-y \\ x-y \end{pmatrix} \end{aligned}$$

b Let S have standard matrix **A** and T have standard matrix **B**.

$$\text{Now} \quad \mathbf{A} = \left(\mathrm{S}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \middle| \mathrm{S}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \middle| \mathrm{S}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \right) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\text{and} \quad \mathbf{B} = \left(\mathrm{T}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \middle| \mathrm{T}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \right) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}$$

$$\begin{aligned} \therefore \quad \mathrm{S} \circ \mathrm{T} \text{ has standard matrix } \quad \mathbf{AB} &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \therefore \quad (\mathrm{S} \circ \mathrm{T})\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= \begin{pmatrix} 3 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 3x-y \\ x-y \end{pmatrix} \end{aligned}$$

INVERSE LINEAR TRANSFORMATIONS

The **identity transformation** is the linear transformation $\mathrm{T} : \mathbb{R}^n \mapsto \mathbb{R}^n$ where $\mathrm{T}(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

The standard matrix for the identity transformation is $\mathbf{I}_n$.

Proof:

$$\begin{aligned} \text{The standard matrix} \quad \mathbf{A} &= (\mathrm{T}(\mathbf{e}_1) \,|\, \mathrm{T}(\mathbf{e}_2) \,|\, ... \,|\, \mathrm{T}(\mathbf{e}_n)) \\ &= (\mathbf{e}_1 \,|\, \mathbf{e}_2 \,|\, ... \,|\, \mathbf{e}_n) \qquad \{\text{since } \mathrm{T}(\mathbf{v}) = \mathbf{v} \text{ for all } \mathbf{v} \in \mathbb{R}^n\} \\ &= \mathbf{I}_n \end{aligned}$$

A linear transformation $\mathrm{T} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is **invertible** if there exists a linear transformation $\mathrm{S} : \mathbb{R}^n \mapsto \mathbb{R}^n$ where $(\mathrm{T} \circ \mathrm{S})(\mathbf{v}) = (\mathrm{S} \circ \mathrm{T})(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$. S is called the **inverse linear transformation** of T.

If the linear transformation T has standard matrix **A**, then the standard matrix of S is $\mathbf{A}^{-1}$.

Proof: Let the standard matrix of S be $\mathbf{B}$.

$$\begin{aligned} \text{Now} \quad (\mathrm{T}\circ\mathrm{S})(\mathbf{v}) &= (\mathrm{S}\circ\mathrm{T})(\mathbf{v}) = \mathbf{v} \\ \therefore \quad \mathbf{ABv} &= \mathbf{BAv} \quad = \mathbf{I}_n\mathbf{v} \\ \therefore \quad \mathbf{AB} &= \mathbf{BA} \quad = \mathbf{I}_n \\ \therefore \quad \mathbf{B} &= \mathbf{A}^{-1} \quad \text{by definition of matrix inverse.} \end{aligned}$$

Example 50

Suppose $\mathrm{T}:\mathbb{R}^2 \mapsto \mathbb{R}^2$ is defined by $\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3x+y \\ 4x+y \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

Find $\mathrm{S}:\mathbb{R}^2 \mapsto \mathbb{R}^2$ such that S is the inverse of T.

$$\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$ T has standard matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}$

Now $\mathbf{A}^{-1} = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}$

$$\therefore \quad \mathrm{S}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \quad \mathrm{S}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -x+y \\ 4x-3y \end{pmatrix}$$

EXERCISE 1J.3

1 Consider $\mathrm{S}:\mathbb{R}^3 \mapsto \mathbb{R}^2$ where $\mathrm{S}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x \\ z+x-y \end{pmatrix}$

and $\mathrm{T}:\mathbb{R}^2 \mapsto \mathbb{R}^3$ where $\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y-x \\ y+x \end{pmatrix}$.

Find $\mathrm{T}\circ\mathrm{S}$ using:

a the definition of $\mathrm{T}\circ\mathrm{S}$ **b** standard matrices.

2 Consider $\mathrm{S}:\mathbb{R}^3 \mapsto \mathbb{R}^3$ where $\mathrm{S}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y \\ -x \\ y+z \end{pmatrix}$

and $\mathrm{T}:\mathbb{R}^3 \mapsto \mathbb{R}^3$ where $\mathrm{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x \\ -y \\ z-x \end{pmatrix}$.

Find $\mathrm{S}\circ\mathrm{T}$ using:

a the definition of $\mathrm{S}\circ\mathrm{T}$ **b** standard matrices.

3 Suppose $\mathrm{T}:\mathbb{R}^2 \mapsto \mathbb{R}^2$ is defined by $\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x-2y \\ 2x-3y \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

Find $\mathrm{S}:\mathbb{R}^2 \mapsto \mathbb{R}^2$ such that S is the inverse of T.

4 Suppose $\mathrm{T} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is defined by $\mathrm{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y-z \\ z+2x \\ y-z-x \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

Find $\mathrm{S} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that S is the inverse of T.

5 Let T and S be linear transformations defined by $\mathrm{T} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ where $\mathrm{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ z \\ 2y \end{pmatrix}$ and $\mathrm{S} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ where $\mathrm{S}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ 0 \\ -y \end{pmatrix}$.

- **a** Show that $\mathrm{T} \circ \mathrm{S} \neq \mathrm{S} \circ \mathrm{T}$.
- **b**
  - **i** Find **A** and **B**, the standard matrices for T and S respectively.
  - **ii** Calculate **AB** and **BA**.
  - **iii** What is the significance of your result in **ii**?

6 Prove that the composition of two linear transformations is linear.

APPLICATIONS TO SOLVING $\mathbf{Ax} = \mathbf{b}$

We have seen previously that for an $m \times n$ matrix **A**:

- the **row rank** of **A** is the dimension of the row space of **A**
- the **column rank** of **A** is the dimension of the column space of **A**
- rank (T) is the dimension of the range $\mathscr{R}(\mathrm{T})$.

For a linear transformation T where $\mathrm{T}(\mathbf{v}) = \mathbf{Av}$,
row rank of **A** = column rank of **A** = rank (T).

Example 51

Verify for $\mathrm{T}(\mathbf{v}) = \mathbf{Av} = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & -1 \\ 1 & 3 & -1 & 7 \end{pmatrix}\mathbf{v}$, that

row rank of **A** = column rank of **A** = rank (T).

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & -1 \\ 1 & 3 & -1 & 7 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \boxed{1} & -\frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis of the row space is $\left\{\begin{pmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 1 & -\frac{1}{2} & \frac{5}{2} \end{pmatrix}\right\}$, so row rank = 2.

A basis of the column space is $\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}\right\}$, so column rank = 2.

But $\mathscr{R}(\mathrm{T})$ is the column space of **A**, so rank (T) = column rank = 2. ✓

The system of linear equations $\mathbf{Ax} = \mathbf{b}$ is consistent and thus has a solution
$\Leftrightarrow$ rank $(\mathbf{A})$ = rank $((\mathbf{A} \mid \mathbf{b}))$

Example 52

Use rank to check the system $\begin{cases} x+y+2z=4 \\ x-y-z=3 \\ 2x+4y+7z=11 \end{cases}$ for consistency.

The system has the form $\mathbf{Ax}=\mathbf{b}$ where $\mathbf{A}=\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & -1 \\ 2 & 4 & 7 \end{pmatrix}$, $\mathbf{b}=\begin{pmatrix} 4 \\ 3 \\ 11 \end{pmatrix}$.

Now $\mathbf{A}=\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & -1 \\ 2 & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 0 & \frac{1}{2} \\ 0 & \boxed{1} & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$ so rank $(\mathbf{A})=2$.

$(\mathbf{A}\,|\,\mathbf{b})=\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & -1 & -1 & 3 \\ 2 & 4 & 7 & 11 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & 0 & \frac{1}{2} & 0 \\ 0 & \boxed{1} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$ so rank $(\mathbf{A}\,|\,\mathbf{b})=3$

Since rank $(\mathbf{A}) \neq$ rank $(\mathbf{A}\,|\,\mathbf{b})$, there are no solutions to $\mathbf{Ax}=\mathbf{b}$.

Theorem on a known solution:

Let $\mathbf{x}_0$ be a particular solution of $\mathbf{Ax}=\mathbf{b}$, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots., \mathbf{v}_r\}$ be a basis for the null space of $\mathbf{A}$.

(1) Every solution of $\mathbf{Ax}=\mathbf{b}$ can be written in the form $\mathbf{x}=\mathbf{x}_0+c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots.+c_r\mathbf{v}_r$ where $c_1, c_2, \ldots., c_r$ are constants.

(2) Every vector of the form $\mathbf{x}=\mathbf{x}_0+c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots.+c_r\mathbf{v}_r$ where $c_1, c_2, \ldots., c_r$ are constants, is a solution to $\mathbf{Ax}=\mathbf{b}$.

Proof:

(1) If $\mathbf{x}_0$ is a solution to $\mathbf{Ax}=\mathbf{b}$, then $\mathbf{Ax}_0=\mathbf{b}$

$$\therefore\ \mathbf{Ax}_0=\mathbf{Ax}$$
$$\therefore\ \mathbf{Ax}-\mathbf{Ax}_0=\mathbf{0}$$
$$\therefore\ \mathbf{A}(\mathbf{x}-\mathbf{x}_0)=\mathbf{0}$$

$\therefore$ $\mathbf{x}-\mathbf{x}_0$ is in the null space of $\mathbf{A}$

$\therefore$ there exist scalars $c_1, c_2, \ldots., c_r$ such that $\mathbf{x}-\mathbf{x}_0=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots.+c_r\mathbf{v}_r$

$$\therefore\ \mathbf{x}=\mathbf{x}_0+c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots.+c_r\mathbf{v}_r$$

(2) If there are scalars $c_1, c_2, \ldots., c_r$ such that

$$\mathbf{x}=\mathbf{x}_0+c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots.+c_r\mathbf{v}_r,$$

then $\mathbf{Ax}=\mathbf{A}(\mathbf{x}_0+c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots.+c_r\mathbf{v}_r)$

$$\therefore\ \mathbf{Ax}=\mathbf{Ax}_0+c_1\mathbf{Av}_1+c_2\mathbf{Av}_2+\ldots.+c_r\mathbf{Av}_r$$
$$\therefore\ \mathbf{Ax}=\mathbf{b}+c_1\mathbf{0}+c_2\mathbf{0}+\ldots.+c_r\mathbf{0}$$
$$\therefore\ \mathbf{Ax}=\mathbf{b}$$

Example 53

Use the Theorem on a known solution to solve
$$\begin{cases} x_1 - x_2 - 2x_3 + 3x_4 + x_5 = 1 \\ x_1 - x_2 - 2x_3 + x_5 = 1 \\ \phantom{x_1 -{}} x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 - x_3 + 2x_5 = 2\,. \end{cases}$$

By inspection, $x_1 = 1,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,\ x_5 = 0$ is a particular solution.

The corresponding homogeneous system of equations has augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & -1 & -2 & 3 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} \boxed{1} & 0 & -1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Letting $x_3 = s$ and $x_5 = t$, we find $x_4 = 0$, $x_2 = -s$, and $x_1 = s - t$.

$$\therefore\quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{where } s, t \in \mathbb{R}.$$

Note that if we solve the system in **Example 53** directly, we have $(\mathbf{A}\,|\,\mathbf{b}) \sim \left(\begin{array}{ccccc|c} \boxed{1} & 0 & -1 & 0 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$

Letting $x_3 = s$ and $x_5 = t$, we find $x_4 = 0$, $x_2 = -s$, and $x_1 = 1 + s - t$.

Hence $\mathbf{x} = \begin{pmatrix} 1+s-t \\ -s \\ s \\ 0 \\ t \end{pmatrix}$ which is the same form as in **Example 53**.

However, for a different choice of particular solution, we may end up with the same set of solutions but in a different form.

For example, suppose the particular solution we chose was $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_5 = 1$. The homogeneous solutions would be the same, but the final solution would have the form

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} s-t \\ -s \\ s \\ 0 \\ 1+t \end{pmatrix} \quad \text{where } s, t \in \mathbb{R}.$$

EXERCISE 1J.4

1 Verify for each linear transformation $\text{T}(\mathbf{v}) = \mathbf{A}\mathbf{v}$, that row rank of $\mathbf{A}$ = column rank of $\mathbf{A}$ = rank $(\mathbf{A})$:

a $\text{T}(\mathbf{v}) = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}\mathbf{v}$

b $\text{T}(\mathbf{v}) = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 0 & -1 & 1 \end{pmatrix}\mathbf{v}$

c $\text{T}(\mathbf{v}) = \begin{pmatrix} 1 & -1 & -2 & 3 & 1 \\ 1 & -1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 3 \end{pmatrix}\mathbf{v}$

2 Use rank to check each system for consistency:

a $\begin{cases} x - y + z = 4 \\ 2x + y + z = 6 \\ 3x + 3y + z = 8 \end{cases}$

b $\begin{cases} x + 2y + z = 2 \\ x + y + z = -1 \\ 3x + 4y + 3z = 10 \end{cases}$

3 Use the Theorem on a known solution to solve:

a $\begin{cases} x_1 + x_2 - x_3 + x_4 = 3 \\ x_1 - x_2 + x_3 + 2x_4 = 3 \end{cases}$

b $\begin{cases} x_1 - x_2 + x_3 - x_4 + x_5 = 1 \\ x_1 + x_2 - x_3 - x_4 - x_5 = -1 \\ 2x_1 + x_2 + 3x_3 - 2x_4 + x_5 = 3 \end{cases}$

K GEOMETRIC TRANSFORMATIONS

In this Section we apply linear transformations to functions and relations in the Cartesian plane $\mathbb{R}^2$.

In $\mathbb{R}^2$, suppose $P(x, y)$ moves to $P'(x', y')$ under the linear transformation

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \quad \text{where } a, b, c, d \in \mathbb{R}.$$

We say that:

- P has been subjected to a **geometric linear transformation**
- P is the **object** point and P′ is the **image** of P.

In matrix form, we write $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ or $\mathbf{v}' = \mathbf{A}\mathbf{v}$

where $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the **transformation matrix**.

Example 54

A linear transformation T has equations $\begin{cases} x' = 3x - y \\ y' = x + y. \end{cases}$ Under T, find the image of:

a the point $(3, 2)$

b the line with equation $y = 2x - 1$.

a $x' = 3x - y = 3(3) - (2) = 7$

$y' = x + y = 3 + 2 = 5$

$\therefore \ (3, 2) \xrightarrow{T} (7, 5)$

b We need to write the equation of the image line in terms of x' and y'.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \ \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \ \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{4}x' + \frac{1}{4}y' \\ -\frac{1}{4}x' + \frac{3}{4}y' \end{pmatrix}$$

$\therefore \ y = 2x - 1$ becomes

$$-\tfrac{1}{4}x' + \tfrac{3}{4}y' = 2\left(\tfrac{1}{4}x' + \tfrac{1}{4}y'\right) - 1$$

$$\Rightarrow \ -x' + 3y' = 2x' + 2y' - 4$$

$$\Rightarrow \ 3x' - y' = 4$$

Hence $y = 2x - 1 \xrightarrow{T} 3x - y = 4$

DISCUSSION

Under a linear transformation, will the shape of an object be preserved in its image?

For example, will a line always remain a line, a circle always remain a circle, and so on?

EXERCISE 1K.1

1 A linear transformation T has equations $\begin{cases} x' = 2x + y \\ y' = x - y. \end{cases}$ Under T, find the image of:

- **a** the point $(0, 1)$
- **b** the point $(-1, -3)$
- **c** the line $y = 3x + 2$
- **d** the circle $x^2 + y^2 = 1$
- **e** the parabola $y = x^2 + 1$.

2 **a** Find the equations of the linear transformation S which maps $(2, 1)$ onto $(4, 1)$ and $(-1, 3)$ onto $(-7, 3)$.

b Under S, a point is transformed to $(3, -1)$. Find the object point.

3 Show that under the linear transformation $\mathbf{v}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{v} = \mathbf{A}\mathbf{v}$, the line with equation $y = mx + k$ maps onto another line provided $|\mathbf{A}| \neq 0$.

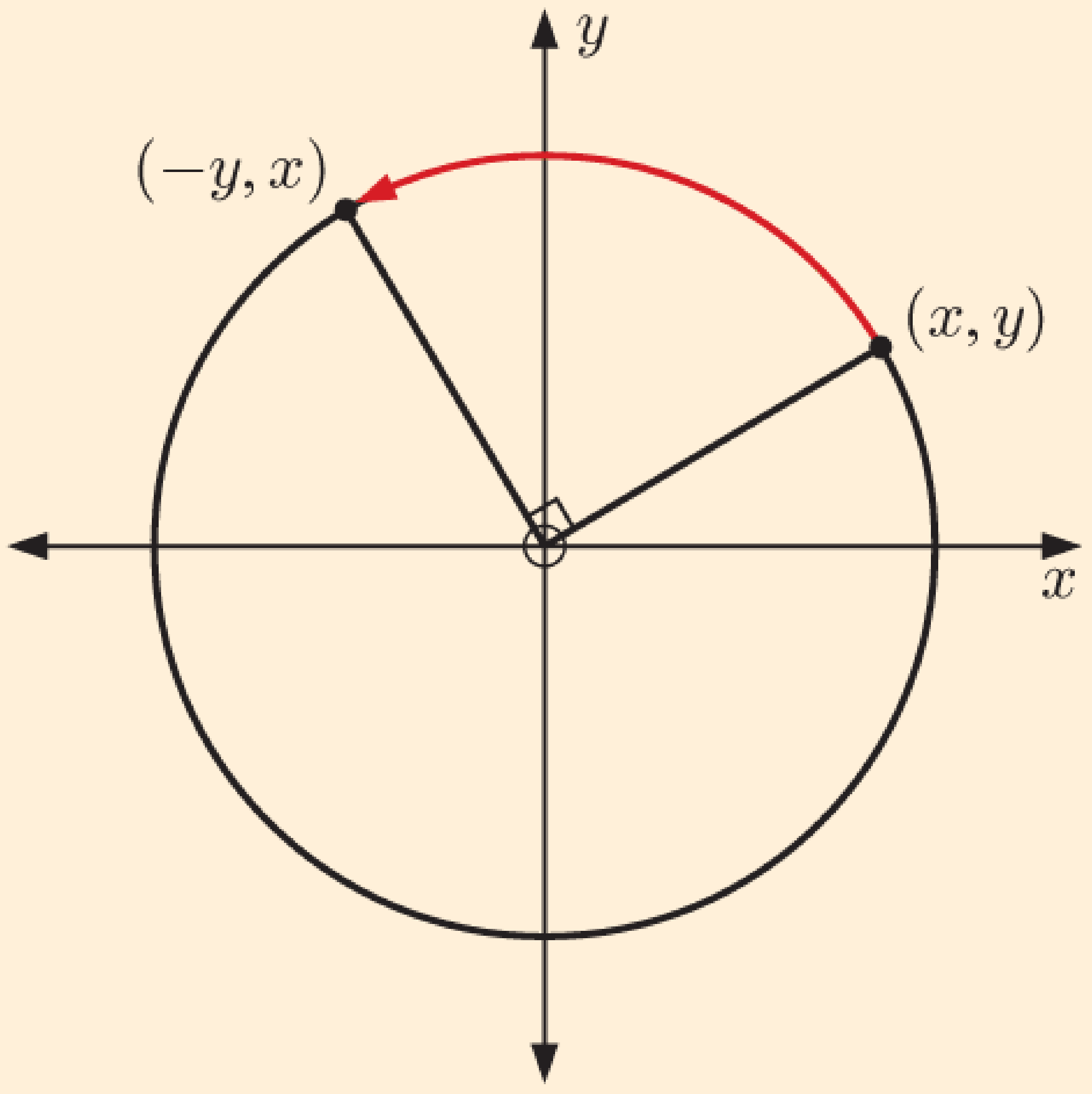
4 Under what conditions does the linear transformation $\mathbf{v}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{v} = \mathbf{A}\mathbf{v}$ transform the circle $x^2 + y^2 = 1$ into:

- **a** an ellipse
- **b** a circle?

ROTATIONS AND REFLECTIONS

In the following **Investigation** we consider common rotations and reflections which you would have studied in previous years, but this time in terms of linear transformations.

INVESTIGATION 4 ROTATIONS AND REFLECTIONS



Under an anticlockwise rotation about O through $\frac{\pi}{2}$, $(3, 1) \to (-1, 3)$, and in general $(x, y) \to (-y, x)$.

$\therefore \begin{cases} x' = -y \\ y' = x \end{cases}$ are the transformation equations for an anticlockwise rotation about O through $\frac{\pi}{2}$.

In matrix form, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

with transformation matrix $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $|\mathbf{A}| = 1$.

What to do:

1 Copy and complete:

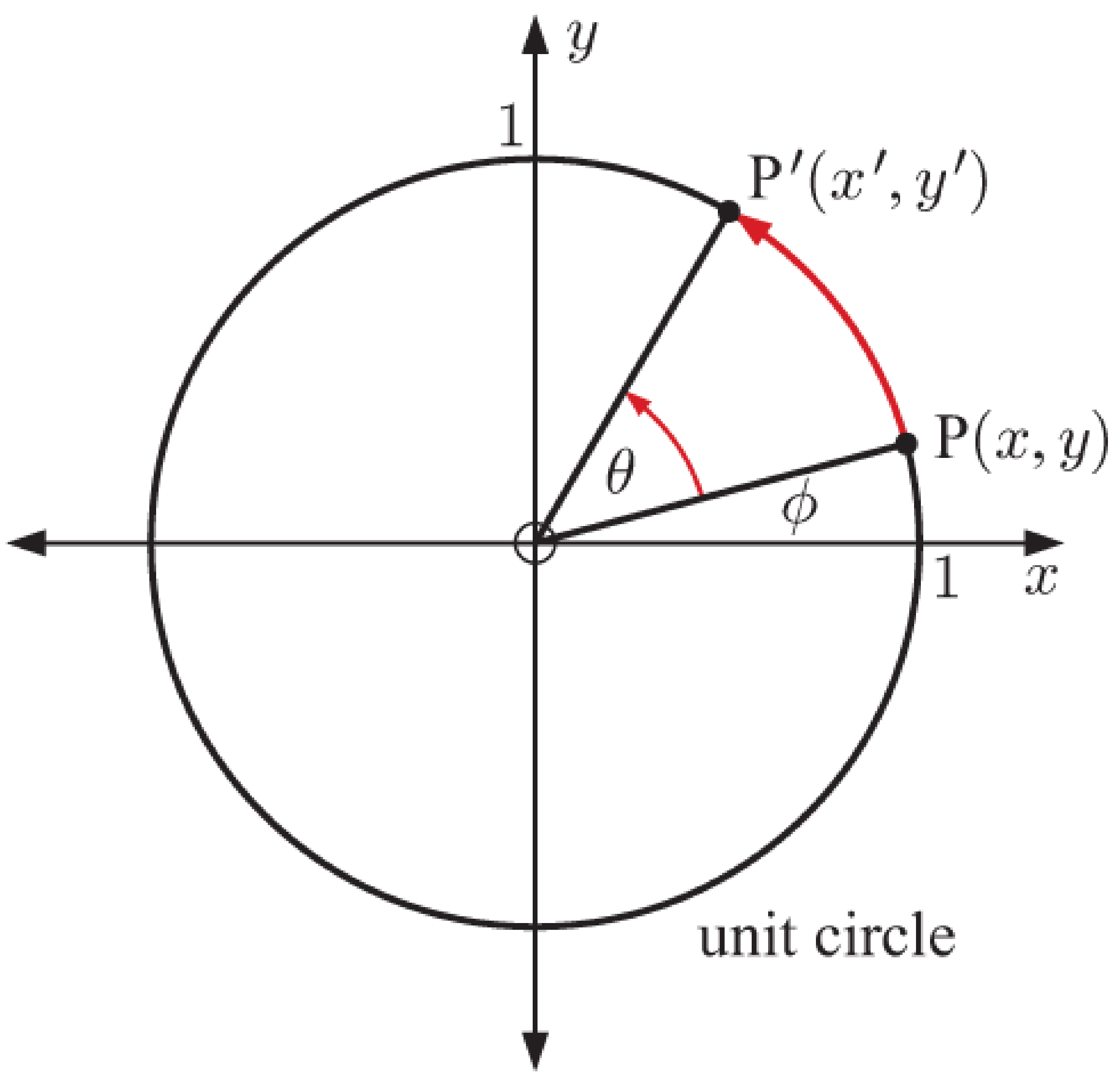
Transformation	$\mathbf{A}$	$\lvert\mathbf{A}\rvert$	*Transformation*	$\mathbf{A}$	$\lvert\mathbf{A}\rvert$
Anticlockwise rotation about O through $\frac{\pi}{2}$.	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	1	Reflection in the x-axis.		
Clockwise rotation about O through $\frac{\pi}{2}$.			Reflection in the y-axis.		
Rotation about O through π.			Reflection in the line $y = x$.		
Rotation about O through 0.			Reflection in the line $y = -x$.		

2 Record any observations from your results in **1**.

ROTATIONS ABOUT O(0, 0) THROUGH θ

We now consider a more general rotation about O(0, 0), this time anticlockwise through an arbitrary angle θ.

Consider the following traditional method for finding the corresponding transformation matrix:



Let $P(x, y)$ be a point on the unit circle which makes an angle ϕ with the positive x-axis.

$\therefore$ [OP′] makes angle $\phi + \theta$ with the x-axis.

Since the unit circle has radius 1,
P is $(\cos\phi, \sin\phi)$ and P′ is $(\cos(\phi+\theta), \sin(\phi+\theta))$.

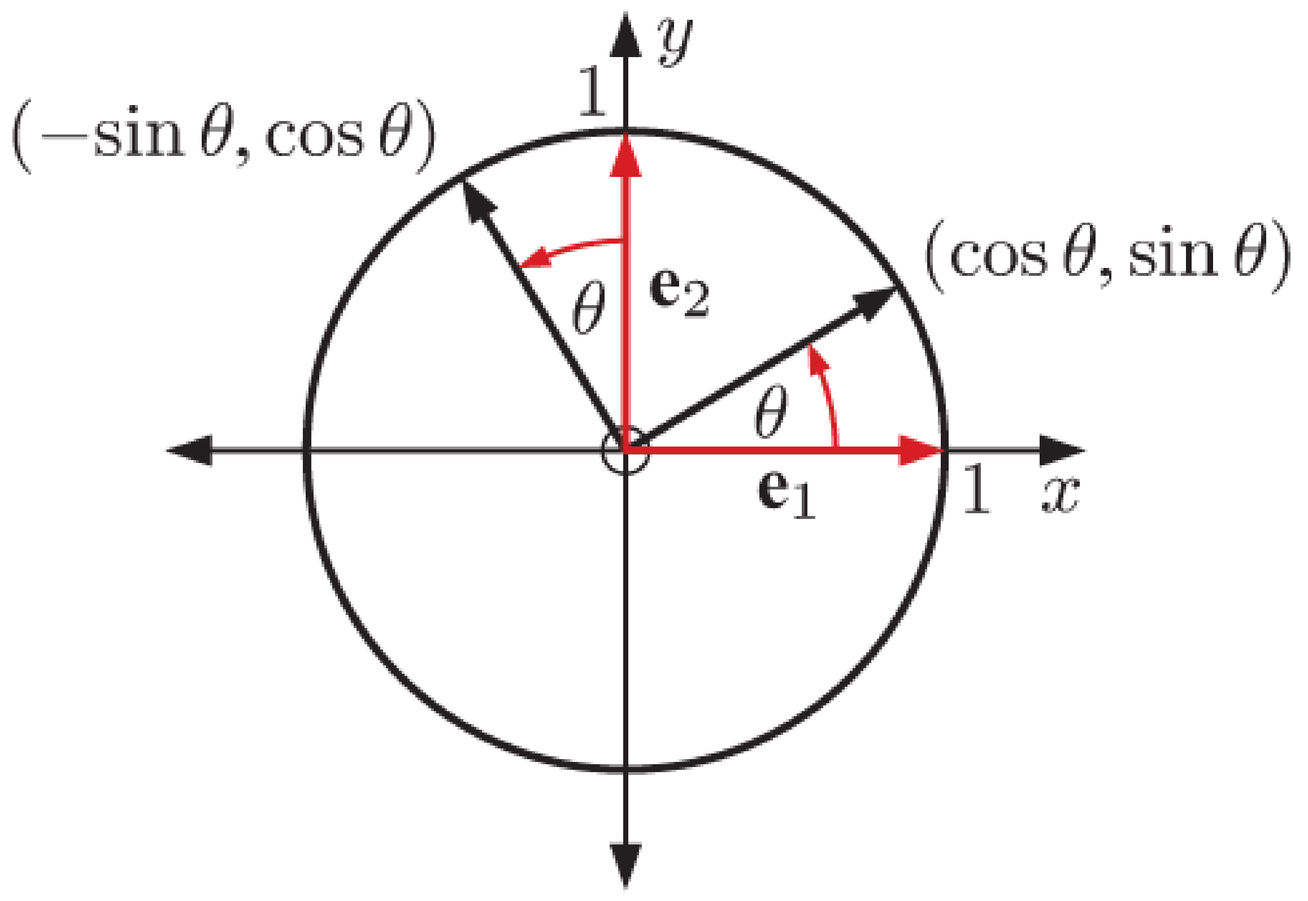
$$\begin{aligned} \text{Thus}\quad x' &= \cos(\phi+\theta) \\ &= \cos\phi\cos\theta - \sin\phi\sin\theta \\ &= x\cos\theta - y\sin\theta \\ \text{and}\quad y' &= \sin(\phi+\theta) \\ &= \sin\phi\cos\theta + \cos\phi\sin\theta \\ &= y\cos\theta + x\sin\theta \end{aligned}$$

Hence, the transformation equations are: $\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{cases}$

with $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

We notice that $\lvert\mathbf{A}\rvert = \cos^2\theta + \sin^2\theta = 1$.

An alternative method for generating the transformation matrix is to use vectors:


y
1
$(-\sin\theta, \cos\theta)$
θ
$\mathbf{e}_2$
$(\cos\theta, \sin\theta)$
θ
$\mathbf{e}_1$
1
x

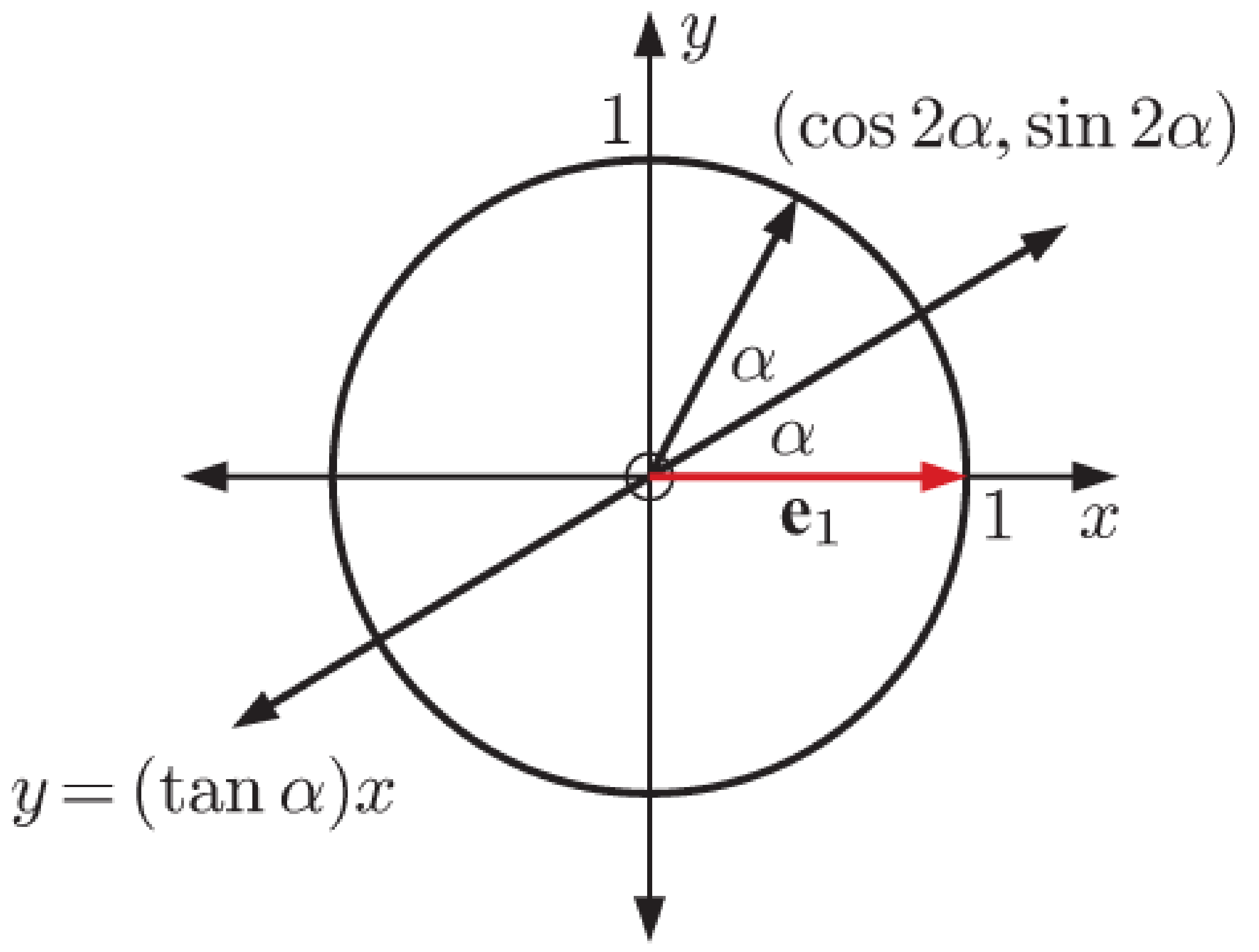
The standard matrix for T, a rotation anticlockwise about O(0, 0) through θ, is:

$$\mathbf{A} = (\mathrm{T}(\mathbf{e}_1) \,|\, \mathrm{T}(\mathbf{e}_2))$$
$$= \left(\mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right) \middle| \mathrm{T}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)\right)$$
$$= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

For a rotation anticlockwise about O(0, 0) through θ, the transformation matrix is

$$\mathbf{A} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \quad \text{with} \quad |\mathbf{A}| = 1.$$

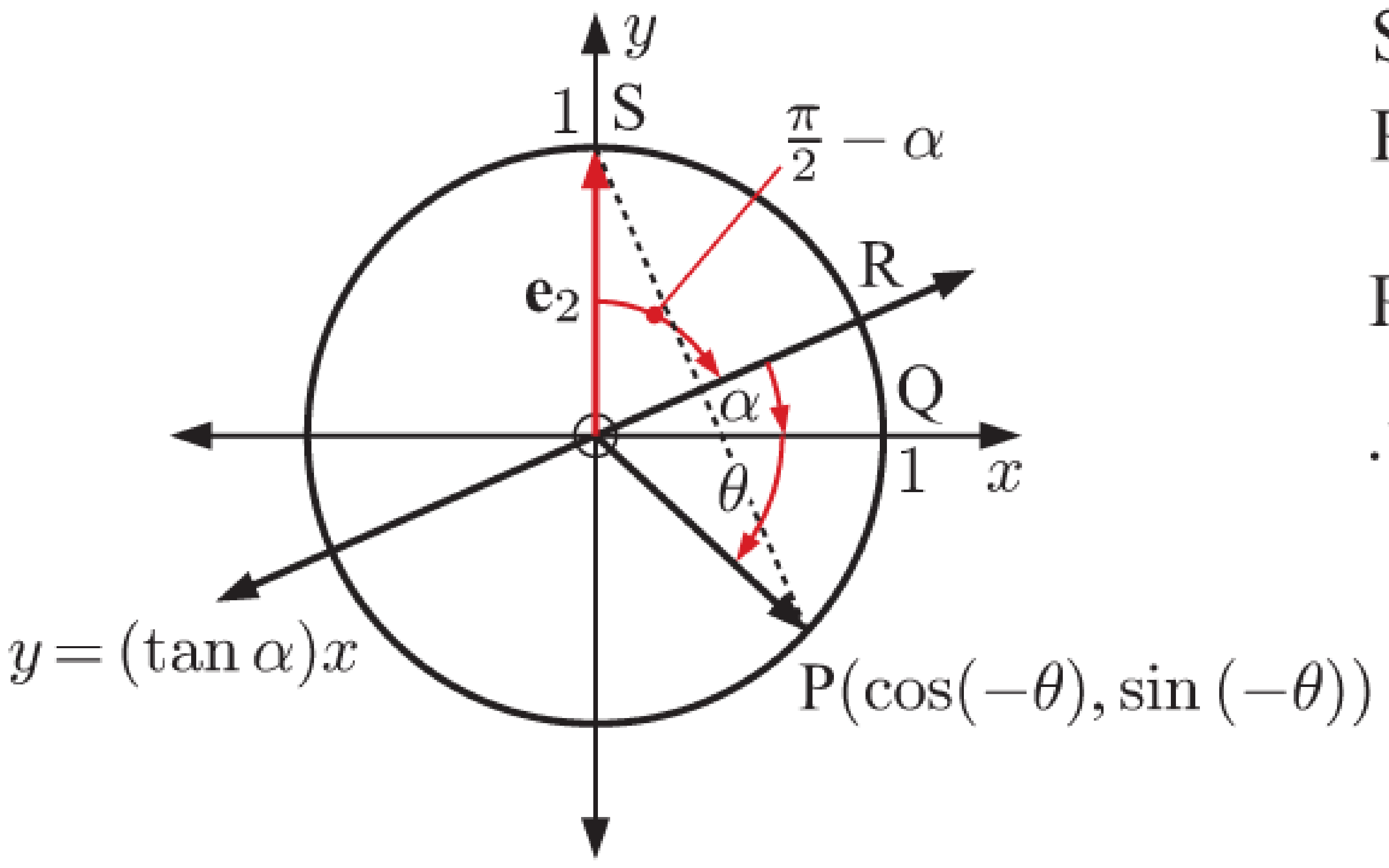
REFLECTION IN THE LINE $y = (\tan\alpha)x$


y
1
$(\cos 2\alpha, \sin 2\alpha)$
α
α
$\mathbf{e}_1$
1
x
$y = (\tan\alpha)x$

Consider a mirror line which makes an angle α with the x-axis.

We use vector methods to find the transformation matrix corresponding to a reflection in this line.

$$\mathrm{T}(\mathbf{e}_1) = \mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}\cos 2\alpha\\ \sin 2\alpha\end{pmatrix}$$


y
1
S
$\frac{\pi}{2} - \alpha$
R
$\mathbf{e}_2$
α
Q
θ
1
x
$y = (\tan\alpha)x$
$\mathrm{P}(\cos(-\theta), \sin(-\theta))$

Suppose the point S(0, 1) is reflected to point P(cos(−θ), sin(−θ)).

$$\widehat{\mathrm{POQ}} = \widehat{\mathrm{POR}} - \alpha$$
$$\therefore \quad \theta = \widehat{\mathrm{SOR}} - \alpha \qquad \{\text{reflection}\}$$
$$= \left(\tfrac{\pi}{2} - \alpha\right) - \alpha$$
$$= \tfrac{\pi}{2} - 2\alpha \quad \text{and is the magnitude of } \widehat{\mathrm{POQ}}.$$

Thus $$\mathrm{T}(\mathbf{e}_2) = \mathrm{T}\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}\cos\left(2\alpha - \frac{\pi}{2}\right)\\ \sin\left(2\alpha - \frac{\pi}{2}\right)\end{pmatrix}$$
$$= \begin{pmatrix}\sin 2\alpha\\ -\cos 2\alpha\end{pmatrix}$$

Hence $\mathbf{A} = (\mathrm{T}(\mathbf{e}_1) \,|\, \mathrm{T}(\mathbf{e}_2)) = \begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix}$ and $|\mathbf{A}| = -\cos^2 2\alpha - \sin^2 2\alpha = -1$.

For a reflection in the mirror line $y = (\tan\alpha)x$, the transformation matrix is

$$\mathbf{A} = \begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix} \quad \text{with} \quad |\mathbf{A}| = -1.$$

The following formulae are useful for working with the transformation matrix for reflections:

If $m = \tan\alpha$, then:

- $\cos 2\alpha = \dfrac{1-m^2}{1+m^2}$
- $\sin 2\alpha = \dfrac{2m}{1+m^2}$
- $\tan 2\alpha = \dfrac{2m}{1-m^2}$.

Proof:

- $$\begin{aligned}&\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\\ &= \frac{1-\frac{\sin^2\alpha}{\cos^2\alpha}}{1+\frac{\sin^2\alpha}{\cos^2\alpha}}\\ &= \frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha+\sin^2\alpha}\\ &= \frac{\cos 2\alpha}{1}\\ &= \cos 2\alpha\end{aligned}$$
- $$\begin{aligned}&\frac{2\tan\alpha}{1+\tan^2\alpha}\\ &= \frac{\frac{2\sin\alpha}{\cos\alpha}}{1+\frac{\sin^2\alpha}{\cos^2\alpha}}\\ &= \frac{2\cos\alpha\sin\alpha}{\cos^2\alpha+\sin^2\alpha}\\ &= \frac{\sin 2\alpha}{1}\\ &= \sin 2\alpha\end{aligned}$$
- $$\begin{aligned}&\tan 2\alpha\\ &= \frac{\sin 2\alpha}{\cos 2\alpha}\\ &= \frac{\frac{2m}{1+m^2}}{\frac{1-m^2}{1+m^2}}\\ &= \frac{2m}{1-m^2}\end{aligned}$$

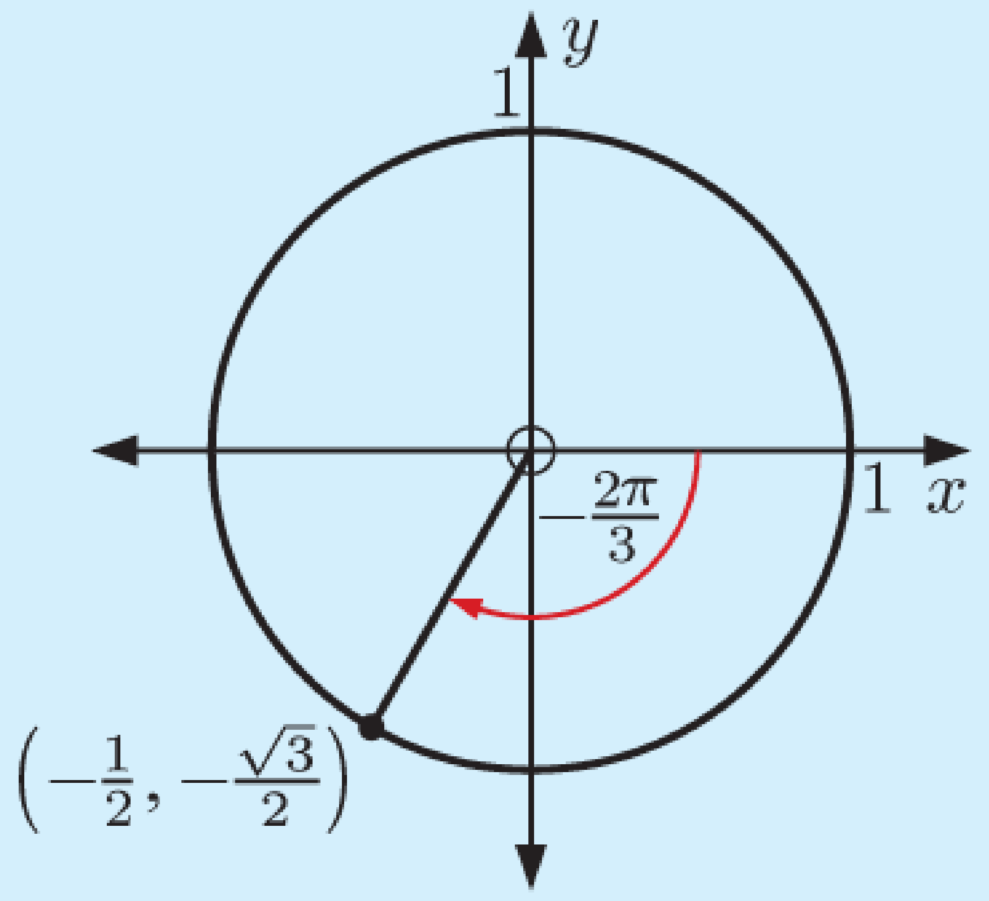
Example 55

Find the transformation matrix $\mathbf{A}$ for:

a a clockwise rotation about O through $\frac{2\pi}{3}$

b a reflection in the line $y = 3x$.

a



$\cos\theta = -\frac{1}{2}$, $\sin\theta = -\frac{\sqrt{3}}{2}$

$$\therefore \quad \mathbf{A} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$
$$= \begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$$

b $m = \tan\alpha = 3$

Now $\cos 2\alpha = \dfrac{1-m^2}{1+m^2} = \frac{-8}{10} = -\frac{4}{5}$

and $\sin 2\alpha = \dfrac{2m}{1+m^2} = \frac{6}{10} = \frac{3}{5}$

$$\therefore \quad \mathbf{A} = \begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix}$$
$$= \begin{pmatrix}-\frac{4}{5} & \frac{3}{5}\\ \frac{3}{5} & \frac{4}{5}\end{pmatrix}$$

Example 56

Find the nature of the transformation with equations:

a $\begin{cases} x' = \dfrac{-3x-4y}{5} \\ y' = \dfrac{-4x+3y}{5} \end{cases}$

b $\begin{cases} x' = \dfrac{3x-4y}{5} \\ y' = \dfrac{4x+3y}{5} \end{cases}$

a $\mathbf{A} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$ where $|\mathbf{A}| = -1$.

$\mathbf{A}$ has form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ so $\mathbf{A}$ is a reflection matrix where $\cos 2\alpha = -\frac{3}{5}$ and $\sin 2\alpha = -\frac{4}{5}$

$\therefore \quad \tan 2\alpha = \frac{4}{3}$ and $-\pi < 2\alpha < 0$.

If $m = \tan\alpha$ then

$\dfrac{2m}{1-m^2} = \frac{4}{3}$ which simplifies to

$$2m^2 + 3m - 2 = 0$$

$$\therefore \quad (2m-1)(m+2) = 0$$

$$\therefore \quad m = \tfrac{1}{2} \text{ or } -2$$

But $-\frac{\pi}{2} < \alpha < 0$ so $m < 0$

$\therefore \quad \tan\alpha = -2$

$\therefore$ the transformation is a reflection in the line $y = -2x$.

b $\mathbf{A} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ where $|\mathbf{A}| = 1$.

Since $\mathbf{A}$ has form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $\mathbf{A}$ is a rotation matrix.

If the angle of rotation is θ, $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$

$\Rightarrow \quad \tan\theta = \frac{4}{3}$

$\Rightarrow \quad \theta = \arctan\left(\frac{4}{3}\right)$

(or $\theta \approx 0.927^c$)

$\therefore$ the transformation is an anticlockwise rotation about O(0, 0) through $\arctan\left(\frac{4}{3}\right)$.

EXERCISE 1K.2

1 Find the transformation matrix $\mathbf{A}$ for:

- **a** a reflection in the y-axis
- **b** an anticlockwise rotation about O through $\frac{\pi}{2}$
- **c** an anticlockwise rotation about O through $\frac{\pi}{3}$
- **d** a reflection in the line $y = -x$
- **e** a clockwise rotation about O through $\frac{\pi}{4}$
- **f** a reflection in the line $y = 5x$
- **g** a clockwise rotation about O through $\frac{5\pi}{6}$
- **h** a reflection in the line $y = \sqrt{3}x$
- **i** an anticlockwise rotation about O through $\frac{5\pi}{4}$.

2 Find the nature of the transformation with equations:

a $\begin{cases} x' = \frac{1}{\sqrt{2}}(x+y) \\ y' = \frac{1}{\sqrt{2}}(y-x) \end{cases}$

b $\begin{cases} x' = \frac{1}{\sqrt{2}}(x+y) \\ y' = \frac{1}{\sqrt{2}}(x-y) \end{cases}$

c $\begin{cases} x' = \dfrac{-5x+12y}{13} \\ y' = \dfrac{-12x-5y}{13} \end{cases}$

d $\begin{cases} x' = \dfrac{8y-15x}{17} \\ y' = \dfrac{8x+15y}{17} \end{cases}$

3 **a** Show that the matrix of a linear transformation which maps $(\sqrt{2}, -\sqrt{2})$ onto $(0, 2)$ has the form $\mathbf{A} = \begin{pmatrix} s & s \\ t+\sqrt{2} & t \end{pmatrix}$ for some $s, t \in \mathbb{R}$.

b Discuss whether the linear transformation in **a** can be

i a rotation about O **ii** a reflection in the line $y = (\tan\alpha)x$.

4 Let $\mathbf{A}$ be the transformation matrix for an anticlockwise rotation about O through angle θ. Show that $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$, and explain the significance of this result.

5 Let $\mathbf{A}$ be the transformation matrix for a reflection in the line $y = (\tan\alpha)x$. Show that $\mathbf{A}^{-1} = \mathbf{A}$, and explain the significance of this result.

6 Let T be an anticlockwise rotation about O through $-\frac{2\pi}{3}$. Under T, find the image of:

a the point $(5, -1)$ **b** the line $y = 3x - 1$ **c** the rectangular hyperbola $y = \dfrac{1}{x}$.

7 Let S be a reflection in the line $y = -2x$. Under S, find the image of:

a the point $(-4, 2)$ **b** the line $y = 2 - x$ **c** the parabola $y = x^2$.

8 Suppose that $y = mx + c$ under a reflection in $y = \frac{2}{3}x$, becomes $32x + 43y = 13$. Find the equation of the object line.

9 A straight line under an anticlockwise rotation about O through $\frac{3\pi}{4}$, becomes $x - 3y = -\sqrt{2}$. Find the equation of the object line.

SENSE AND AREA

INVESTIGATION 5 SENSE AND AREA

The square OABC with vertices O(0, 0), A(1, 0), B(1, 1), and C(0, 1), is called the "unit square".

What to do:

1 Illustrate OABC and its image O′A′B′C′ under a linear transformation with matrix $\mathbf{A}$, where $\mathbf{A}$ is:

a $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ **d** $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ **e** $\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$

2 Find $\det(\mathbf{A})$ for each matrix $\mathbf{A}$ in **1**.

3 The labelling of square OABC is anticlockwise. If the labelling of O′A′B′C′ is also anticlockwise, we say that **sense** has been preserved; otherwise, we say that sense has been reversed.

What is the connection between $\det(\mathbf{A})$ and sense for a linear transformation with matrix $\mathbf{A}$?

4 For each of the matrices in **1**, find $\dfrac{\text{area O}'\text{A}'\text{B}'\text{C}'}{\text{area OABC}}$.

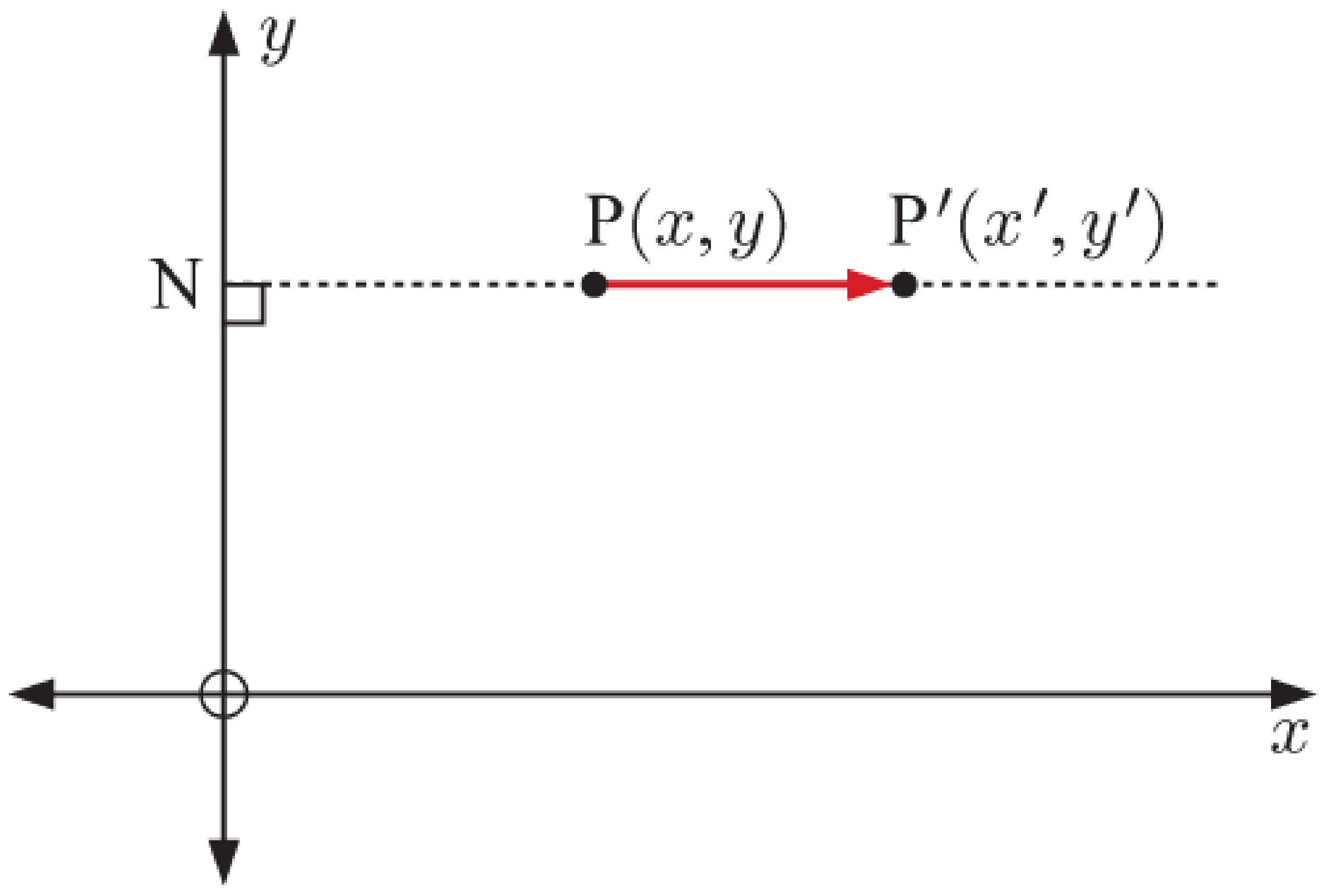
What is the relationship between this fraction and $\det(\mathbf{A})$?

5 Discuss the effect on sense and area for: **a** rotations **b** reflections.

STRETCHES

Consider a stretch which is parallel to the x-axis with scale factor k. The point P is moved to P$'$ where [PP$'$] is parallel to the x-axis, and if (PP$'$) meets the y-axis at N then $\text{NP}' = k\text{NP}$, $k > 0$.

We therefore find: $\begin{cases} x' = \text{NP}' = k\text{NP} = kx \\ y' = y \end{cases}$

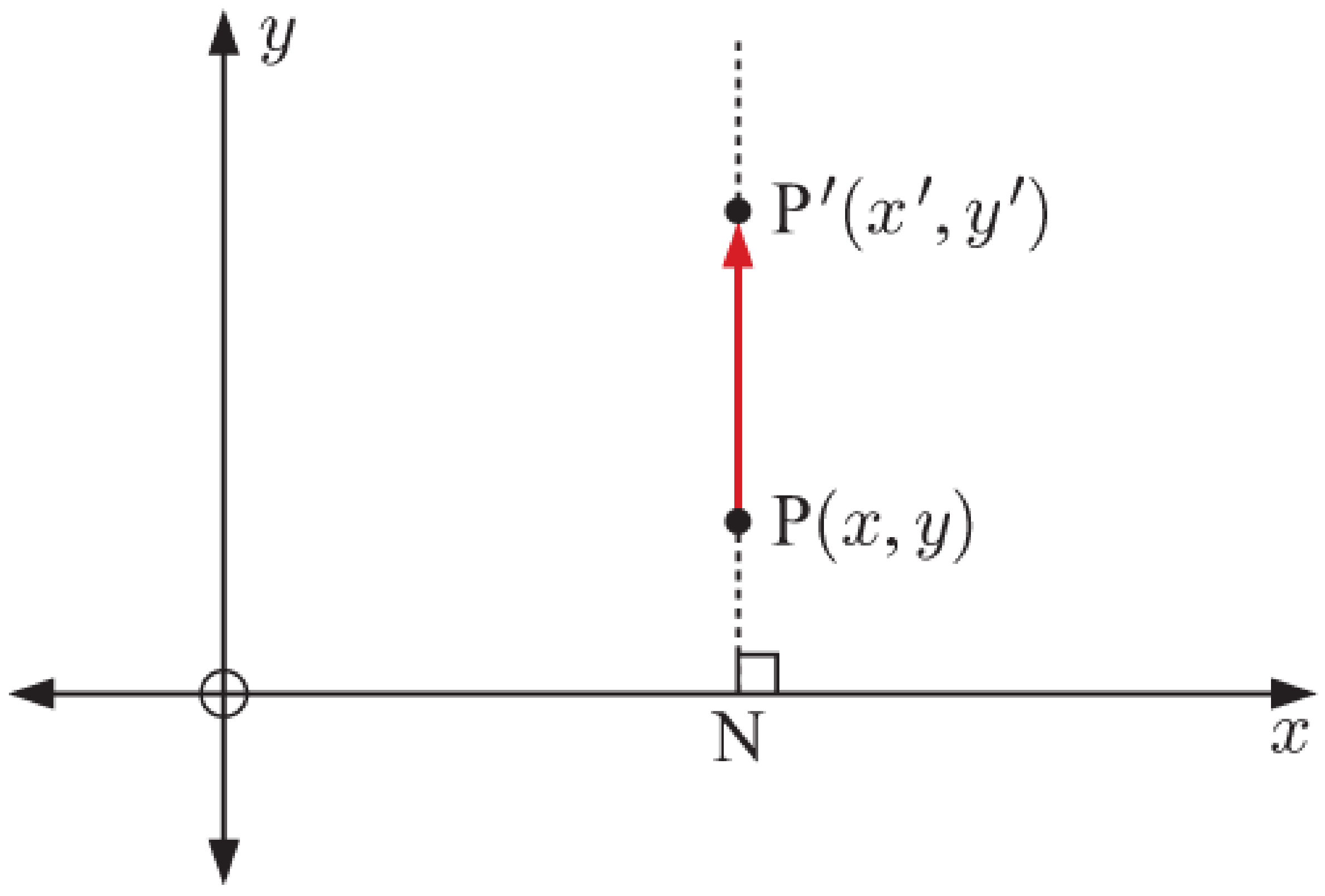


For a **stretch parallel to the x-axis** with **scale factor** k, the transformation matrix is

$\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ and $|\mathbf{A}| = k$.

Now consider a stretch which is parallel to the y-axis with scale factor k. The point P is moved to P$'$ where [PP$'$] is parallel to the y-axis, and if (PP$'$) meets the x-axis at N then $\text{NP}' = k\text{NP}$, $k > 0$.

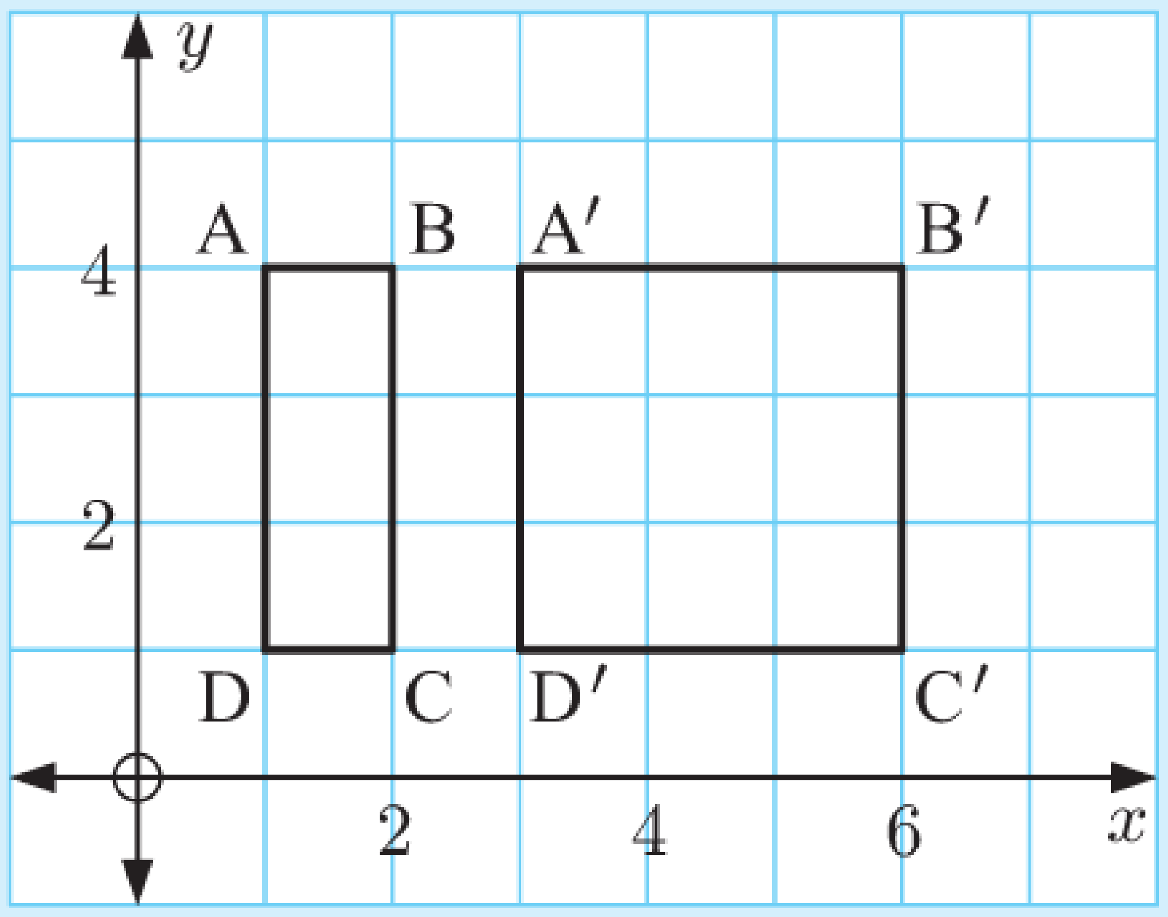
We therefore find: $\begin{cases} x' = x \\ y' = \text{NP}' = k\text{NP} = ky \end{cases}$



For a **stretch parallel to the y-axis** with **scale factor** k, the transformation matrix is

$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ and $|\mathbf{A}| = k$.

Example 57



A linear transformation maps rectangle ABCD onto square A$'$B$'$C$'$D$'$.

a Identify the geometric linear transformation T.

b Write down the transformation matrix.

c Verify that $\dfrac{\text{area A}'\text{B}'\text{C}'\text{D}'}{\text{area ABCD}} = ||\mathbf{A}||$.

a T is a stretch, parallel to the x-axis with scale factor $k = 3$.

b $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

c $\dfrac{\text{area A}'\text{B}'\text{C}'\text{D}'}{\text{area ABCD}}$

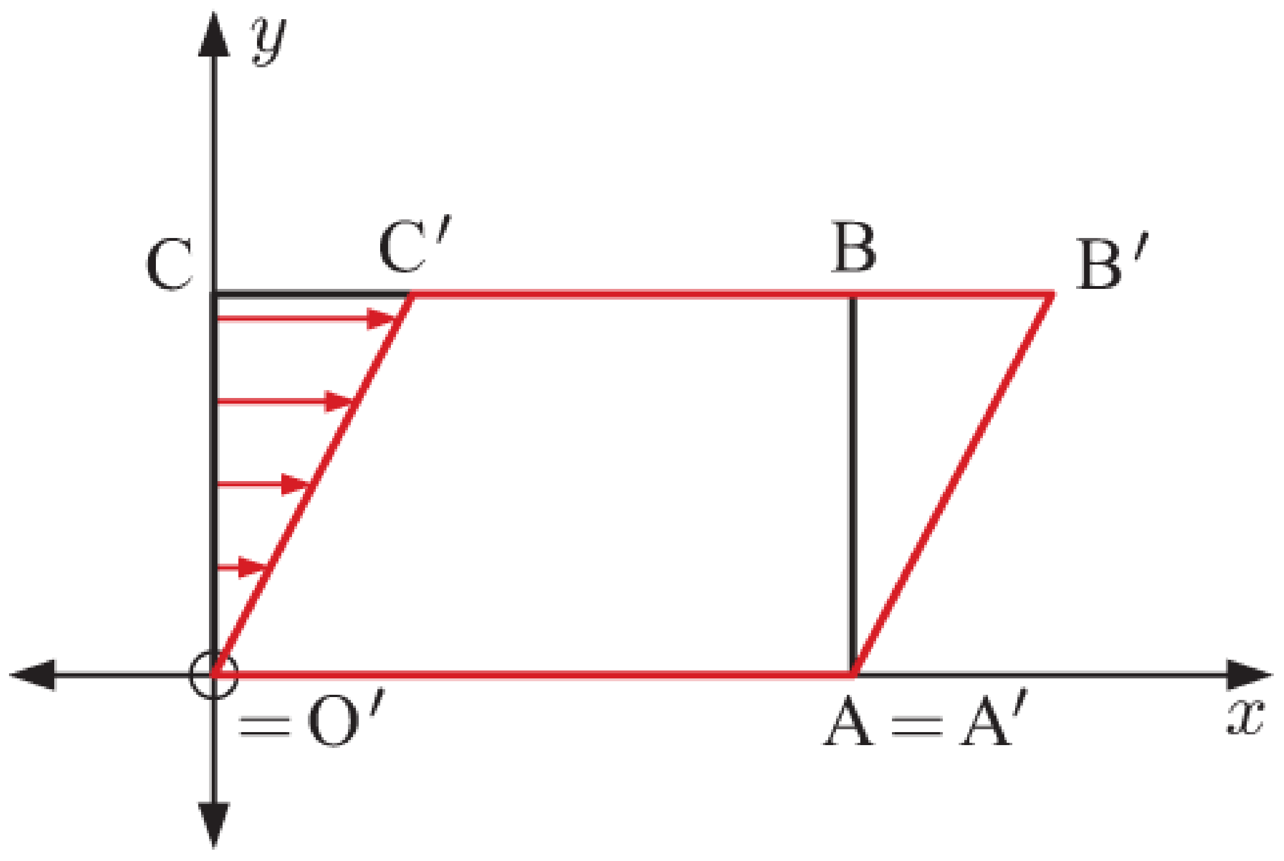
$= \frac{9}{3}$

$= 3$

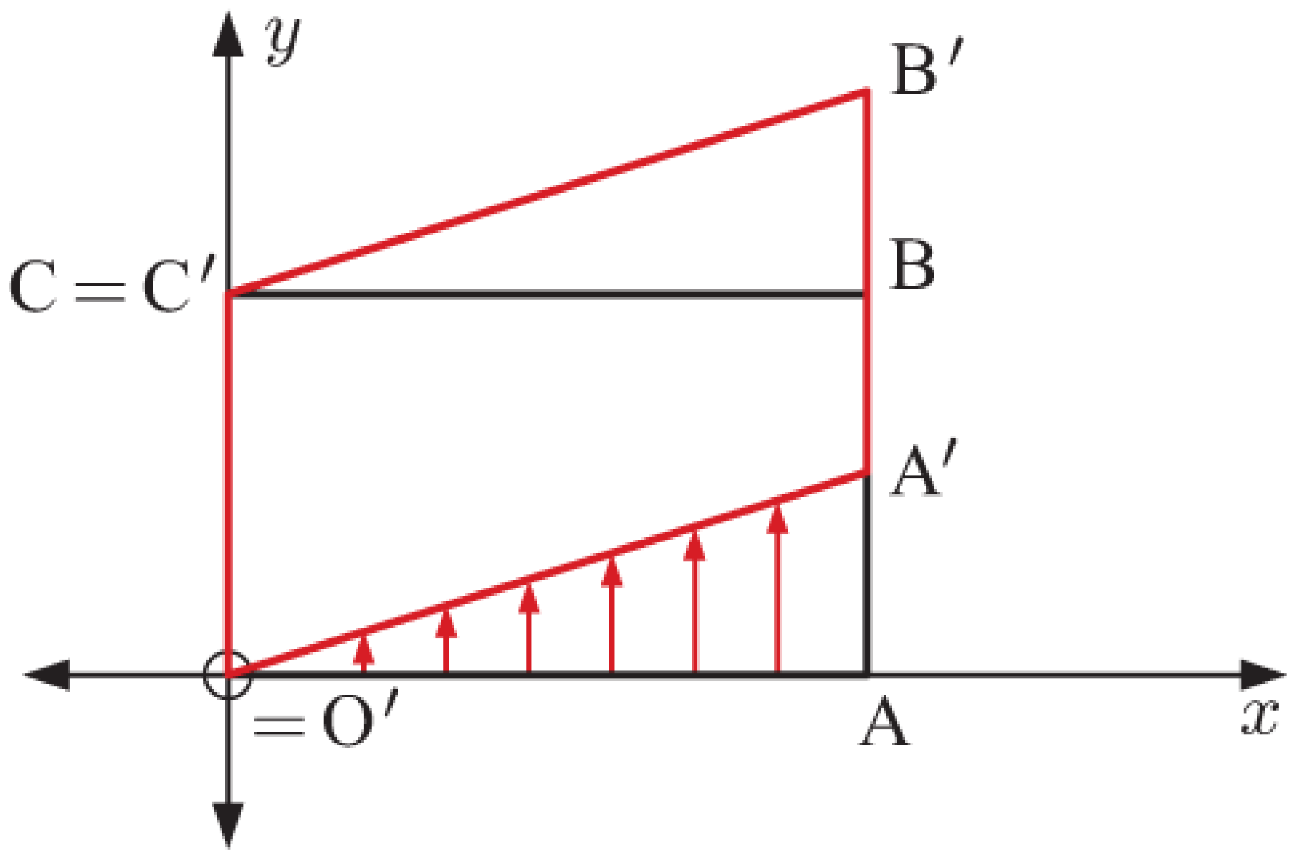
$= ||\mathbf{A}||$

SHEARS

The two diagrams below show rectangle OABC subjected to **shears** in different directions. In each case a parallelogram O′A′B′C′ results.



shear parallel to the x-axis



shear parallel to the y-axis

For a shear parallel to the x-axis with scale factor k, the point P is moved to P′ where [PP′] is parallel to the x-axis, and if N is the foot of the perpendicular from P to the x-axis then $\text{PP}' = k\text{PN}$, $k > 0$.

We therefore find: $\begin{cases} x' = x + \text{PP}' = x + k\text{PN} = x + ky \\ y' = y \end{cases}$

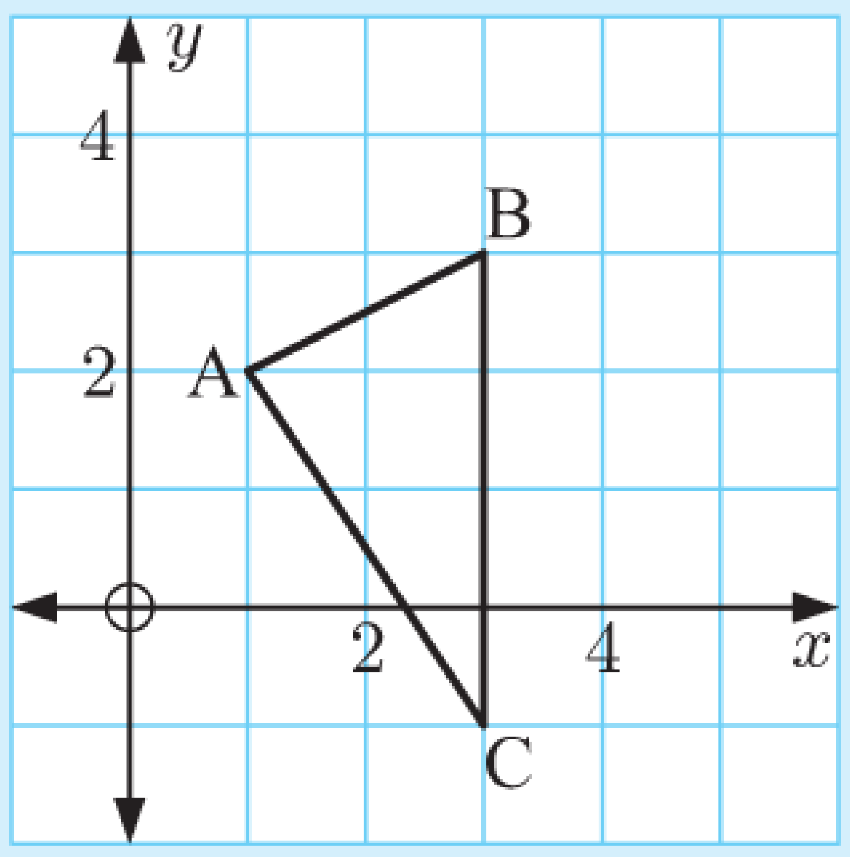
For a **shear parallel to the x-axis** with **scale factor** k, the transformation matrix is $\mathbf{A} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and $|\,\mathbf{A}\,| = 1$.

For a shear parallel to the y-axis with scale factor k, the point P is moved to P′ where [PP′] is parallel to the y-axis, and if N is the foot of the perpendicular from P to the y-axis then $\text{PP}' = k\text{PN}$, $k > 0$.

We therefore find: $\begin{cases} x' = x \\ y' = y + \text{PP}' = y + k\text{PN} = y + kx \end{cases}$

For a **shear parallel to the y-axis** with **scale factor** k, the transformation matrix is $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ and $|\,\mathbf{A}\,| = 1$.

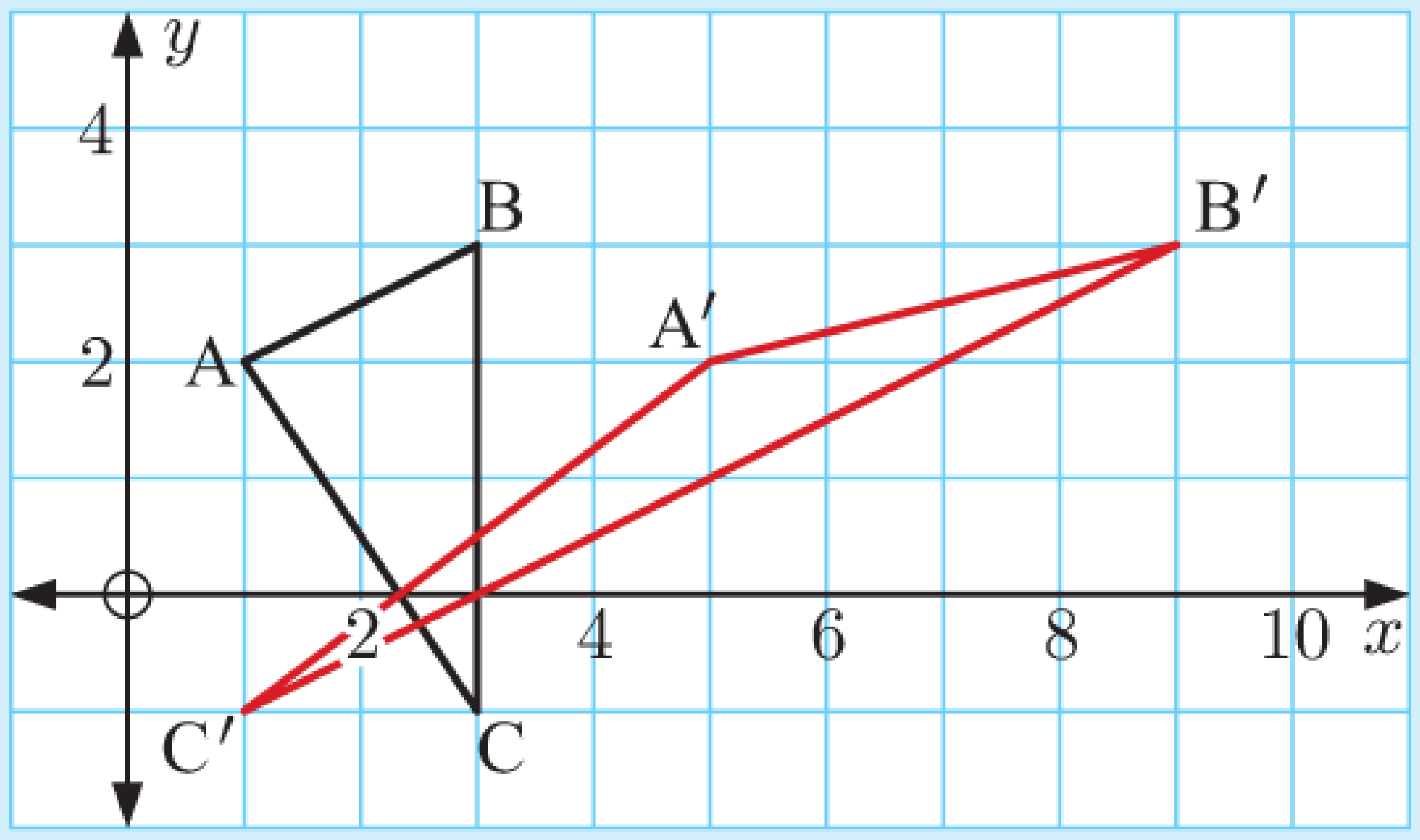
Example 58



The linear transformation T is a shear parallel to the x-axis with scale factor 2. Triangle ABC is moved to A′B′C′ under T.

a Find A′, B′, and C′, and illustrate the transformation.

b Find the area of △ABC and *hence* find the area of △A′B′C′.



a $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ so we have $\begin{cases} x' = x + 2y \\ y' = y. \end{cases}$

$A(1, 2) \to A'(5, 2)$
$B(3, 3) \to B'(9, 3)$
$C(3, -1) \to C'(1, -1)$

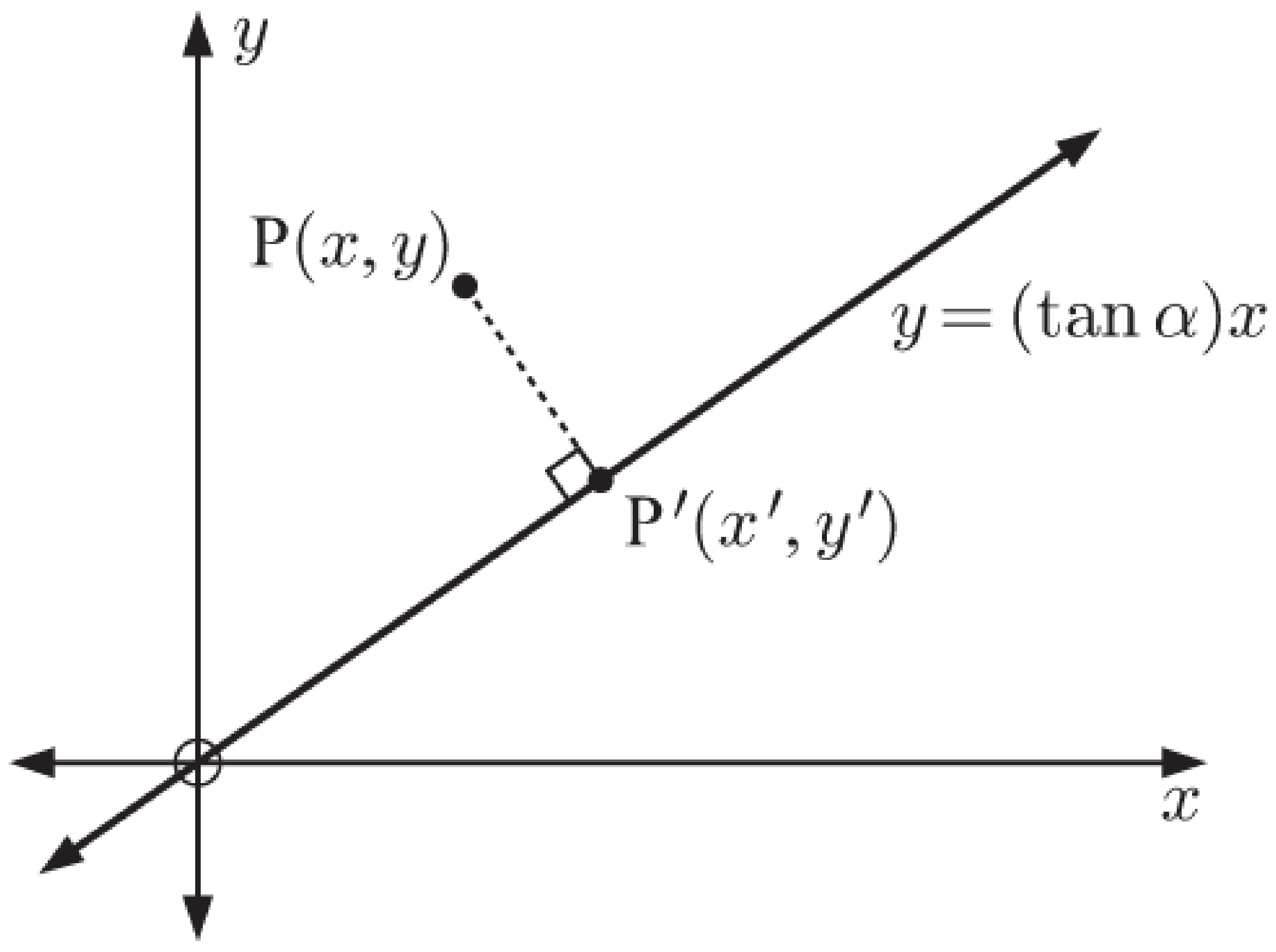
b Area $\triangle ABC = \frac{1}{2} \times 4 \times 2 = 4$ units^2

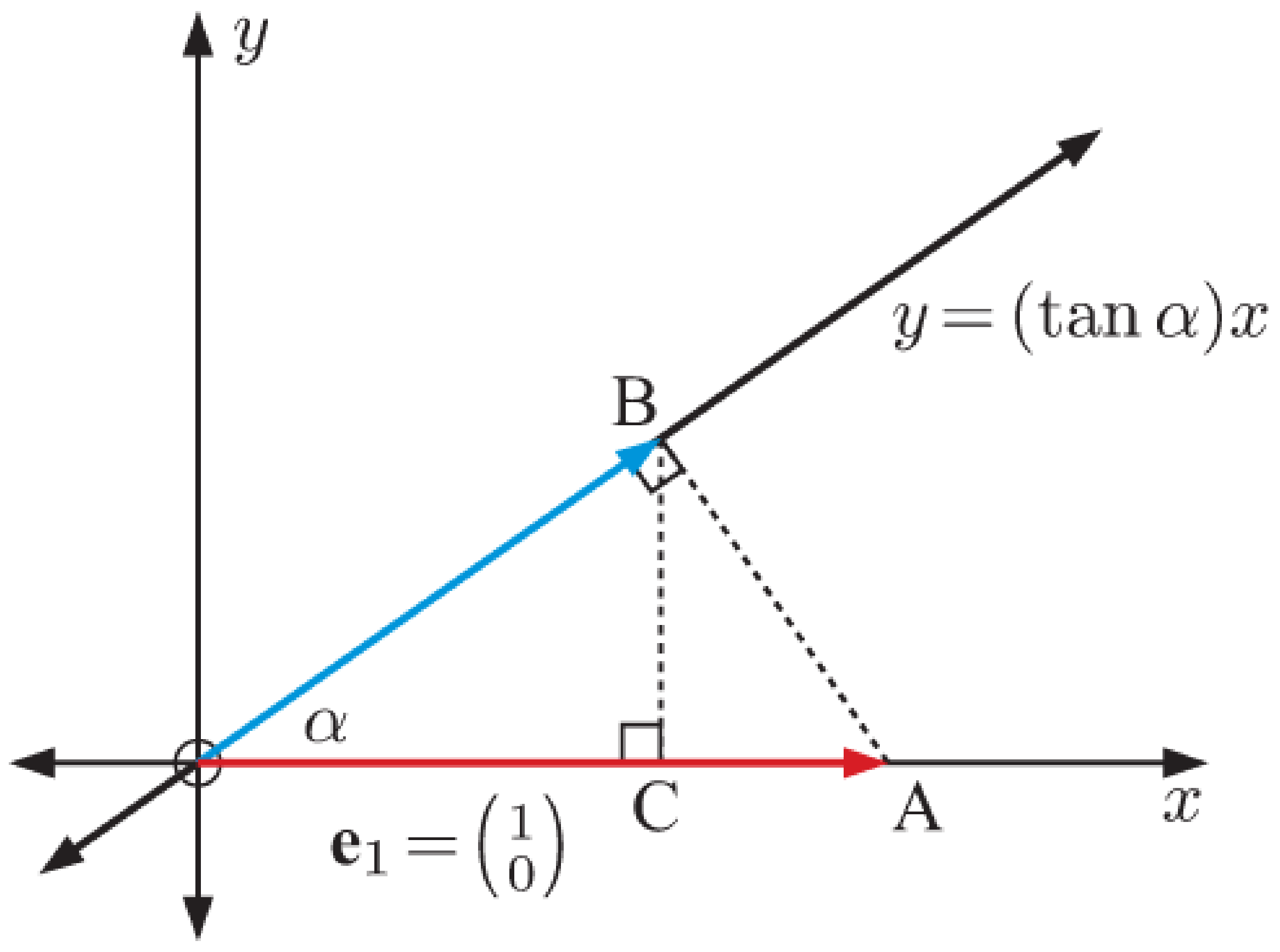
$\therefore$ area $\triangle A'B'C' = ||\,\mathbf{A}\,|| \times$ area $\triangle ABC$
$= 1 \times 4$ units^2
$= 4$ units^2

PROJECTING A POINT ONTO THE LINE $y = (\tan\alpha)x$

When a point $P(x, y)$ is projected onto the line $y = (\tan\alpha)x$, it is moved to the point $P'(x', y')$ on the line such that $[PP']$ is perpendicular to the line.

To find the transformation matrix for such a projection, we return to vector methods.



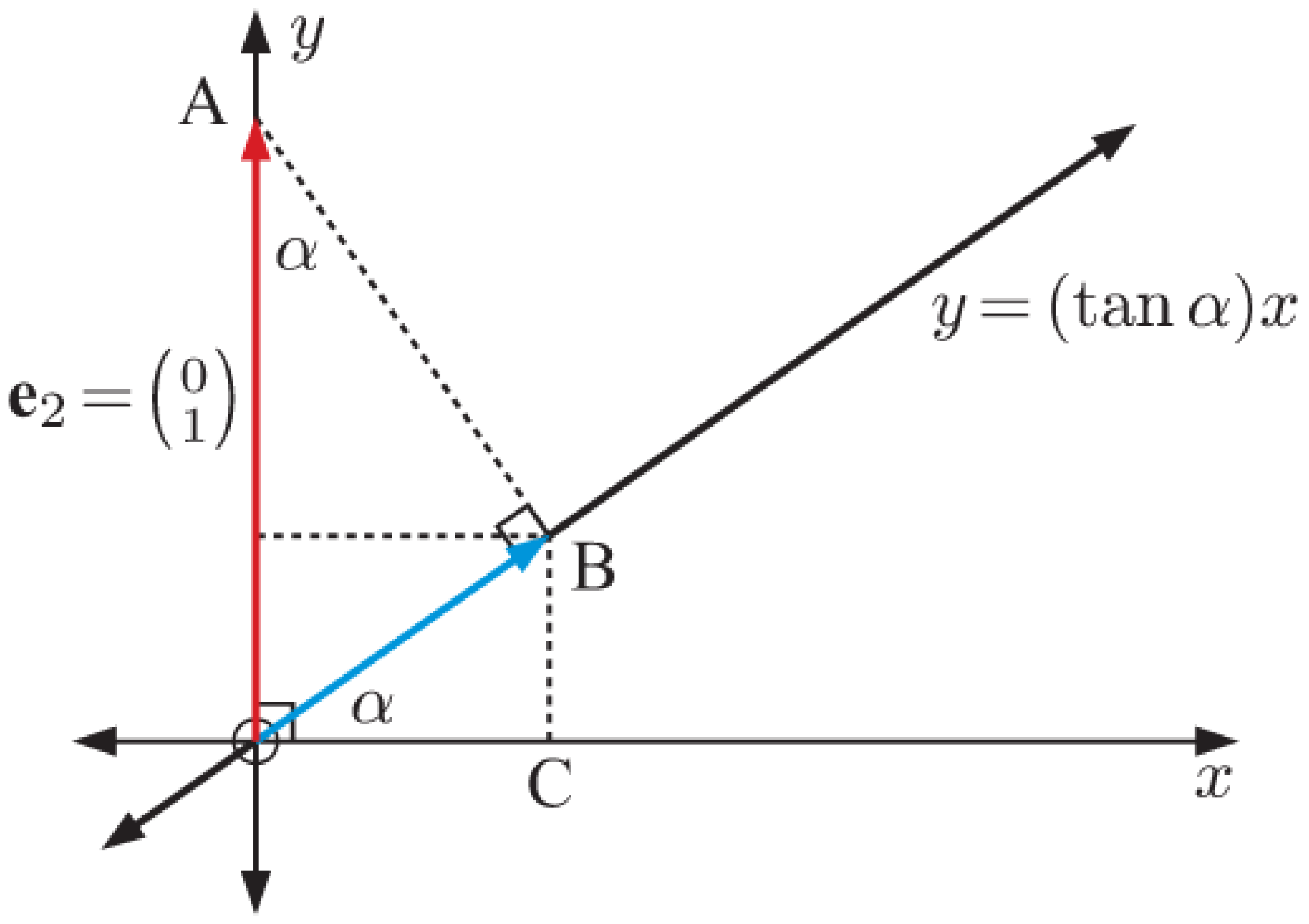


For $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T(\mathbf{e}_1) = \begin{pmatrix} OC \\ BC \end{pmatrix}$.

Using $\triangle OAB$, $\cos\alpha = \dfrac{OB}{OA} = OB$ .... (1) $\{OA = 1\}$

Using $\triangle OCB$, $OC = OB\cos\alpha = \cos^2\alpha$ {using (1)}
and $BC = OB\sin\alpha = \sin\alpha\cos\alpha$ {using (1)}

$\therefore\ T(\mathbf{e}_1) = \begin{pmatrix} \cos^2\alpha \\ \sin\alpha\cos\alpha \end{pmatrix}$



For $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T(\mathbf{e}_2) = \begin{pmatrix} OC \\ BC \end{pmatrix}$.

Now $A\widehat{O}B = \frac{\pi}{2} - \alpha$

$\therefore\ O\widehat{A}B = \alpha$

Using $\triangle OAB$, $\sin\alpha = \dfrac{OB}{OA} = OB$ .... (1) $\{OA = 1\}$

Using $\triangle OBC$, $OC = OB\cos\alpha = \sin\alpha\cos\alpha$ {using (1)}
and $BC = OB\sin\alpha = \sin^2\alpha$ {using (1)}

$\therefore\ T(\mathbf{e}_2) = \begin{pmatrix} \sin\alpha\cos\alpha \\ \sin^2\alpha \end{pmatrix}$

For a **projection onto the line $y = (\tan\alpha)x$**, the transformation matrix is

$$\mathbf{A} = \begin{pmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix} \quad \text{and} \quad |\mathbf{A}| = 0.$$

DISCUSSION

For a projection onto a line, how many possible object points could correspond to a particular image point?

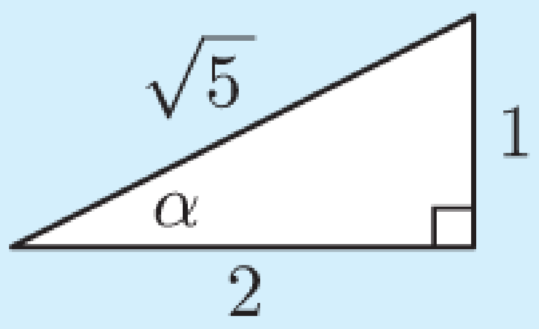
Can you use this to explain why $|\mathbf{A}| = 0$ for this transformation?

Example 59

a Find the projection of the point (3, 4) onto the line $y = \frac{1}{2}x$.

b Hence find the shortest distance from (3, 4) to the line $y = \frac{1}{2}x$.

a $\tan\alpha = \frac{1}{2}$



$\therefore \sin\alpha = \frac{1}{\sqrt{5}}$

and $\cos\alpha = \frac{2}{\sqrt{5}}$

The transformation matrix $\mathbf{A} = \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$.

$\therefore \quad x' = \frac{4}{5}x + \frac{2}{5}y$ and $y' = \frac{2}{5}x + \frac{1}{5}y$

Hence when $x = 3$, $y = 4$

$x' = \frac{4}{5}(3) + \frac{2}{5}(4) = 4$ and

$y' = \frac{2}{5}(3) + \frac{1}{5}(4) = 2$

$\therefore$ the projection of (3, 4) onto $y = \frac{1}{2}x$ is (4, 2).

b The shortest distance from (3, 4) to $y = \frac{1}{2}x$

= the distance between (3, 4) and (4, 2)

$= \sqrt{(4-3)^2 + (2-4)^2}$

$= \sqrt{5}$ units

EXERCISE 1K.3

1 A stretch parallel to the y-axis has scale factor $2\frac{1}{2}$. For this stretch, find the image of:

- **a** P(3, 1)
- **b** the line $y = 1 - 4x$.

2 A shear parallel to the x-axis has scale factor $1\frac{1}{2}$. For this shear, find the image of:

- **a** Q(−2, 6)
- **b** the circle $x^2 + y^2 = 10$.

3 Find the projection of R(4, −1) onto the line with equation $3x + y = 0$.

4 A shear parallel to the y-axis has scale factor 4. For this shear, find the image of:

- **a** S(−1, −3)
- **b** the parabola $y = -2x^2$.

5 A stretch parallel to the x-axis has scale factor $3\frac{1}{2}$. For this stretch, find the image of:

- **a** T(−2, 4)
- **b** the line $3x - 4y = 6$.

6 The projection of $(4, -\frac{1}{2})$ onto the line $y = mx$ is $(1, 1\frac{1}{2})$. Find m.

7 $A(-1, 0)$, $B(2, -2)$, $C(5, 1)$, and $D(2, 7)$ are the vertices of quadrilateral ABCD.
A shear parallel to the y-axis with scale factor $1\frac{1}{2}$ is applied to ABCD.

- **a** Find and illustrate the image $A'B'C'D'$.
- **b** Find the areas of ABCD and $A'B'C'D'$.

8 The circle $x^2 + y^2 = 9$ is subjected to a vertical stretch with scale factor $\frac{5}{3}$.

- **a** Illustrate the object and image.
- **b** Find the equation and area of the image.

9 Find the shortest distance from:

- **a** the point $(-1, 3)$ to the line $y = 4x$
- **b** the point (h, k) to the line $y = mx$.

10 The circle $x^2 + y^2 = a^2$ is transformed by a stretch parallel to the x-axis with scale factor $\dfrac{b}{a}$.

- **a** Illustrate the object and image.
- **b** Prove that the image has area πab.

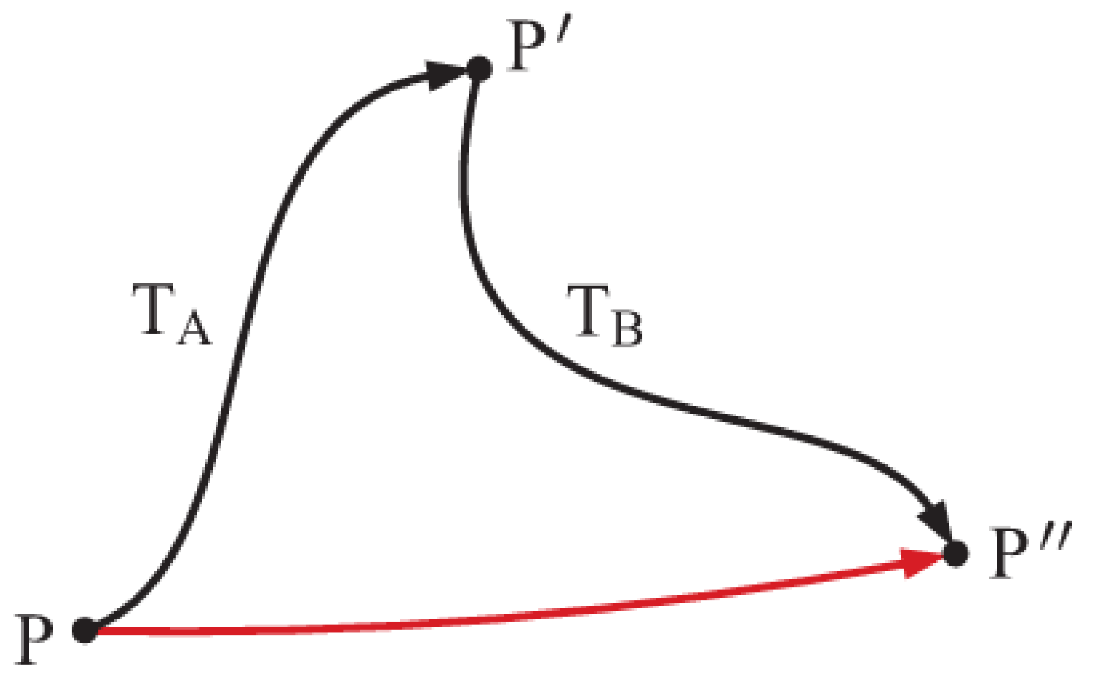
COMPOSITIONS OF TRANSFORMATIONS

Suppose point P is transformed to point P′ under a linear transformation T_A with matrix **A**, and then P′ is transformed to P″ under a second linear transformation T_B with matrix **B**.

We can call this "T_A followed by T_B" or "T_B follows T_A".

Under T_A, $\mathbf{X}' = \mathbf{AX}$, and under T_B, $\mathbf{X}'' = \mathbf{BX}'$

$$\therefore \quad \mathbf{X}'' = \mathbf{B}(\mathbf{AX}) = (\mathbf{BA})\mathbf{X}.$$



The single linear transformation which maps P directly onto P″ is the composition $T_B \circ T_A$ with transformation matrix **BA**.

Example 60

Find the single transformation equivalent to an anticlockwise rotation about O through $\frac{\pi}{2}$, followed by a reflection in the line $y = 2x$.

If T_A is an anticlockwise rotation of $\frac{\pi}{2}$ about O, then $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

If T_B is a reflection in the line $y = 2x$ then $\cos 2\alpha = \dfrac{1-m^2}{1+m^2} = \dfrac{1-4}{1+4} = -\frac{3}{5}$ and

$$\sin 2\alpha = \frac{2m}{1+m^2} = \frac{4}{5}$$

$$\therefore \quad \mathbf{B} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

Thus $\mathbf{BA} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$

BA has form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $|\mathbf{BA}| = |\mathbf{B}|\,|\mathbf{A}| = (-1)(1) = -1$

$\therefore$ T_A followed by T_B is a reflection on a line through $(0, 0)$.

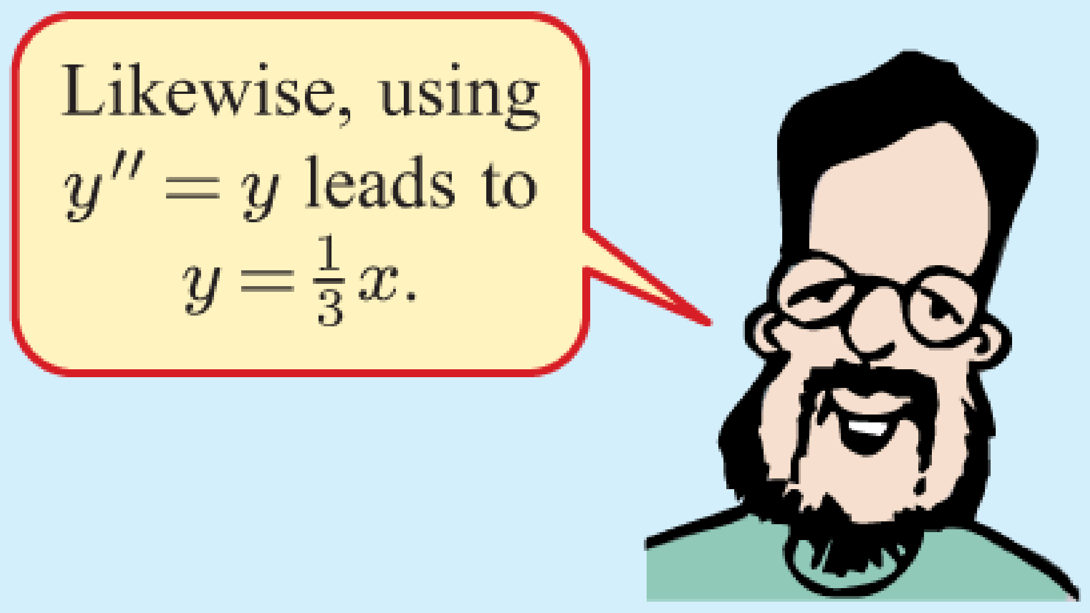
For a point on the mirror line, $x'' = x$

$$\therefore\ \tfrac{4}{5}x + \tfrac{3}{5}y = x$$

$$\therefore\ 4x + 3y = 5x$$

$$\therefore\ y = \tfrac{1}{3}x$$

Hence T_A followed by T_B is a reflection in the line $y = \frac{1}{3}x$.


Likewise, using $y'' = y$ leads to $y = \frac{1}{3}x$.

EXERCISE 1K.4

1 Find the single transformation equivalent to:

- **a** a reflection in the x-axis followed by an anticlockwise rotation of $\frac{\pi}{2}$ about O
- **b** an anticlockwise rotation through $\frac{2\pi}{3}$ about O followed by a reflection in the line $y = -x$
- **c** a reflection in the line $y = \sqrt{3}x$ followed by a reflection in the y-axis
- **d** a reflection in the line $y = x$ followed by a reflection in the line $y = 3x$.

2 Suppose T_1 is a reflection in the line $y = x$ and T_2 is an anticlockwise rotation about O through $\frac{\pi}{3}$.

- **a** Find the transformation matrices for T_1 and T_2.
- **b** Determine the nature of the composition: **i** $T_1 \circ T_2$ **ii** $T_2 \circ T_1$
- **c** Does $T_1 \circ T_2 = T_2 \circ T_1$?

3 Find the single transformation equivalent to:

- **a** a rotation about O through θ followed by a rotation about O through ϕ
- **b** a reflection in the line $y = (\tan\alpha)x$ followed by a reflection in the line $y = (\tan\beta)x$.

4 Prove that a reflection in a line through O, followed by a rotation about O, is equivalent to a reflection in another line through O.

5 Giving reasons for your answers, discuss the combined effect of:

- **a** an even number of reflections
- **b** an odd number of reflections.

6 What transformation is needed before a clockwise rotation through $\frac{\pi}{2}$ about O, in order to give a reflection in the line $y = \frac{1}{2}x$?

7 The line with equation $y = 2x - 1$ is subjected to a reflection in the line $y = -x$ followed by a rotation of $\frac{3\pi}{4}$ about O. Find the equation of the image.

8
- **a** Find the transformation matrix for a vertical stretch with scale factor k.
- **b** Find the single transformation matrix for a vertical stretch with scale factor 2, followed by a reflection in the line $y = -2x$.
- **c** What effect does the composition of transformations in **b** have on sense and area?

L EIGENVALUES AND EIGENVECTORS

In the final Section of this topic, we consider the eigenvalues and eigenvectors of 2×2 matrices. These have useful applications in geometric transformations, molecular and quantum physics, vibrations, geology, and glaciology.

HISTORICAL NOTE

One of the first known applications of eigenvectors came from **Leonhard Euler's** study of the rotational motion of rigid bodies. The Italian mathematician **Joseph-Louis Lagrange** recognised that the principal axes for the rotation correspond to the eigenvectors of the inertia matrix.

In the 19th century, numerous mathematicians and physicists considered the properties of eigenvalues and eigenvectors in their studies. However, it was not until 1904 that the Prussian mathematician **David Hilbert** gave them the German description *eigen*, meaning "own".

David Hilbert

In general, if we are given a 2×2 matrix $\mathbf{A}$ and 2×1 non-zero vector $\mathbf{x}$, we will not be able to find a constant λ such that $\mathbf{Ax} = \lambda\mathbf{x}$.

For example, if $\mathbf{A} = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then $\mathbf{Ax} = \begin{pmatrix} 14 \\ 16 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for any $\lambda \in \mathbb{R}$.

However, for some 2×2 matrices $\mathbf{A}$, we can find non-zero vectors $\mathbf{x}$ and corresponding constants λ such that $\mathbf{Ax} = \lambda\mathbf{x}$.

For example, if $\mathbf{A} = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$, then $\mathbf{Ax} = \begin{pmatrix} 60 \\ 40 \end{pmatrix} = 10\mathbf{x}$ where $\lambda = 10$.

Let $\mathbf{A}$ be a square matrix. If $\mathbf{x}$ is a non-zero vector and λ is a constant such that $\mathbf{Ax} = \lambda\mathbf{x}$, then λ is an **eigenvalue** of $\mathbf{A}$ and $\mathbf{x}$ is its corresponding **eigenvector**.

Note that we demand $\mathbf{x}$ to be non-zero, since clearly if $\mathbf{x} = \mathbf{0}$ then $\mathbf{Ax} = \lambda\mathbf{x} = \mathbf{0}$ for all $\lambda \in \mathbb{R}$.

Now, if $\mathbf{Ax} = \lambda\mathbf{x}$ then $\lambda\mathbf{x} - \mathbf{Ax} = \mathbf{0}$

$$\therefore \quad (\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$$

This equation has non-zero solutions for $\mathbf{x}$ $\Leftrightarrow$ $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$. We therefore conclude that:

The eigenvalues of $\mathbf{A}$ can be found by solving $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$.

Example 61

Find the eigenvalues and their corresponding eigenvectors for $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$.

If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda & -3 \\ -1 & \lambda+2 \end{vmatrix} = 0$

$$\therefore \quad \lambda^2 + 2\lambda - 3 = 0$$
$$\therefore \quad (\lambda - 1)(\lambda + 3) = 0$$
$$\therefore \quad \lambda = 1 \text{ or } -3$$

Thus the eigenvalues are $\lambda = 1, -3$.

For $\lambda = 1$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} 1 & -3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \quad a - 3b = 0$$

Letting $b = t$, $t \neq 0$,
$a = 3t$

$\therefore \quad \mathbf{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} t, \quad t \neq 0$

For $\lambda = -3$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} -3 & -3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \quad -a - b = 0$$

Letting $b = t$, $t \neq 0$,
$a = -t$

$\therefore \quad \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$

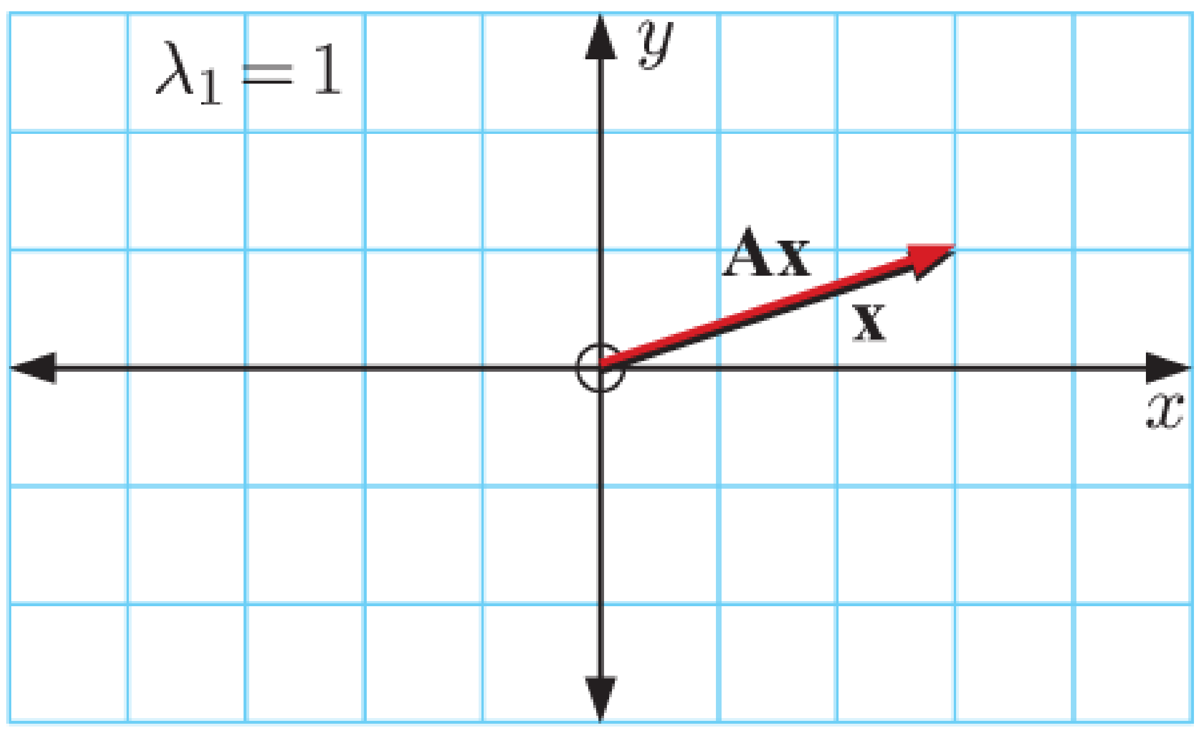
Any vector of the form $\begin{pmatrix} 3 \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 1.

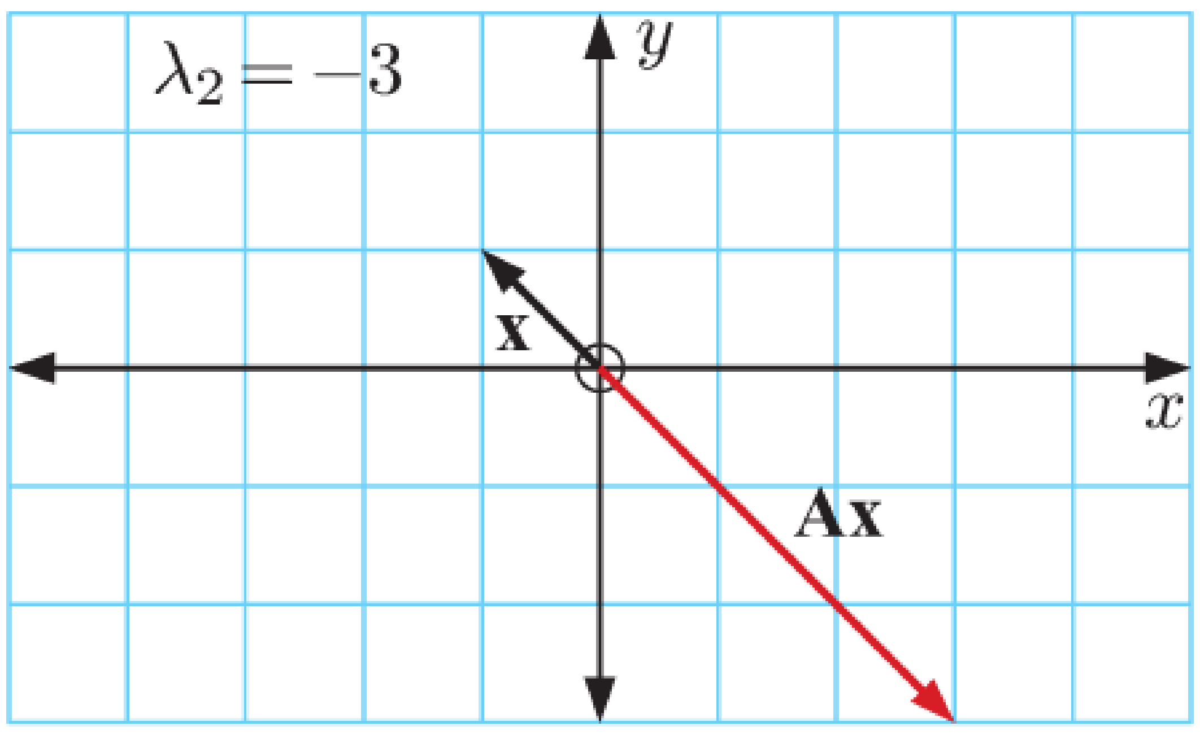
Any vector of the form $\begin{pmatrix} -1 \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue -3.

GEOMETRIC INTERPRETATION

In the previous **Example** where $\mathbf{A}$ is the matrix $\begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$, we see that:

- for $\lambda_1 = 1$, the basic eigenvector is $\mathbf{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, and
  $\mathbf{Ax} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
- for $\lambda_2 = -3$, the basic eigenvector is $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and
  $\mathbf{Ax} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$





From examples like these, we can conclude the following:

Suppose the matrix $\mathbf{A}$ has eigenvalue λ with corresponding eigenvector $\mathbf{x}$.

- If $\lambda = 1$, $\mathbf{Ax}$ does not change $\mathbf{x}$.
- If $\lambda > 0$, the effect of $\mathbf{A}$ on $\mathbf{x}$ is to increase its length by factor λ, and preserve its direction.
- If $\lambda < 0$, the effect of $\mathbf{A}$ on $\mathbf{x}$ is to increase its length by factor $|\lambda|$, and reverse its direction.

EIGENSPACES

If $\mathbf{A}$ is a square matrix and λ is an eigenvalue of $\mathbf{A}$, then $E_\lambda = \{\mathbf{x} \mid \mathbf{x} = \mathbf{0} \cup \mathbf{Ax} = \lambda\mathbf{x}\}$ is the **eigenspace** of $\mathbf{A}$ corresponding to eigenvalue λ.

If $\mathbf{A}$ is $n \times n$ then E is a subspace of $\mathbb{R}^n$.

Proof:

If $\mathbf{x} \in E$,

$$\begin{aligned} \mathbf{A}(t\mathbf{x}) &= t(\mathbf{Ax}) \\ &= t(\lambda\mathbf{x}) \\ &= \lambda(t\mathbf{x}) \end{aligned}$$

$\Rightarrow$ $t\mathbf{x} \in E$ for all $t \in \mathbb{R}$

$\Rightarrow$ E is closed under scalar multiplication.

and

if $\mathbf{x}_1, \mathbf{x}_2, \ldots., \mathbf{x}_n \in E$

$$\begin{aligned} &\mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2 + \ldots. + \mathbf{x}_n) \\ &= \mathbf{Ax}_1 + \mathbf{Ax}_2 + \ldots. + \mathbf{Ax}_n \\ &= \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2 + \ldots. + \lambda\mathbf{x}_n \\ &= \lambda(\mathbf{x}_1 + \mathbf{x}_2 + \ldots. + \mathbf{x}_n) \end{aligned}$$

$\Rightarrow$ $\mathbf{x}_1 + \mathbf{x}_2 + \ldots. + \mathbf{x}_n \in E$

$\Rightarrow$ E is closed under vector addition.

Since E is closed under scalar multiplication and vector addition, it is a subspace of $\mathbb{R}^n$.

If $\mathbf{x}_1, \mathbf{x}_2, \ldots., \mathbf{x}_r$ are linearly independent eigenvectors of $\mathbf{A}$ then $E = \{\mathbf{x}_1, \mathbf{x}_2, \ldots., \mathbf{x}_r\}$ is called the **eigenbasis** of $\mathbf{A}$.

The eigenspace includes all vectors which are linear combinations of eigenvectors.

For example, for the matrix $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$ in **Example 61**:

- $E_1 = \left\{\mathbf{x} \,\middle|\, \mathbf{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} t\right\}$ is the eigenspace for $\mathbf{A}$ corresponding to $\lambda = 1$
- $E_{-3} = \left\{\mathbf{x} \,\middle|\, \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t\right\}$ is the eigenspace for $\mathbf{A}$ corresponding to $\lambda = -3$
- $E = \left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$ is the eigenbasis for $\mathbf{A}$.

CHARACTERISTIC POLYNOMIAL

The **characteristic polynomial** of an $n \times n$ matrix $\mathbf{A}$ is $\det(\lambda \mathbf{I} - \mathbf{A})$.

For 2×2 matrices, the characteristic polynomial is quadratic.

For 3×3 matrices, the characteristic polynomial is cubic.

DISCUSSION

How does the existence of real solutions to the characteristic polynomial affect the eigenvalues and eigenvectors of a matrix?

What happens in the case of a repeated root?

For a 2×2 matrix $\mathbf{A}$ with eigenvalues λ_1 and λ_2:

- $\lambda_1 + \lambda_2$ is the sum of the elements on the leading diagonal of $\mathbf{A}$, commonly called **trace** $(\mathbf{A})$ or $\text{tr}(\mathbf{A})$
- $\lambda_1 \lambda_2 = |\mathbf{A}|$

Proof:

For eigenvalues λ_1 and λ_2, $\det(\lambda \mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$ .... (1)

For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(\lambda \mathbf{I} - \mathbf{A}) = \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix}$

$$= \lambda^2 - a\lambda - d\lambda + ad - bc$$

$$= \lambda^2 - (a + d)\lambda + |\mathbf{A}| \quad .... (2)$$

Comparing (1) and (2), $\lambda_1 + \lambda_2 = a + d = \text{tr}(\mathbf{A})$ and $\lambda_1\lambda_2 = |\mathbf{A}|$.

Example 62

Find the eigenvalues for:

a $\mathbf{A} = \begin{pmatrix} 0 & -3 \\ -4 & -1 \end{pmatrix}$

b $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}$

a $\text{tr}(\mathbf{A}) = -1$ and $|\mathbf{A}| = -12$

$\therefore \; |\lambda \mathbf{I} - \mathbf{A}| = \lambda^2 + \lambda - 12$

$\therefore \; |\lambda \mathbf{I} - \mathbf{A}| = (\lambda - 3)(\lambda + 4)$

$\therefore \; |\lambda \mathbf{I} - \mathbf{A}| = 0 \;\Leftrightarrow\; \lambda = 3, -4$

$\therefore$ the eigenvalues are $3, -4$.

b $\text{tr}(\mathbf{A}) = 2$ and $|\mathbf{A}| = -5$

$\therefore \; |\lambda \mathbf{I} - \mathbf{A}| = \lambda^2 - 2\lambda - 5$

$$\therefore \; |\lambda \mathbf{I} - \mathbf{A}| = 0 \;\Leftrightarrow\; \lambda = \frac{2 \pm \sqrt{4 + 20}}{2}$$

$$= 1 \pm \sqrt{6}$$

$\therefore$ the eigenvalues are $1 + \sqrt{6}$, $1 - \sqrt{6}$.

EXERCISE 1L.1

1 Find the eigenvalues and corresponding eigenvectors for each of the following matrices:

a $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

d $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ **e** $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$

2 Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$.

a Find the eigenvalues of $\mathbf{A}$ and the corresponding eigenvectors.

b Describe the geometric effect of matrix $\mathbf{A}$ on the eigenvectors.

c State the eigenspaces of $\mathbf{A}$ and an eigenbasis.

3 For any 2×2 matrix $\mathbf{A}$, show that $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ have the same eigenvalues. Discuss whether they would have the same corresponding eigenvectors.

4 Consider $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ -5 & 3 \end{pmatrix}$.

a Find $\mathbf{A}^2$ and $\mathbf{A}^{-1}$.

b Find the characteristic polynomials of $\mathbf{A}$, $\mathbf{A}^2$, and $\mathbf{A}^{-1}$.

c Discuss the connection between the eigenvalues of:

i $\mathbf{A}$ and $\mathbf{A}^2$ **ii** $\mathbf{A}$ and $\mathbf{A}^{-1}$.

5 Suppose $\mathbf{A}$ is an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$.

a Prove that $\mathbf{A}^2$ has eigenvalue λ^2 with corresponding eigenvector $\mathbf{x}$.

b If $\mathbf{A}^{-1}$ exists, prove that $\mathbf{A}^{-1}$ has eigenvalue $\dfrac{1}{\lambda}$ with corresponding eigenvector $\mathbf{x}$.

c Prove by mathematical induction that $\mathbf{A}^n$ has eigenvalue λ^n with corresponding eigenvector $\mathbf{x}$ for all $n \in \mathbb{Z}^+$.

6 Prove that if a 2×2 matrix has distinct eigenvalues, then the corresponding eigenvectors are linearly independent.

7 **a** Prove that if λ is an eigenvalue of matrix $\mathbf{A}$, then $\lambda + k$ is an eigenvalue of $\mathbf{A} + k\mathbf{I}$. Discuss the connection between the eigenvectors of $\mathbf{A}$ and $\mathbf{A} + k\mathbf{I}$.

b Suppose $\mathbf{A}$ is a square matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$. Write the eigenvalue of $\mathbf{A}^2 + 4\mathbf{A}$ and the corresponding eigenvector in terms of λ and $\mathbf{x}$.

8 Suppose matrices $\mathbf{A}$ and $\mathbf{B}$ are symmetric. Prove that $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues.

9 If $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} > 0$ for all non-zero vectors $\mathbf{x}$, show that the eigenvalues of $\mathbf{A}$ are positive.

10 Suppose matrix $\mathbf{A}$ is real and symmetric with unequal eigenvalues λ_1, λ_2 and corresponding eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$. Prove that $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

DIAGONALISATION OF 2×2 MATRICES

A 2×2 matrix $\mathbf{A}$ is **diagonalisable** if there exists a 2×2 matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. We say that $\mathbf{P}$ diagonalises $\mathbf{A}$.

In **Example 61**, $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$ has eigenvalues 1, -3 with corresponding eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ respectively.

Consider the matrix $\mathbf{P} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$ made up using the eigenvectors of $\mathbf{A}$.

We observe that
$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \end{aligned}$$

Examples like this lead us to the theorem:

The 2×2 matrix $\mathbf{A}$ with *distinct* eigenvalues λ_1, λ_2 and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$ is diagonalisable. The matrix $\mathbf{P} = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$ diagonalises $\mathbf{A}$, and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Proof: Since $\mathbf{A}$ has distinct eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, these vectors are linearly independent. {proved in the previous **Exercise**}

Consequently $\mathbf{P} = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$ has an inverse $\mathbf{P}^{-1}$ where $\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$.

Now $\mathbf{P}^{-1}\mathbf{P} = \mathbf{P}^{-1}(\mathbf{x}_1 \,|\, \mathbf{x}_2) = \left(\mathbf{P}^{-1}\mathbf{x}_1 \,\middle|\, \mathbf{P}^{-1}\mathbf{x}_2\right)$

and $\mathbf{I} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

$$\therefore \quad \mathbf{P}^{-1}\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{P}^{-1}\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad .... \ (1)$$

So,
$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \mathbf{P}^{-1}(\mathbf{A}\mathbf{x}_1 \,|\, \mathbf{A}\mathbf{x}_2) && \{\text{since } \mathbf{P} = (\mathbf{x}_1 \,|\, \mathbf{x}_2)\} \\ &= \mathbf{P}^{-1}(\lambda_1\mathbf{x}_1 \,|\, \lambda_2\mathbf{x}_2) && \{\text{since } \mathbf{A}\mathbf{x} = \lambda\mathbf{x}\} \\ &= \left(\lambda_1\mathbf{P}^{-1}\mathbf{x}_1 \,\middle|\, \lambda_2\mathbf{P}^{-1}\mathbf{x}_2\right) \\ &= \left(\lambda_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \lambda_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) && \{\text{using } (1)\} \\ &= \left(\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \end{aligned}$$

Example 63

Let $\mathbf{A}=\begin{pmatrix}2&5\\1&0\end{pmatrix}$.

a Find the eigenvalues $\lambda_1,\ \lambda_2$ $(\lambda_1>\lambda_2)$ and corresponding eigenvectors $\mathbf{x}_1,\ \mathbf{x}_2$.

b Find the matrix $\mathbf{P}$ which will diagonalise $\mathbf{A}$.

c Verify that if $\mathbf{P}=(\mathbf{x}_1\,|\,\mathbf{x}_2)$ then $\mathbf{P}^{-1}\mathbf{AP}=\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}$ and also that

if $\mathbf{P}=(\mathbf{x}_2\,|\,\mathbf{x}_1)$ then $\mathbf{P}^{-1}\mathbf{AP}=\begin{pmatrix}\lambda_2&0\\0&\lambda_1\end{pmatrix}$.

---

a
$$|\,\lambda\mathbf{I}-\mathbf{A}\,|=\begin{vmatrix}\lambda-2&-5\\-1&\lambda\end{vmatrix}=\lambda^2-2\lambda-5$$

$$\therefore\ \ |\,\lambda\mathbf{I}-\mathbf{A}\,|=0\ \Leftrightarrow\ \lambda=\frac{2\pm\sqrt{4+20}}{2}$$
$$\Leftrightarrow\ \lambda=1\pm\sqrt{6}$$

For $\lambda_1=1+\sqrt{6}$,
$$(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$$
becomes
$$\begin{pmatrix}\sqrt{6}-1&-5\\-1&1+\sqrt{6}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ \ -a+(1+\sqrt{6})b=0$$
Letting $b=t,\ \ t\neq0$,
$$a=(1+\sqrt{6})t$$
$$\therefore\ \ \begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1+\sqrt{6}\\1\end{pmatrix}t,\ \ t\neq0$$
$$\therefore\ \ \text{let}\ \ \mathbf{x}_1=\begin{pmatrix}1+\sqrt{6}\\1\end{pmatrix}$$
{choosing $t=1$}

For $\lambda_2=1-\sqrt{6}$,
$$(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$$
becomes
$$\begin{pmatrix}-1-\sqrt{6}&-5\\-1&1-\sqrt{6}\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\therefore\ \ -a+b(1-\sqrt{6})=0$$
Letting $b=t,\ \ t\neq0$,
$$a=(1-\sqrt{6})t$$
$$\therefore\ \ \begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1-\sqrt{6}\\1\end{pmatrix}t,\ \ t\neq0$$
$$\therefore\ \ \text{let}\ \ \mathbf{x}_2=\begin{pmatrix}1-\sqrt{6}\\1\end{pmatrix}$$
{choosing $t=1$}

b $\mathbf{P}$ could be $\begin{pmatrix}1+\sqrt{6}&1-\sqrt{6}\\1&1\end{pmatrix}$ or $\begin{pmatrix}1-\sqrt{6}&1+\sqrt{6}\\1&1\end{pmatrix}$.

c For $\mathbf{P}=\begin{pmatrix}1+\sqrt{6}&1-\sqrt{6}\\1&1\end{pmatrix}$, $\mathbf{P}^{-1}\mathbf{AP}\approx\begin{pmatrix}3.449&0\\0&-1.449\end{pmatrix}$ {using technology}
$$=\begin{pmatrix}1+\sqrt{6}&0\\0&1-\sqrt{6}\end{pmatrix}$$

For $\mathbf{P}=\begin{pmatrix}1-\sqrt{6}&1+\sqrt{6}\\1&1\end{pmatrix}$, $\mathbf{P}^{-1}\mathbf{AP}\approx\begin{pmatrix}-1.449&0\\0&3.449\end{pmatrix}$ {using technology}
$$=\begin{pmatrix}1-\sqrt{6}&0\\0&1+\sqrt{6}\end{pmatrix}$$

CALCULATING THE POWER OF A MATRIX

If a matrix $\mathbf{A}$ is diagonalisable, we can calculate its powers using the following theorem:

Matrix $\mathbf{A}$ is diagonalisable such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ if and only if $\mathbf{A}^k = \mathbf{P}\begin{pmatrix} {\lambda_1}^k & 0 \\ 0 & {\lambda_2}^k \end{pmatrix}\mathbf{P}^{-1}$ for all $k \in \mathbb{Z}^+$.

Proof:

$$
\begin{aligned}
(\Rightarrow) \qquad \mathbf{D} &= \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \\
\therefore\ \mathbf{P}\mathbf{D}\mathbf{P}^{-1} &= \mathbf{P}\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{P}^{-1} \\
\therefore\ \mathbf{A} &= \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \\
\therefore\ \mathbf{A}^k &= \underbrace{(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})\ \ldots.\ (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})}_{k \text{ of these}} \\
\therefore\ \mathbf{A}^k &= \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} \qquad \{\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}\} \\
\therefore\ \mathbf{A}^k &= \mathbf{P}\begin{pmatrix} {\lambda_1}^k & 0 \\ 0 & {\lambda_2}^k \end{pmatrix}\mathbf{P}^{-1}
\end{aligned}
$$

$$
\begin{aligned}
(\Leftarrow) \qquad \text{If}\quad \mathbf{A}^k &= \mathbf{P}\begin{pmatrix} {\lambda_1}^k & 0 \\ 0 & {\lambda_2}^k \end{pmatrix}\mathbf{P}^{-1}, \quad k \in \mathbb{Z}^+ \\
\text{then}\quad \mathbf{A}^1 &= \mathbf{P}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{P}^{-1} \qquad \{\text{letting } k = 1\} \\
\therefore\ \mathbf{A} &= \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \\
\therefore\ \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \mathbf{P}^{-1}(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})\mathbf{P} \\
\therefore\ \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \mathbf{D}
\end{aligned}
$$

Example 64

From **Example 61**, the matrix $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$ has eigenvalues 1, -3 with corresponding eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Use the diagonalisation of $\mathbf{A}$ to find $\mathbf{A}^6$.

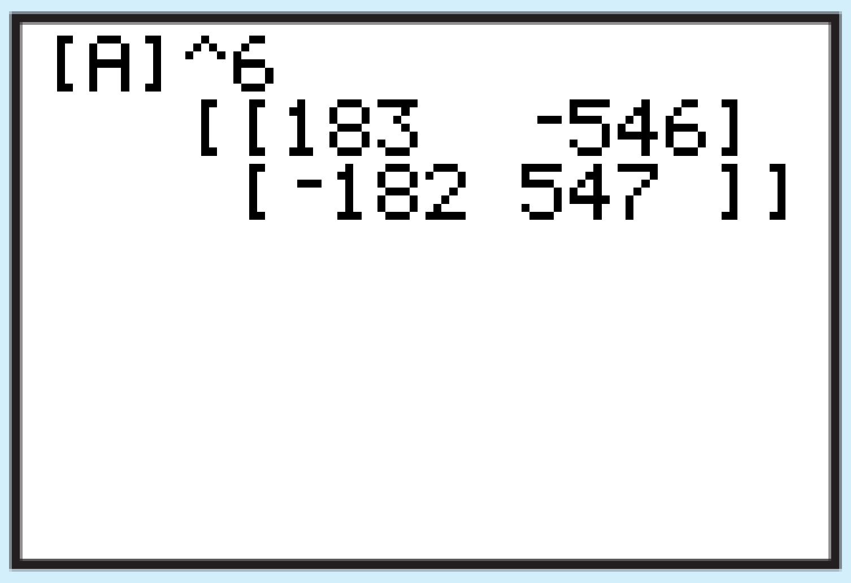
The matrix $\mathbf{P} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$ diagonalises $\mathbf{A}$ with $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$.

Now $\mathbf{A}^k = \mathbf{P}\begin{pmatrix} {\lambda_1}^k & 0 \\ 0 & {\lambda_2}^k \end{pmatrix}\mathbf{P}^{-1}$

$$\therefore\ \mathbf{A}^6 = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-3)^6 \end{pmatrix}\tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

$$\therefore\ \mathbf{A}^6 = \tfrac{1}{4}\begin{pmatrix} 3 & -3^6 \\ 1 & 3^6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$
$$= \tfrac{1}{4}\begin{pmatrix} 3+3^6 & 3-3^7 \\ 1-3^6 & 1+3^7 \end{pmatrix}$$
$$= \begin{pmatrix} 183 & -546 \\ -182 & 547 \end{pmatrix}$$

Check:



```
[A]^6
   [[183  -546]
    [-182 547 ]]
```

Example 65

The matrix $\mathbf{A} = \begin{pmatrix} 3 & -3 \\ -3 & -5 \end{pmatrix}$ has eigenvalues $\lambda_1 = -6$, $\lambda_2 = k$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ respectively.

a Find k. **b** Diagonalise **A**. **c** Find the exact value of $\mathbf{A}^{59}$.

a $\text{tr}(\mathbf{A}) = 3 + (-5) = -2$

$$\therefore\ -6 + k = -2$$
$$\therefore\ k = 4$$

b $\mathbf{P} = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$ diagonalises **A** and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -6 & 0 \\ 0 & 4 \end{pmatrix}$ {Theorem}

c $$\mathbf{A}^{59} = \mathbf{P}\begin{pmatrix} (-6)^{59} & 0 \\ 0 & 4^{59} \end{pmatrix}\mathbf{P}^{-1}$$
$$= \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -6^{59} & 0 \\ 0 & 4^{59} \end{pmatrix}\tfrac{1}{10}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$$
$$= \tfrac{1}{10}\begin{pmatrix} -6^{59} & -3 \times 4^{59} \\ -3 \times 6^{59} & 4^{59} \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$$
$$= \tfrac{1}{10}\begin{pmatrix} 9 \times 4^{59} - 6^{59} & -3 \times 6^{59} - 3 \times 4^{59} \\ -3 \times 6^{59} - 3 \times 4^{59} & 4^{59} - 9 \times 6^{59} \end{pmatrix} \quad \text{\{simplifying\}}$$

The square root of a matrix does not exist. However, when given a 2 × 2 matrix **A**, we can use eigenvalues and eigenvectors to find matrices **B** such that $\mathbf{A} = \mathbf{B}^2$.

Example 66

If $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 0 & 9 \end{pmatrix}$, find all matrices **B** for which $\mathbf{B}^2 = \mathbf{A}$.

First we find the eigenvalues and eigenvectors of **A**:

If $|\lambda\mathbf{I} - \mathbf{A}| = 0$ then $\begin{vmatrix} \lambda - 4 & -1 \\ 0 & \lambda - 9 \end{vmatrix} = 0$

$$\therefore\ (\lambda - 4)(\lambda - 9) = 0$$
$$\therefore\ \lambda = 4,\ 9$$

When $\lambda = 4$,

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$$

$$\therefore \begin{pmatrix} 0 & -1 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ $b = 0,\ a \neq 0$

$\therefore$ an eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

When $\lambda = 9$,

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$$

$$\therefore \begin{pmatrix} 5 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ 5a - b = 0$$

Letting $a = t,\ t \neq 0,$
$b = 5t$

$\therefore$ an eigenvector is $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ {choosing $t = 1$}

$\therefore$ the eigenvalues are $\lambda_1 = 4,\ \lambda_2 = 9$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

Thus $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix}$ will diagonalise $\mathbf{A}$, and

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$$

$$\therefore\ \mathbf{P}^{-1}\mathbf{B}^2\mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$$

$$\therefore\ \mathbf{B}^2 = \mathbf{P}\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}\mathbf{P}^{-1}$$

$$\therefore\ \mathbf{B}^2 = \mathbf{P}\mathbf{C}^2\mathbf{P}^{-1} \quad \text{where} \quad \mathbf{C} = \begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 3 \end{pmatrix}$$

$$\therefore\ \mathbf{B}^2 = (\mathbf{P}\mathbf{C}\mathbf{P}^{-1})(\mathbf{P}\mathbf{C}\mathbf{P}^{-1})$$

$$\therefore\ \mathbf{B} = \mathbf{P}\mathbf{C}\mathbf{P}^{-1}$$

If $\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 5 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & \frac{1}{5} \\ 0 & 3 \end{pmatrix}$

Likewise, using the other possibilities for $\mathbf{C}$, we find other possibilities for $\mathbf{B}$.

The 4 solutions to $\mathbf{B}^2 = \mathbf{A}$ are $\mathbf{B} = \begin{pmatrix} 2 & \frac{1}{5} \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -2 & -\frac{1}{5} \\ 0 & -3 \end{pmatrix}$.

EXERCISE 1L.2

1 Consider matrix $\mathbf{A} = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$.

- **a** Find the eigenvalues of $\mathbf{A}$ and their corresponding eigenvectors.
- **b** If the eigenvalues of $\mathbf{A}$ are λ_1, λ_2 $(\lambda_1 < \lambda_2)$ with corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, and if $\mathbf{P} = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$, verify that:
  - **i** $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
  - **ii** $\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$

2 Consider the matrix $\mathbf{A}=\begin{pmatrix}3&4\\5&2\end{pmatrix}$.

- **a** Find the eigenvalues of $\mathbf{A}$ and their corresponding eigenvectors.
- **b** State a matrix $\mathbf{P}$ which will diagonalise $\mathbf{A}$.
- **c** Hence find the matrix $\mathbf{P}^{-1}\mathbf{A}^3\mathbf{P}$.

3 If $\mathbf{A}=\begin{pmatrix}-1&1\\2&1\end{pmatrix}$, find the exact value of $\mathbf{A}^{60}$.

4 If $\mathbf{C}=\begin{pmatrix}0&2\\1&1\end{pmatrix}$, find the exact value of $\mathbf{C}^{2015}$.

5 For what values of k is $\begin{pmatrix}k&1\\0&k^2\end{pmatrix}$ diagonalisable?

6 Find the possible matrices $\mathbf{B}$ for which $\mathbf{B}^2=\begin{pmatrix}1&0\\2&4\end{pmatrix}$.

7 The **Cayley-Hamilton** theorem is:

> "Every $n\times n$ matrix satisfies its own characteristic equation".

This means that for a 2×2 matrix $\mathbf{A}$, if $|\lambda\mathbf{I}-\mathbf{A}|=\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2=0$ then $\mathbf{A}^2-(\lambda_1+\lambda_2)\mathbf{A}+\lambda_1\lambda_2\mathbf{I}=\mathbf{O}$.

- **a** Prove the Cayley-Hamilton theorem for 2×2 matrices.
- **b** Hence find in the form $a\mathbf{A}+b\mathbf{I}$, the matrices: **i** $\mathbf{A}^3$ **ii** $\mathbf{A}^{-1}$

8 If there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{B}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, prove that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues.

9
- **a** If $\mathbf{A}$ is a 3×3 upper triangular matrix, prove that its eigenvalues are the elements of its main diagonal.
- **b** Does this result hold for a 3×3 lower triangular matrix?

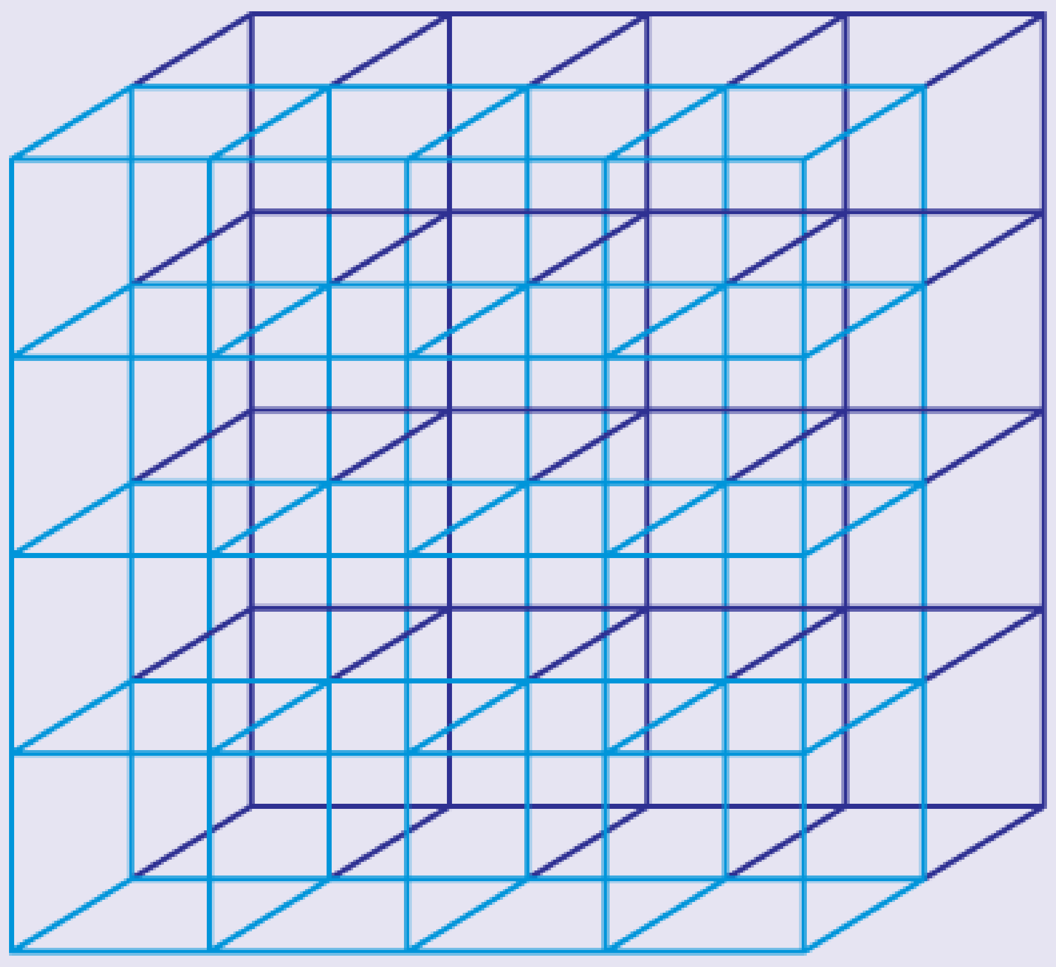
THEORY OF KNOWLEDGE — REPRESENTING SPACE

When we talk about *space*, we commonly refer to a quantity with three physical dimensions.

For example, when we describe the size of an object, we might talk about its length, width, and height.

The dimensions length, width, and height can be described by the orthogonal linear coordinate system called **Euclidean 3-space**. Its basis consists of the linearly independent vectors $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

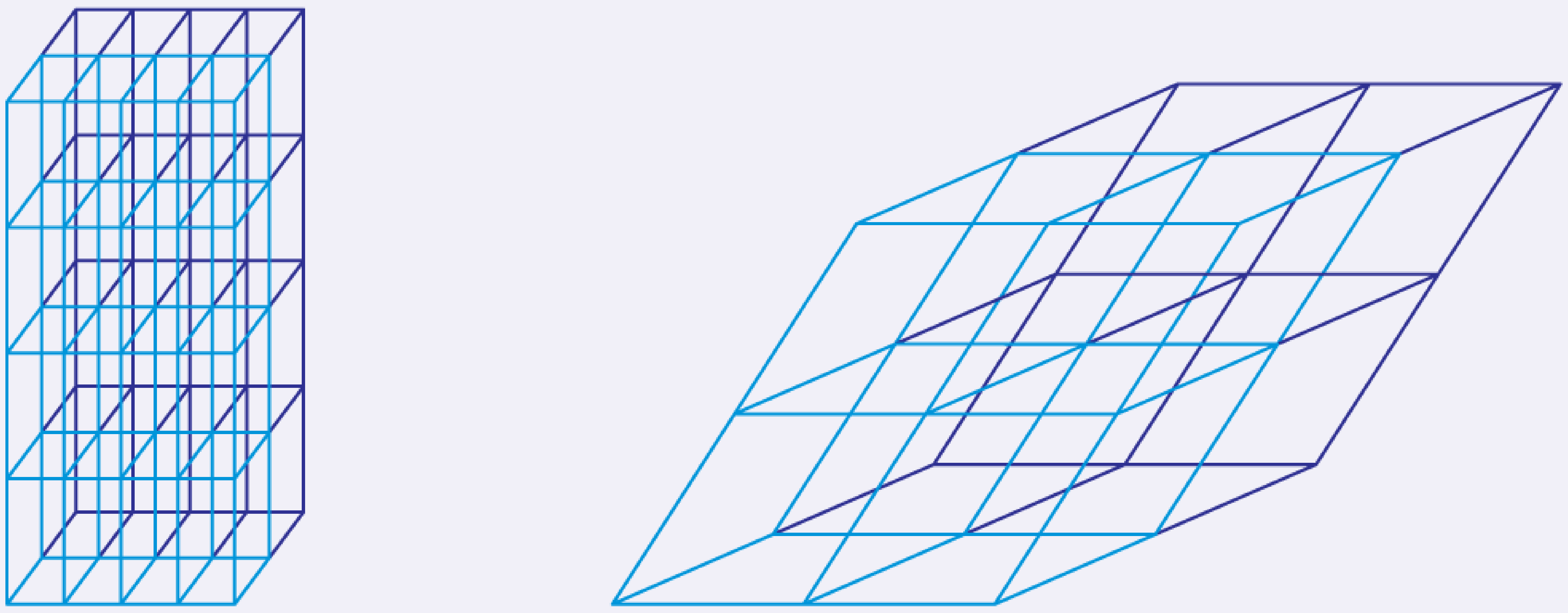
If the coordinate system was physically constructed, we would form a **cubic honeycomb** like the one shown.



1 What *is* space?

2 What does it mean to say that a coordinate system is *independent*? How is independence related to uniqueness?

3 We can use linear transformations to distort the cubic honeycomb into other honeycomb systems such as those shown below:



cuboid honeycomb **parallelepiped honeycomb**

  a Could these other honeycomb systems be used to represent 3-dimensional physical space?

  b What properties would these systems have?

  c Do these systems give us another understanding of space?

  d Research a **bitruncated cubic honeycomb**. Can this honeycomb be used as a coordinate system for 3-dimensional physical space?

4 Does it make sense to use an orthogonal linear coordinate system to describe a sphere?

5 If there are many ways to represent 3-dimensional space, is it reasonable to say that some representations of space are more 'natural' than others?

In the 3rd century BC, **Eratosthenes** proposed a system of latitude and longitude for a map of the world. Measuring latitude was relatively easy since it could be found from the altitude of the sun at noon. However, measurement of longitude was much more complicated, especially at sea, and it was at sea that its use for navigation was fundamentally important. This drove astronomers, physicists, and mathematicians to seek better ways of calculating longitude, and significant advances are credited to

Galileo (1612), **Halley** (1683), and **Maskelyne** (1767) for their lunar methods, and **Harrison** (1773) and **Earnshaw** (1780) for their marine chronometers.

In addition to latitude and longitude, our location on the Earth is more commonly described by including a third dimension which is our elevation above sea level.

6 How does longitude relate to time?

7 Discuss whether the system of longitude, latitude, and elevation is:

a linear **b** independent **c** orthogonal **d** a vector space.

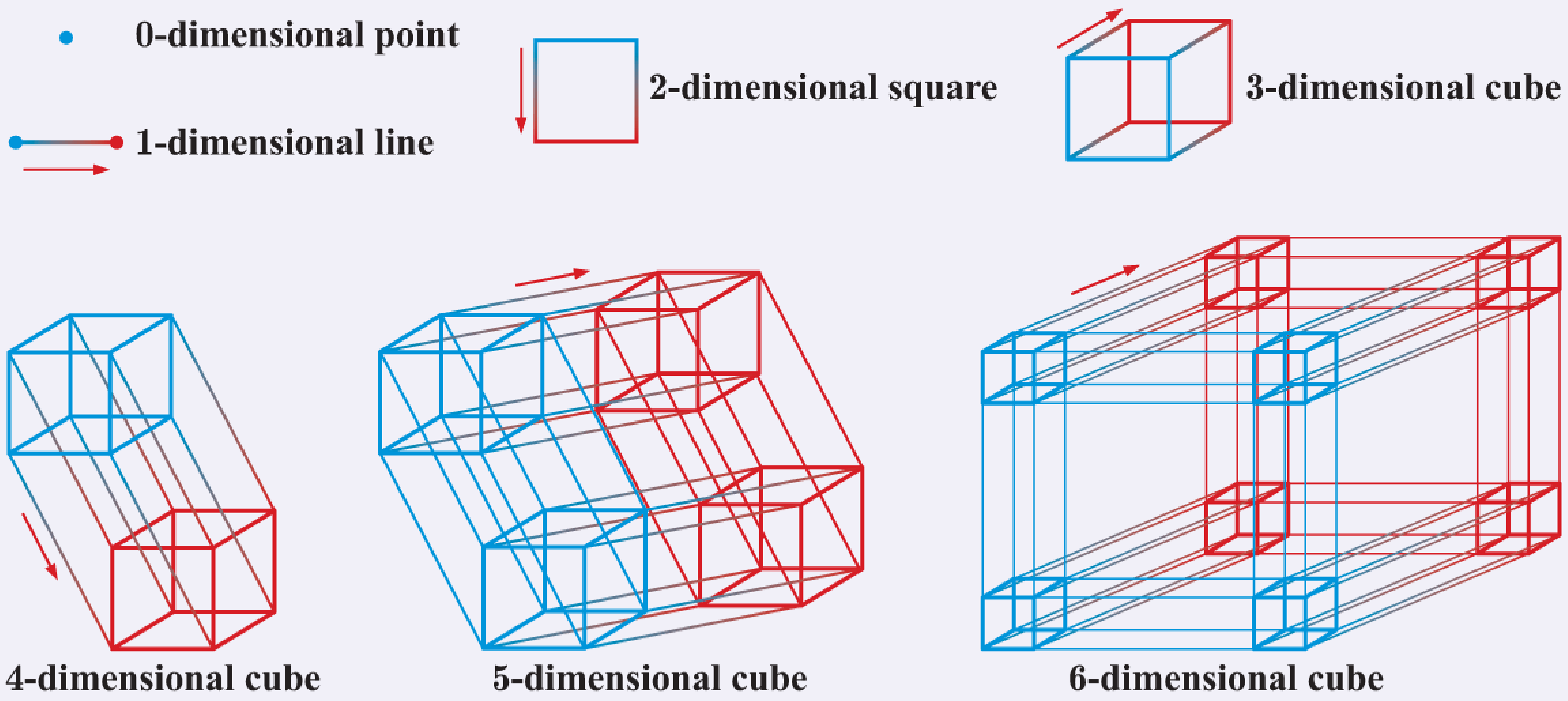
8 Is the system of longitude, latitude, and elevation *equivalent* to 3-dimensional physical space?

In the scientific world today, it is widely agreed that we need more than 3 dimensions to adequately describe the world around us.

9 What would these extra dimensions actually *mean*?

10 Mathematically, it is trivial to extend the **Euclidean 3-space** into a **Euclidean 4-space** with basis $\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$. Does mathematics hold the key to unlocking extra dimensions which are otherwise beyond our comprehension?

11 We are used to representing 3-dimensional objects such as a cubic on 2-dimensional paper. This process can be extended to give visual representation to "cubes" in higher dimensions:



If the extra dimensions do not represent physical space, is there purpose to giving them physical representation?

12 Is time 2-dimensional?

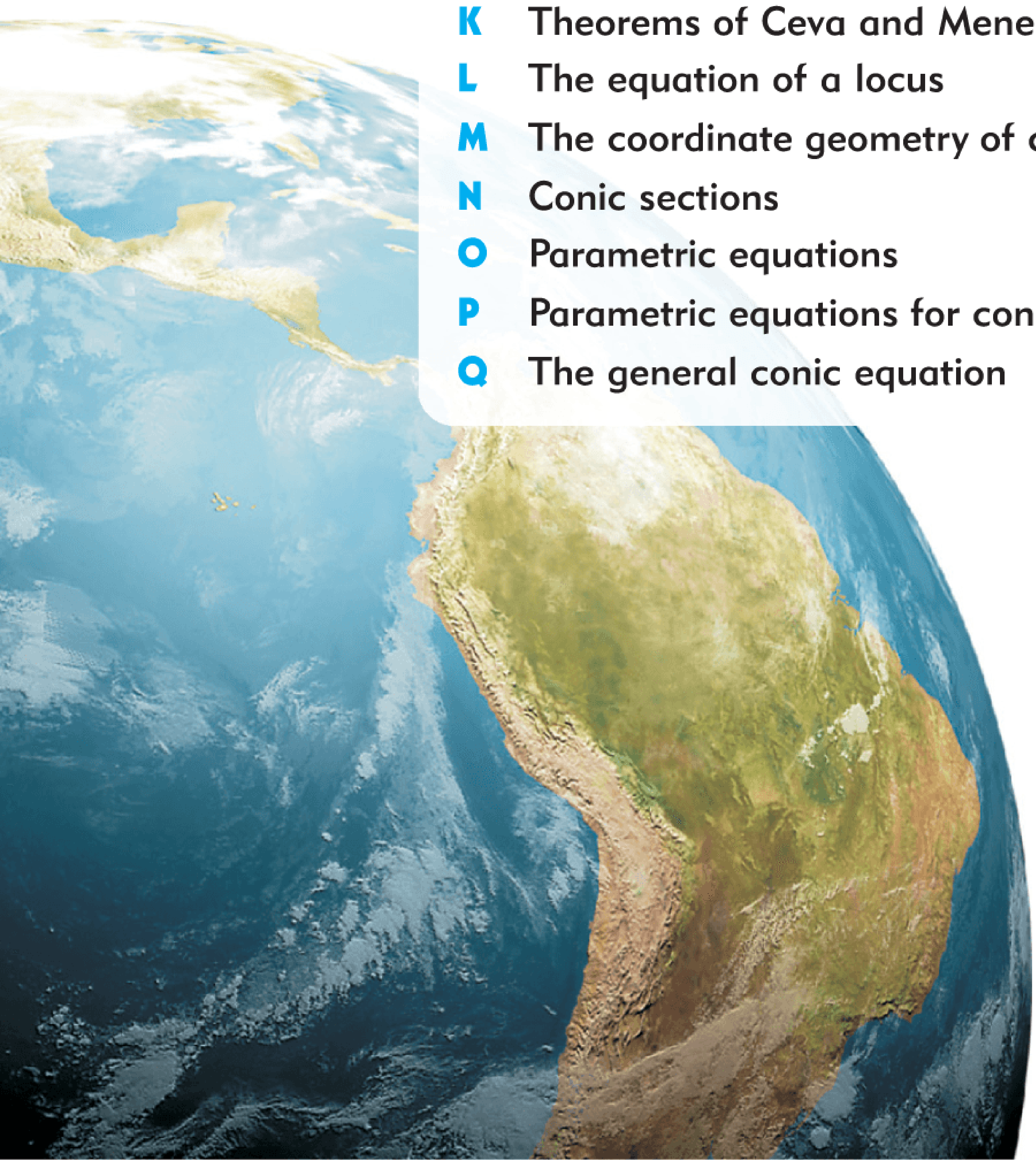
13 Can the universe be completely described by a finite-dimensional space?

Geometry

2

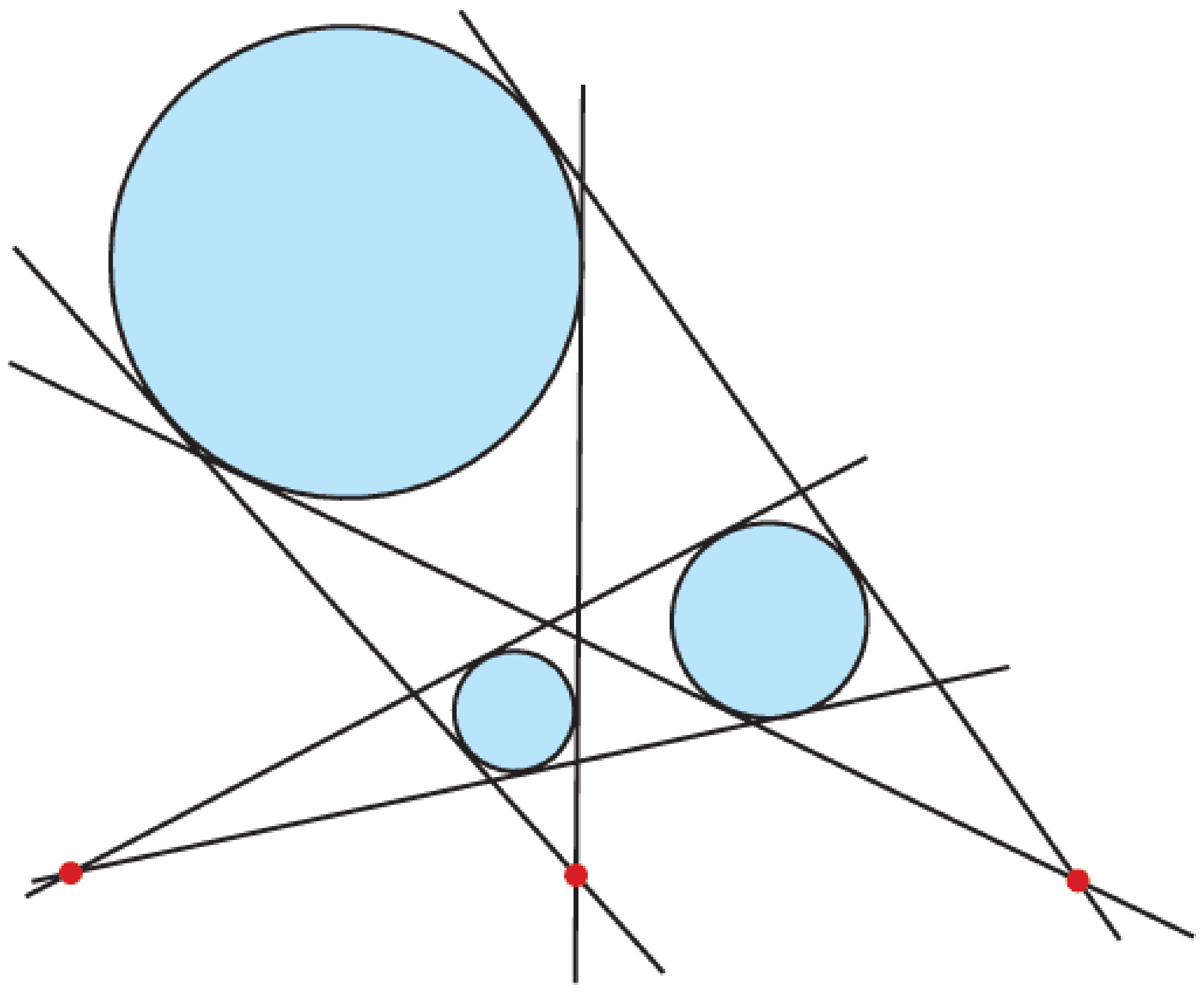
Contents:

A Similar triangles
B Congruent triangles
C Proportionality in right angled triangles
D Circle geometry
E Concyclic points, cyclic quadrilaterals
F Intersecting chords and secants theorems
G Centres of a triangle
H Euclid's angle bisector theorem
I Apollonius' circle theorem
J Ptolemy's theorem for cyclic quadrilaterals
K Theorems of Ceva and Menelaus
L The equation of a locus
M The coordinate geometry of circles
N Conic sections
O Parametric equations
P Parametric equations for conics
Q The general conic equation



Many amazing discoveries have been made by mathematicians and non-mathematicians who were simply drawing figures with rulers and compasses.

For example, this figure consists of three circles of unequal radii. Common external tangents are drawn between each pair of circles and extended until they meet.



Click on the icon to see what interesting fact emerges.

GEOMETRY PACKAGE

This topic is a mix of **Euclidean geometry** and **coordinate geometry**.

The Euclidean geometry is a consistent system of logical thought and deductive reasoning, based on a few simple ideas called **axioms**. The approach is therefore quite formal. Euclidean geometry is felt to possess great mathematical beauty, which is reason enough to justify our study.

This topic deals mainly with ratio properties of figures, and we will concentrate on:

- Apollonius' theorems
- the theorems of Ceva and Menelaus
- Ptolemy's theorem

Coordinate geometry was developed more recently and provides us with an alternative approach to solving geometrical problems.

HISTORICAL NOTE

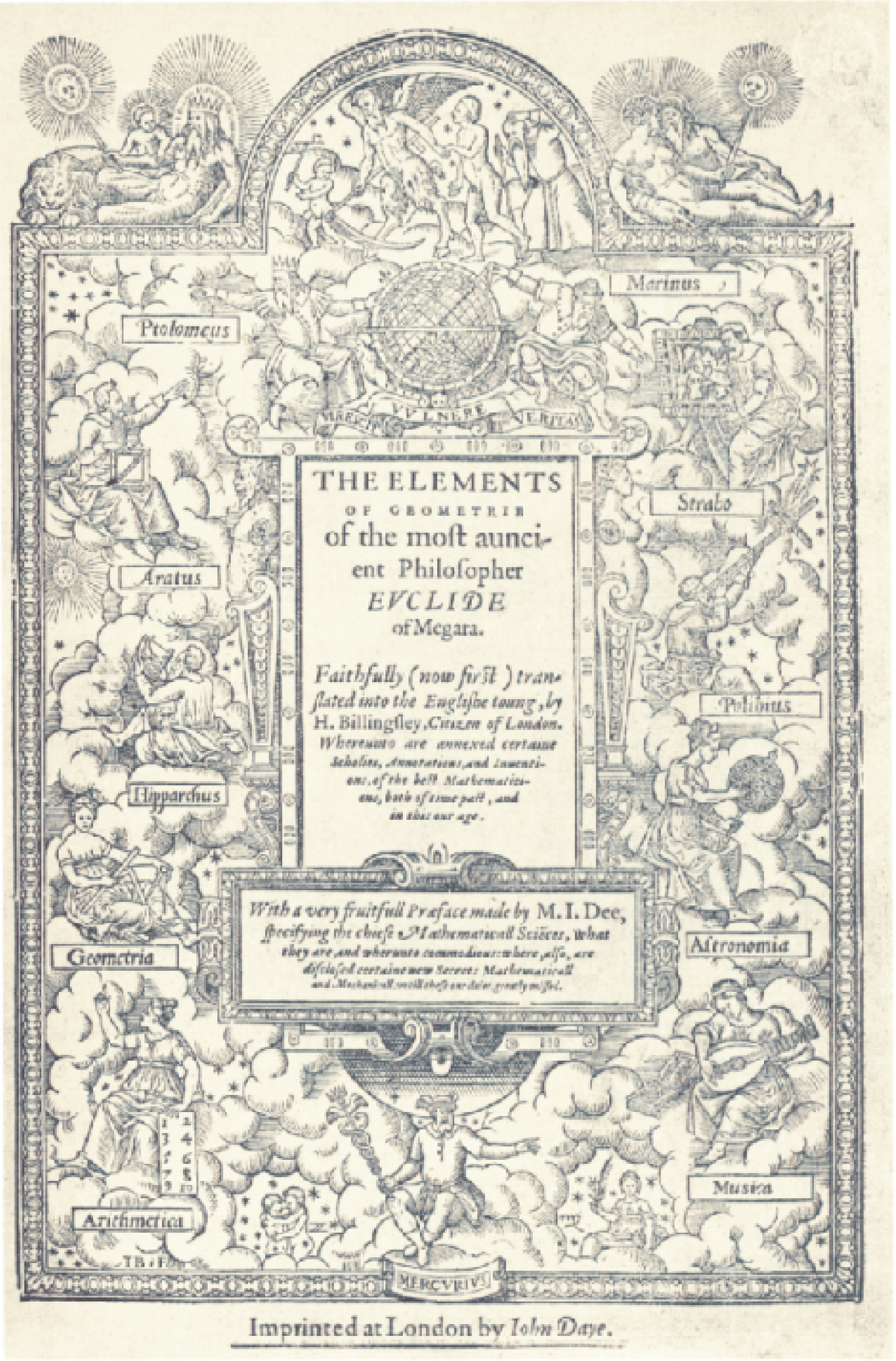
Euclid was one of the great mathematical thinkers of ancient times. He founded a school in Alexandria during the reign of Ptolemy I, which lasted from 323 BC until 284 BC.

Euclid's most famous mathematical writing is called *Elements*. It is the most complete study of geometry ever written, and has been a major source of information for the study of geometric techniques, logic, and reasoning. It was used as a text book for 2000 years until the middle of the 19th century. At this time a number of other texts adapting Euclid's original ideas began to appear.

Like many of the great mathematicians and philosophers, Euclid believed in study and learning for its own merit rather than for the rewards it may bring.

THEORY OF KNOWLEDGE EUCLID'S POSTULATES

Euclid's great work *Elements* is a set of 13 books written in Alexandria around 300 BC. *Elements* is sometimes regarded as the most influential textbook ever written, consisting of definitions, postulates, theorems, constructions, and proofs. It was first printed in 1482 in Venice, making it one of the first mathematical works ever printed.



THE ELEMENTS
OF GEOMETRIE
of the moſt aunci-
ent Philoſopher
EVCLIDE
of Megara.

*Faithfully (now firſt) tran-
ſlated into the Engliſhe toung, by
H. Billingſley, Citizen of London.
Whereunto are annexed certaine
Scholies, Annotations, and Inuenti-
ons, of the beſt Mathematici-
ens, both of time paſt, and
in this our age.*

With a very fruitfull Præface made by M. I. Dee,
*ſpecifying the chiefe Mathematicall Sciēces, what
they are, and wherunto commodious: where, alſo, are
diſcloſed certaine new Secrets Mathematicall
and Mechanicall, vntill theſe our daies, greatly miſſed.*

Imprinted at London by *Iohn Daye*.

The foundation of Euclid's work is his set of postulates, which are the assumptions or **axioms** used to prove further results. Euclid's postulates are:

1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.
4. All right angles are congruent.
5. **Parallel postulate**: If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

1 Can an axiom be proven? Is an axiom necessarily true?

2 Consider the first postulate.

- **a** What is a straight line? How do you know that a line is straight?
- **b** Is straightness more associated with shortest distance or with shortest time? Does light travel in a straight line?
- **c** Is straightness a matter of perception? Does it depend on the reference frame of the observer?

VIDEO

3 For hundreds of years, many people believed the world to be flat. It was then discovered the world was round, so that if you travelled for long enough in a particular direction, you would return to the same place, but at a different time.

- **a** How do we define direction?
- **b** Is a three-dimensional vector sufficient to describe a direction in space-time?
- **c** Can any straight line segment be extended indefinitely in a straight line?

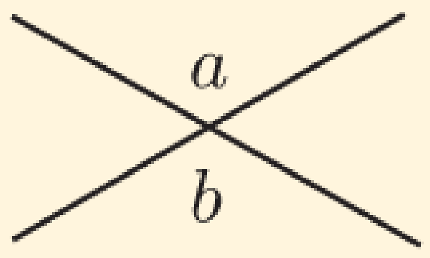
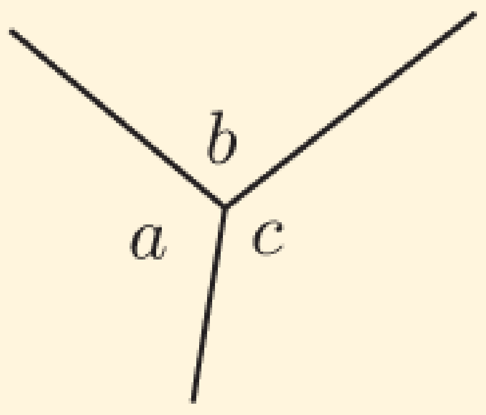
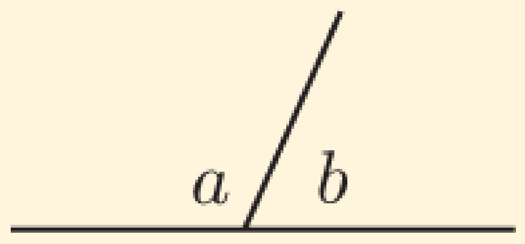
4 Comment on the definition:

A straight line is an infinite set of points in a particular direction.

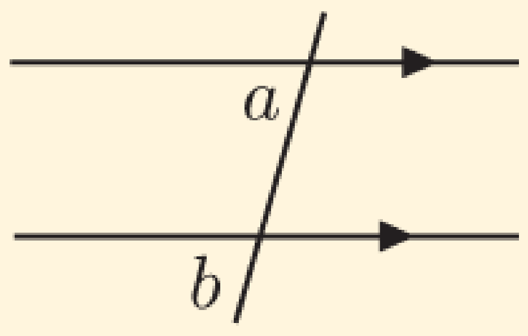
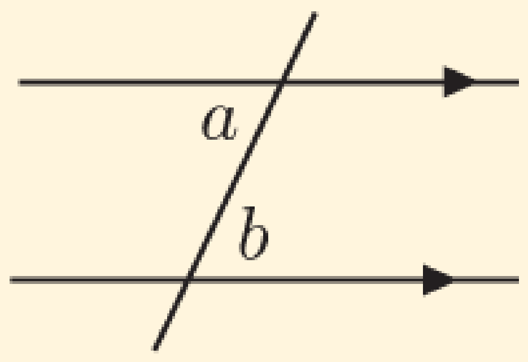
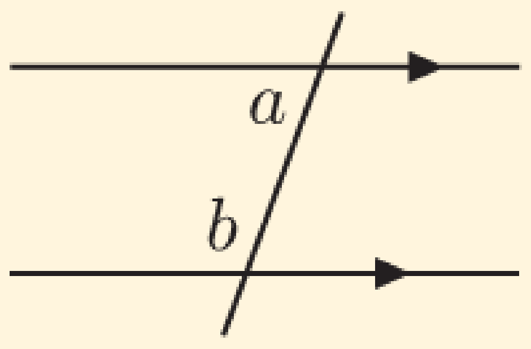
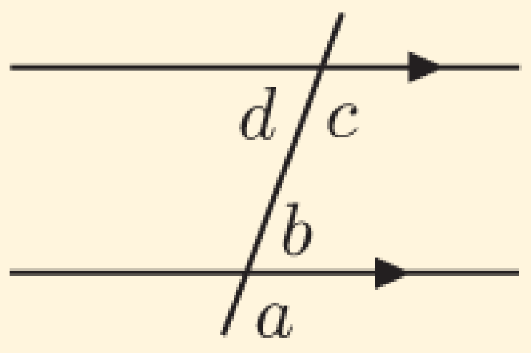
BACKGROUND KNOWLEDGE

From previous courses you should be familiar with the following theorems:

ANGLE THEOREMS

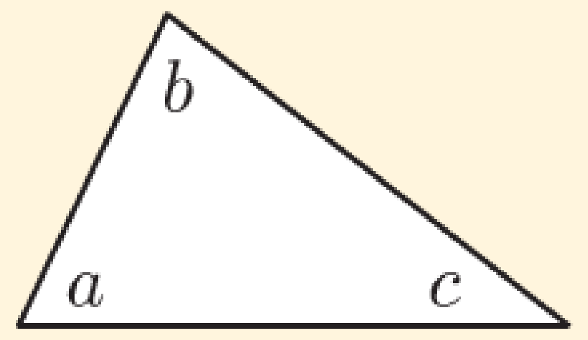
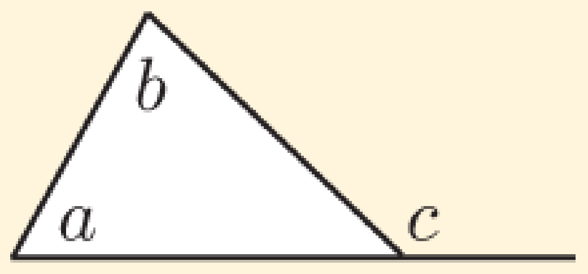
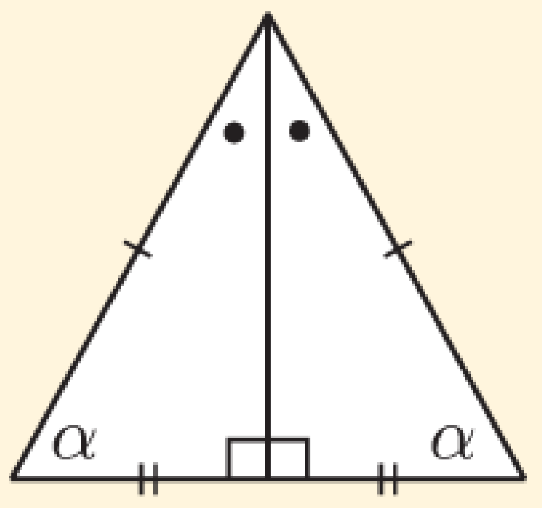
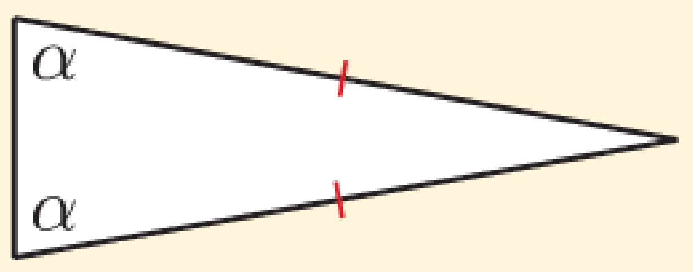
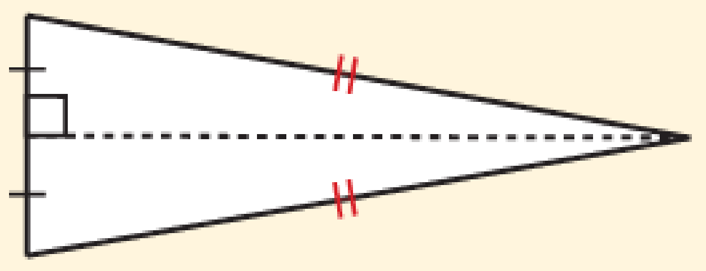
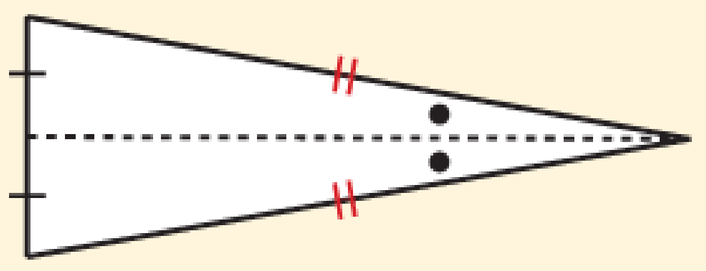
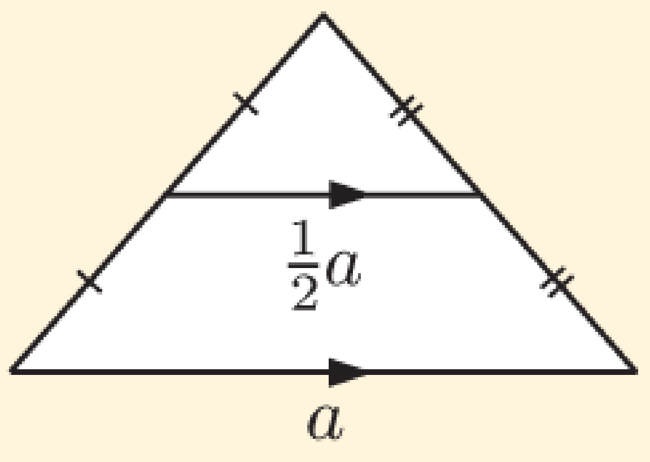
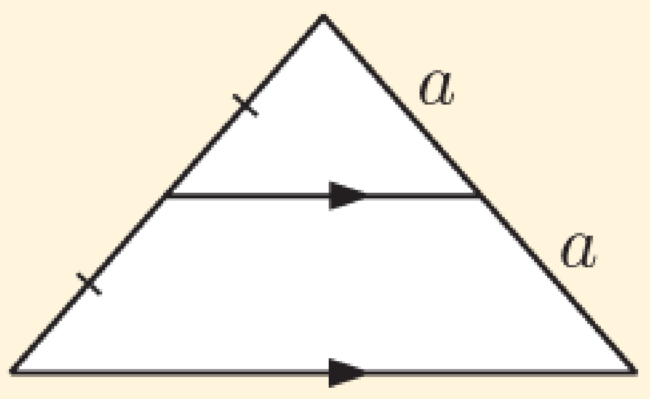
Name	*Theorem*	*Figure*
Vertically opposite angles	Vertically opposite angles are equal.	 $a = b$
Angles at a point	The sum of the angles at a point is 360°.	 $a + b + c = 360°$
Angles on a line	The sum of the angles on a line is 180°.	 $a + b = 180°$

PARALLELISM THEOREMS

Name	*Theorem*	*Figure*
Corresponding angles	When two parallel lines are cut by a third line, the angles in corresponding positions are equal.	 $a = b$
Alternate angles	When two parallel lines are cut by a third line, the angles in alternate positions are equal.	 $a = b$
Allied (or co-interior) angles	When two parallel lines are cut by a third line, the angles in allied positions are supplementary.	 $a + b = 180°$
Converse of parallelism theorems	If two lines are cut by a third line, they are parallel if either corresponding angles are equal, alternate angles are equal, or allied angles are supplementary.	 l_1 is parallel to l_2 if $a = c$ or $b = d$ or $b + c = 180°$.

TRIANGLE THEOREMS

Click on an icon for an interactive demonstration.

Name	*Theorem*	*Figure*
Angles of a triangle	The sum of the interior angles of a triangle is 180°.	 $a + b + c = 180°$
Exterior angle of a triangle	The exterior angle of a triangle is equal to the sum of the interior opposite angles.	 $c = a + b$
Isosceles triangle	In an isosceles triangle: • base angles are equal • the line joining the apex to the midpoint of the base is perpendicular to the base and bisects the angle at the apex.	
Converses of isosceles triangle theorem	• If a triangle has two equal angles, then the triangle is isosceles. • If the third angle of a triangle lies on the perpendicular bisector of its base, then the triangle is isosceles. • If the line joining the midpoint of the base to the apex bisects the angle at the apex, then the triangle is isosceles.	  
Midpoint theorem	The line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.	
Converse of midpoint theorem	The line drawn from the midpoint of one side of a triangle parallel to a second side, bisects the third side.	

GEOMETRY PACKAGE

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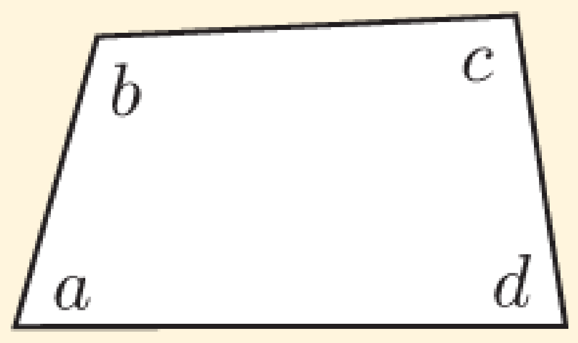
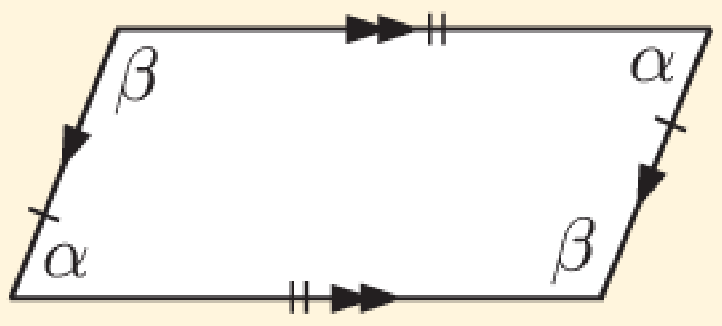
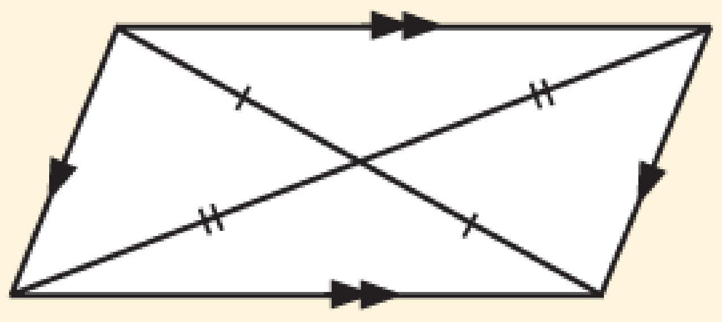
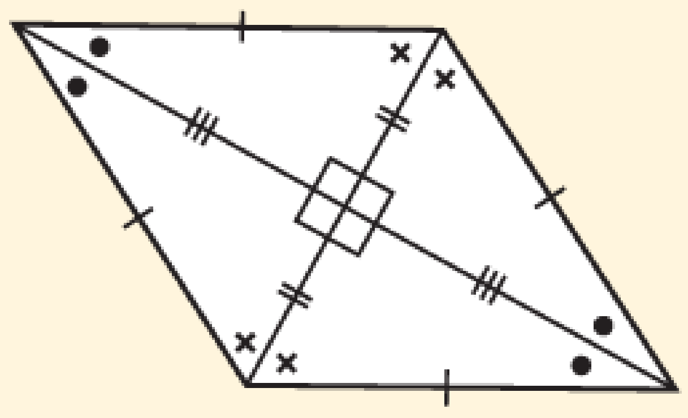
QUADRILATERAL THEOREMS

Name	*Theorem*	*Figure*
Angles of a quadrilateral	The sum of the interior angles of a quadrilateral is 360°.	 $a+b+c+d=360°$
Parallelogram	In a parallelogram: • opposite sides have equal length • opposite angles are equal.	
Diagonals of a parallelogram	The diagonals of a parallelogram bisect each other.	
Diagonals of a rhombus	The diagonals of a rhombus: • bisect each other at right angles • bisect the angles of the rhombus.	

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OTHER IMPORTANT FACTS ABOUT QUADRILATERALS

Any one of the following facts is sufficient to establish that **a quadrilateral is a parallelogram**:

- opposite sides are equal in length

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- one pair of opposite sides is equal in length and parallel

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- opposite angles are equal

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- diagonals bisect each other.

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Any one of the following facts is sufficient to establish that **a quadrilateral is a rhombus**:

- the quadrilateral is a parallelogram with one pair of adjacent sides equal
- the diagonals bisect each other at right angles.

Any one of the following facts is sufficient to prove that **a parallelogram is a rectangle**:

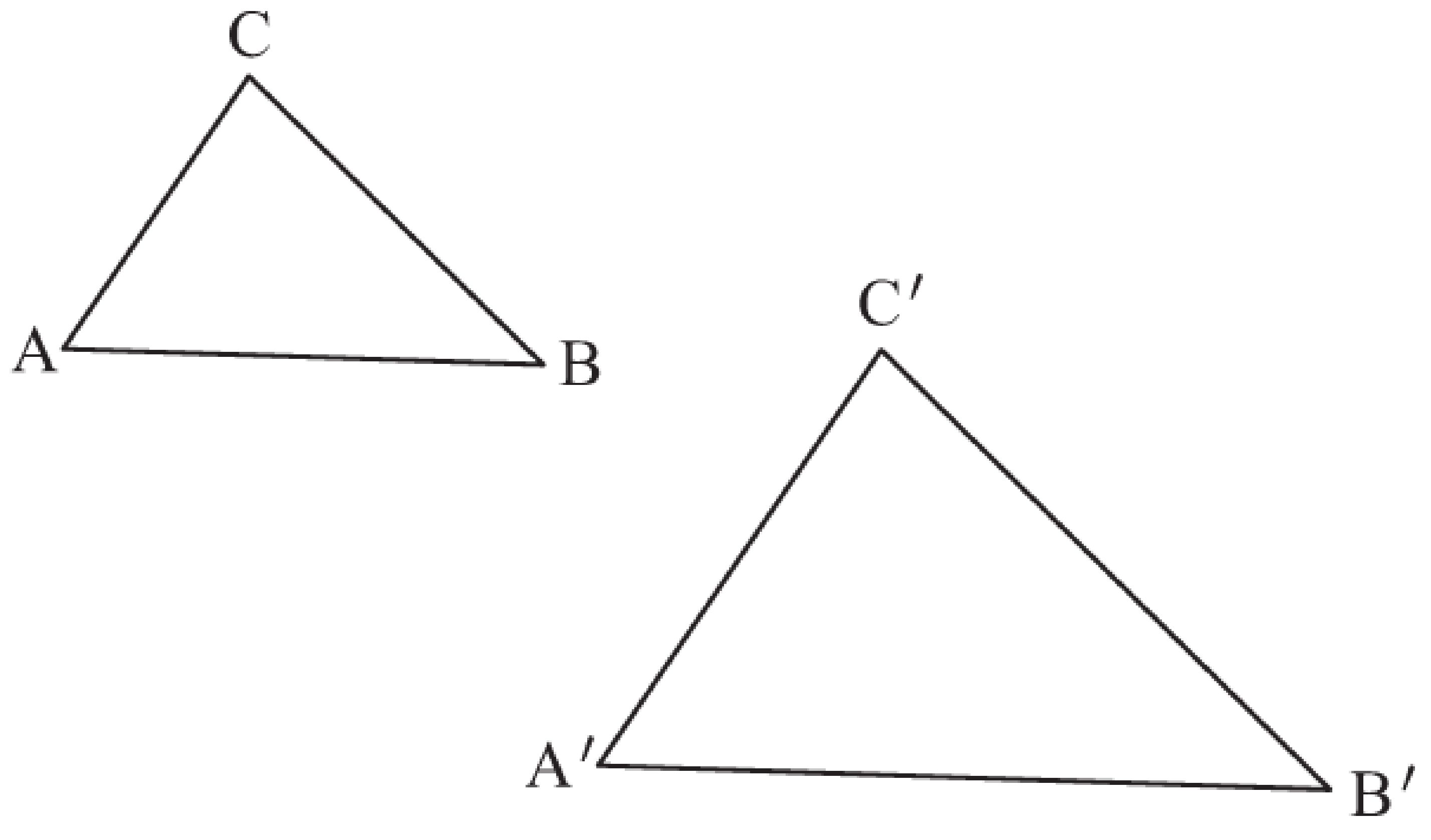
- one angle is a right angle
- the diagonals are equal in length.

Any one of the following facts is sufficient to establish that **a quadrilateral is a square**:

- the quadrilateral is a rhombus with one angle a right angle
- the quadrilateral is a rhombus whose diagonals are equal in length
- the quadrilateral is a rectangle with one pair of adjacent sides equal in length.

A SIMILAR TRIANGLES

Two triangles are **similar** if one is an enlargement of the other.



Consider the two similar triangles shown. Since $\triangle A'B'C'$ is an enlargement of $\triangle ABC$, angles and side ratios are preserved.

For example,

$C\widehat{A}B = C'\widehat{A'}B'$ and $\dfrac{A'C'}{AC} = \dfrac{A'B'}{AB}$.

When triangles are similar, we write the vertices corresponding to equal angles in the same order.

For the above triangles we would write $\triangle ABC$ is similar to $\triangle A'B'C'$.

THEOREMS ON SIMILAR TRIANGLES

Theorem: Two triangles are similar $\Leftrightarrow$ corresponding angles are equal.

This theorem is a direct consequence of the definition, since one triangle is an enlargement of the other.

Theorem:

Two triangles are similar $\Leftrightarrow$ the lengths of corresponding sides are in the same ratio.

Proof:

$\triangle A'B'C'$ is an enlargement of $\triangle ABC$

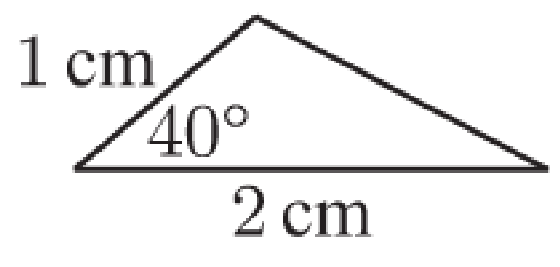
$\Leftrightarrow$ there exists an enlargement factor k such that $A'B' = kAB$, $A'C' = kAC$, $B'C' = kBC$

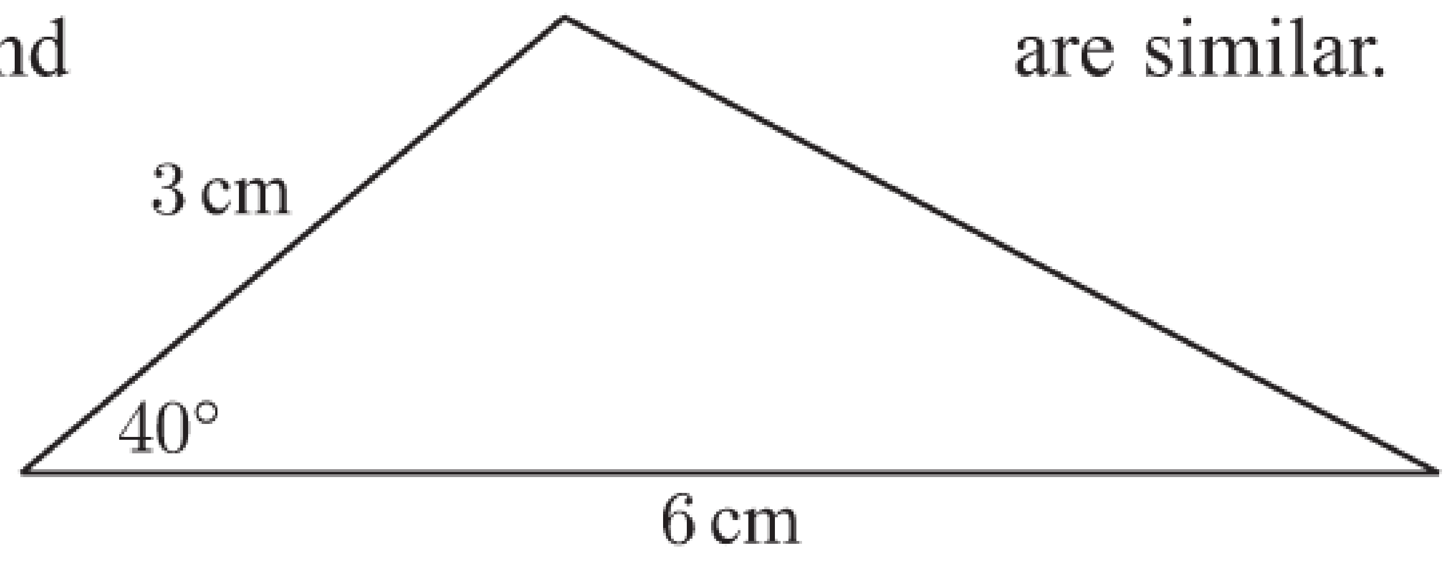
$$\Leftrightarrow \frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC} = k$$

$\Leftrightarrow$ the lengths of corresponding sides are in the same ratio.

Corollary:

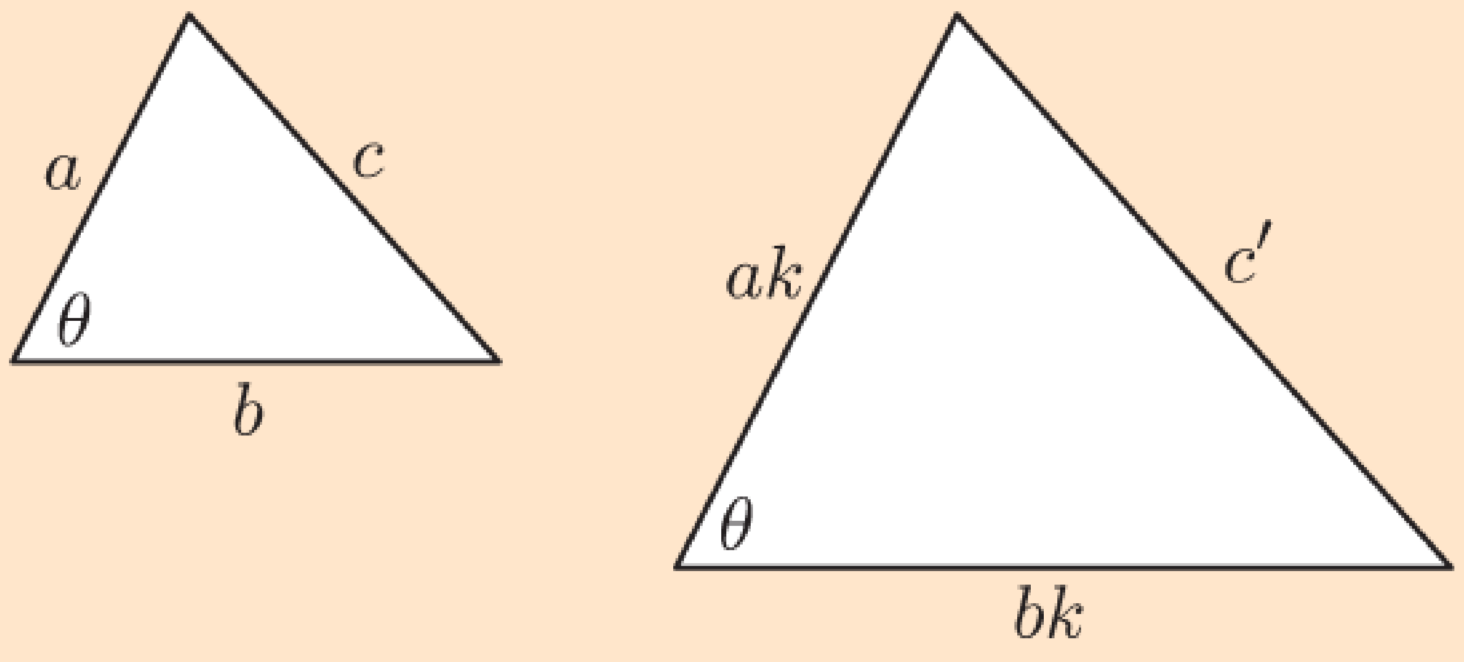
If two triangles are such that two side lengths of each triangle are in the same ratio and the included angles are equal, then the triangles are similar.

For example: 

and 

are similar.

Proof: (modern outline)



Consider the two triangles given.

By the Cosine Rule:

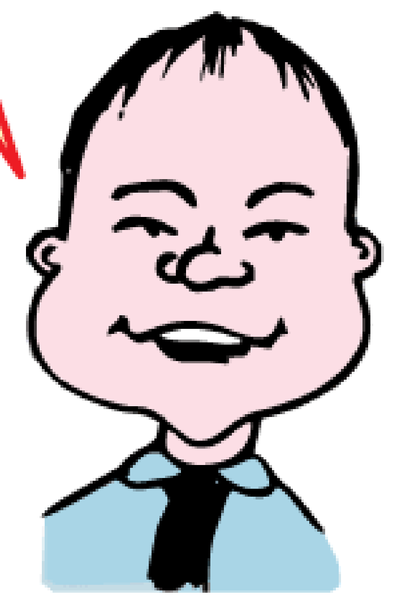
$$\begin{aligned}(c')^2 &= (ka)^2 + (kb)^2 - 2kakb\cos\theta \\ &= k^2(a^2 + b^2 - 2ab\cos\theta) \\ &= k^2c^2 \qquad \{\text{Cosine Rule}\} \\ \Rightarrow \quad c' &= kc\end{aligned}$$

We can repeat this process to show that $a' = ka$ and $b' = kb$.

NECESSARY AND SUFFICIENT CONDITIONS FOR SIMILAR TRIANGLES

- If two triangles are similar then:
  - the triangles are equiangular
  - the corresponding sides are in the same ratio.
- A pair of triangles is similar if any one of the following is true:
  - the triangles are equiangular
  - the corresponding sides of the triangle are in the same ratio
  - two sides of each triangle are in the same ratio and the included angles are equal.

Similarity of triangles is an **equivalence relation**, as studied in the Sets, Relations and Groups topic.



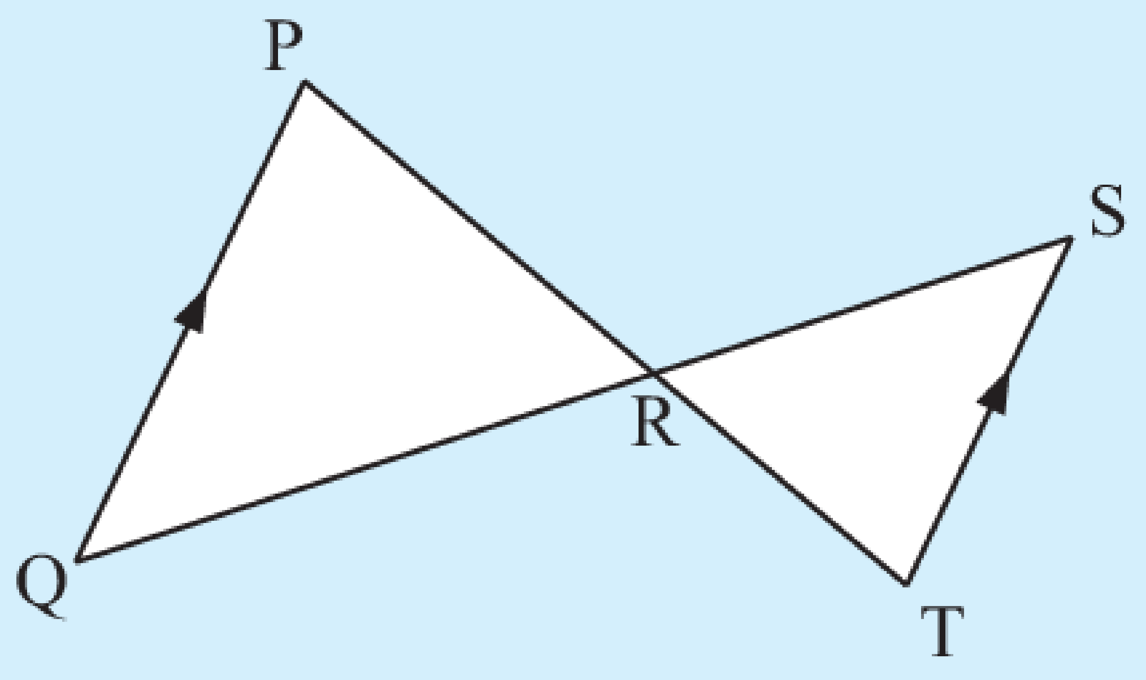
Example 1

For each of the following figures:

i Identify similar triangles and prove that they are similar.

ii Write an equation connecting the lengths of corresponding sides.

a



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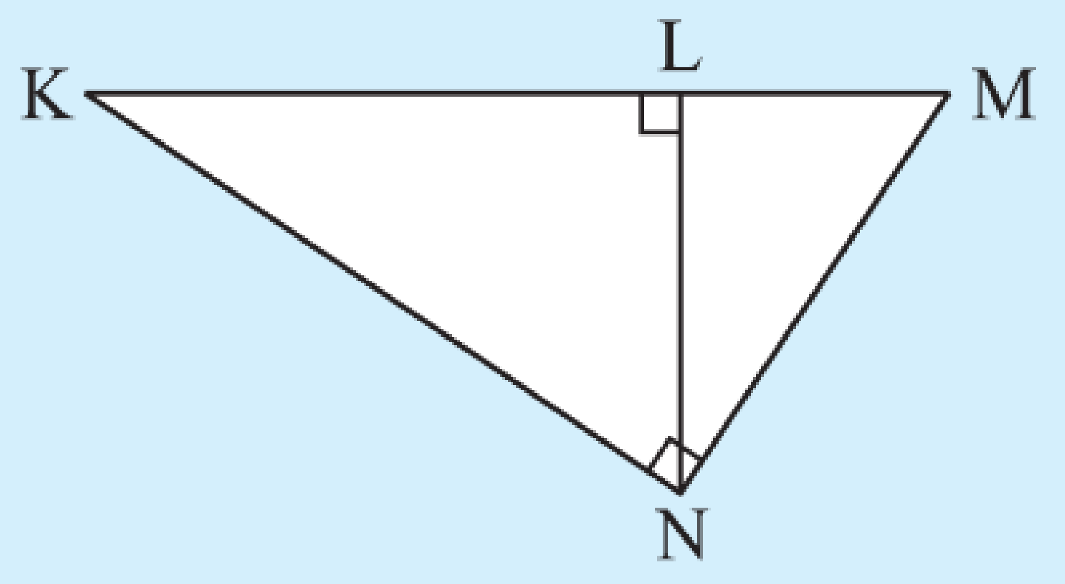
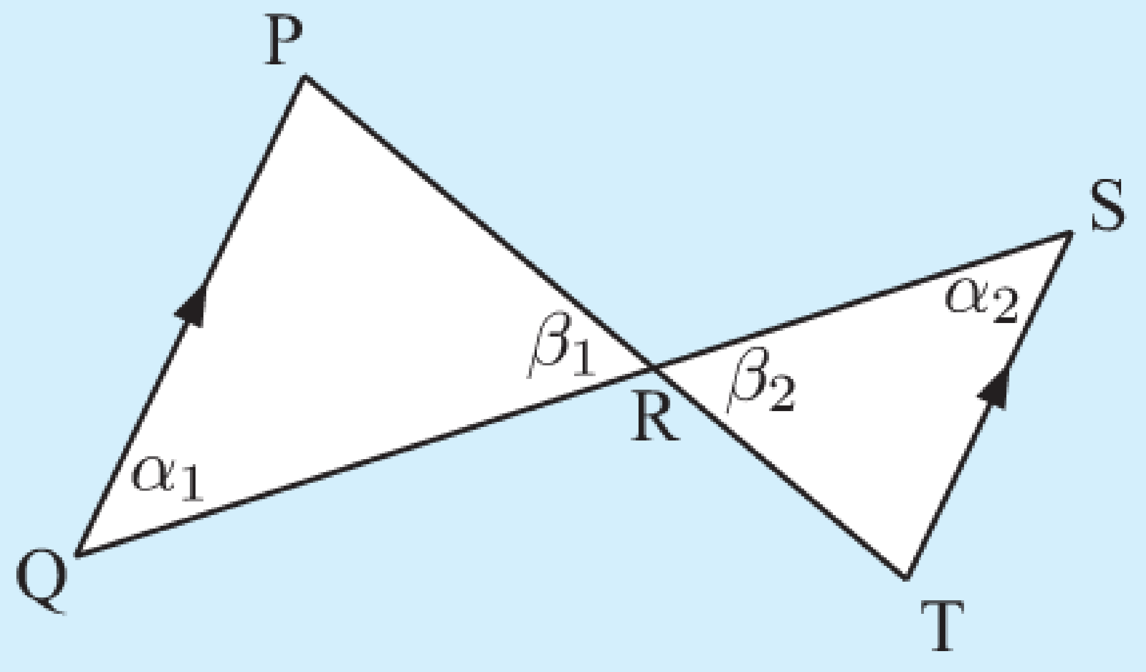
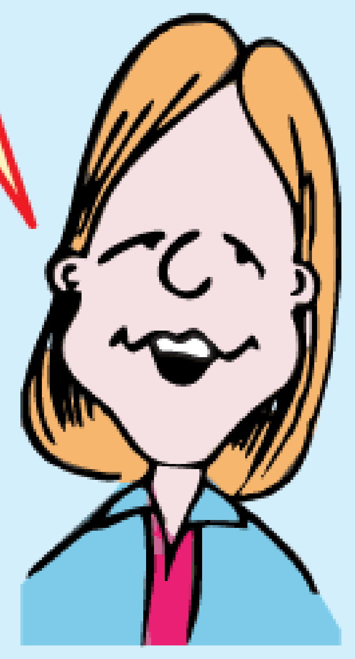


a



Instead of writing $\widehat{PQR} = \widehat{TSR}$ it is often more convenient to write $\alpha_1 = \alpha_2$. The subscripts 1 and 2 show the location of the angle.



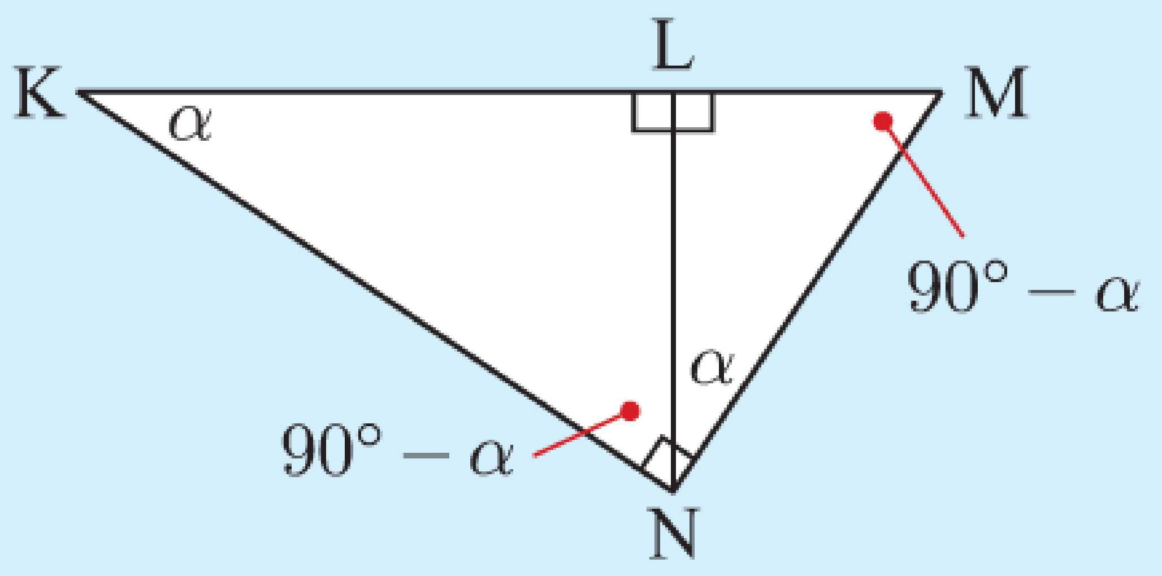
i $\quad \alpha_1 = \alpha_2 \quad$ {alternate angles}

and $\quad \beta_1 = \beta_2 \quad$ {vertically opposite angles}

$\therefore$ $\triangle$s PQR and TSR are equiangular, and therefore similar.

ii $\dfrac{PQ}{TS} = \dfrac{PR}{TR} = \dfrac{QR}{SR}$

b



Consider triangles KLN, NLM, and KMN.

i Let $L\widehat{K}N = \alpha$

$\therefore$ $K\widehat{N}L = 90^\circ - \alpha$ {angles of a △}

$\therefore$ $L\widehat{N}M = \alpha$ {$K\widehat{N}M$ is 90°}

$\therefore$ $L\widehat{M}N = 90^\circ - \alpha$ {angles of a △}

Thus the three triangles are equiangular and therefore similar.

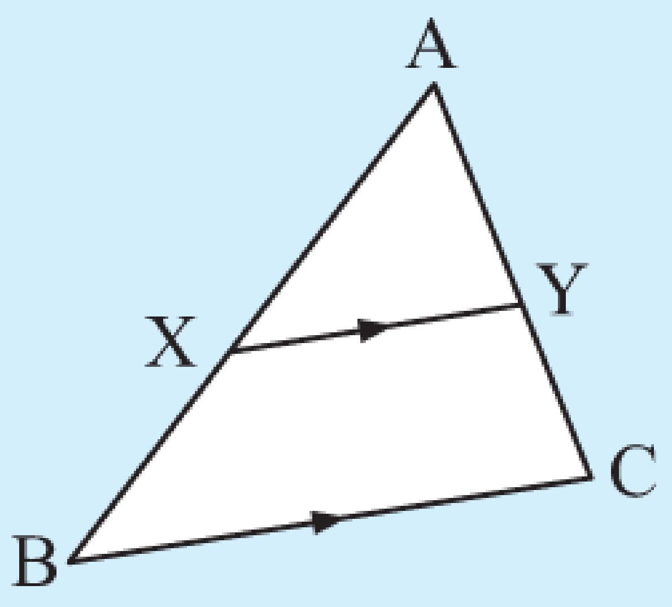
$\therefore$ △s KLN, NLM, and KNM are similar.

ii $\dfrac{KL}{NL} = \dfrac{LN}{LM} = \dfrac{KN}{NM}$ and

$\dfrac{NL}{KN} = \dfrac{LM}{NM} = \dfrac{NM}{KM}$ and

$\dfrac{KL}{KN} = \dfrac{LN}{NM} = \dfrac{KN}{KM}$

PARALLEL LINES WITHIN A TRIANGLE THEOREM



If [XY] is parallel to [BC] then $\dfrac{AX}{XB} = \dfrac{AY}{YC}$.

Proof:

$B\widehat{A}C = X\widehat{A}Y$

and $\alpha_1 = \alpha_2$ {equal corresponding angles}

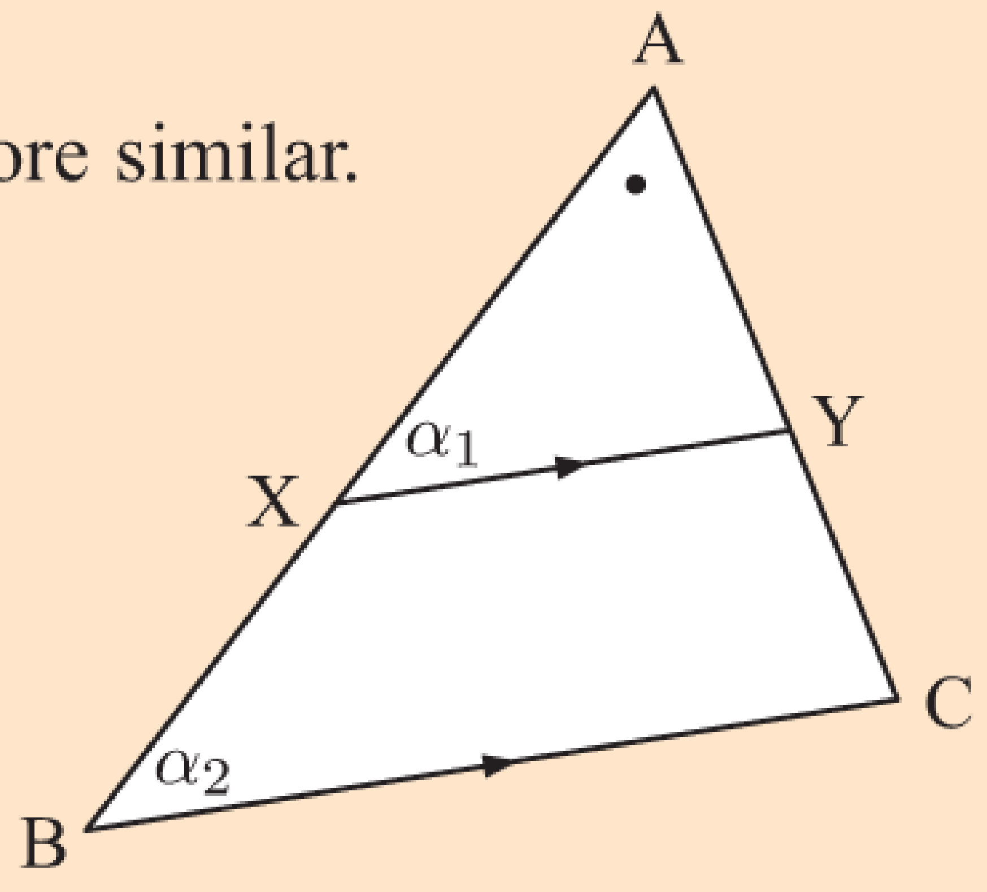
$\therefore$ △s AXY and ABC are equiangular and therefore similar.

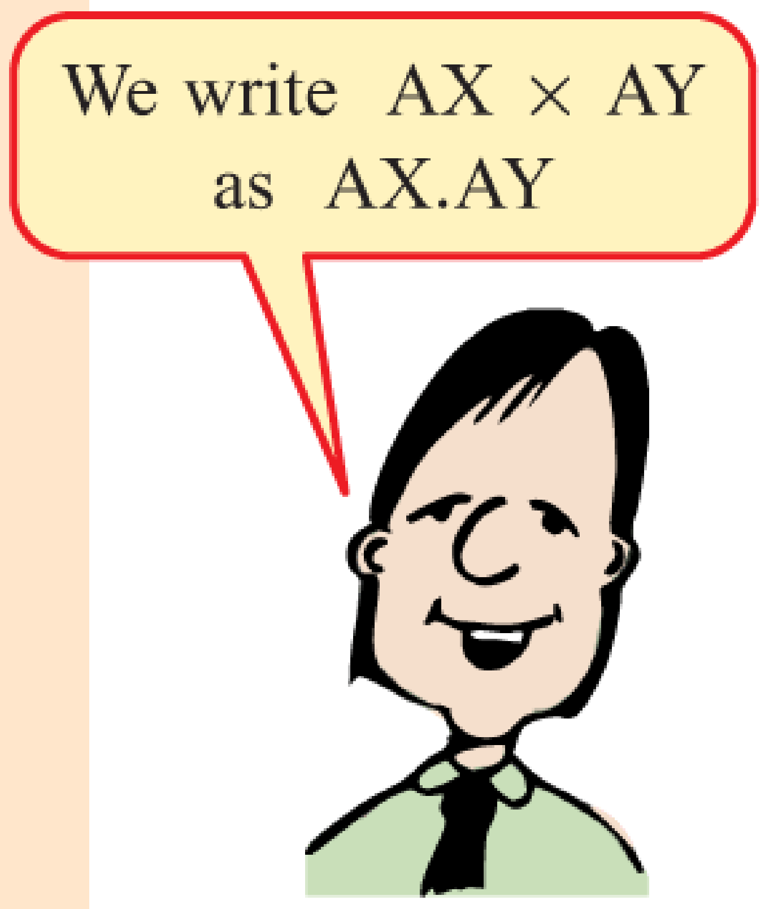
$$\therefore \quad \frac{AX}{AY} = \frac{AB}{AC}$$

$$\therefore \quad \frac{AX}{AY} = \frac{AX + XB}{AY + YC}$$

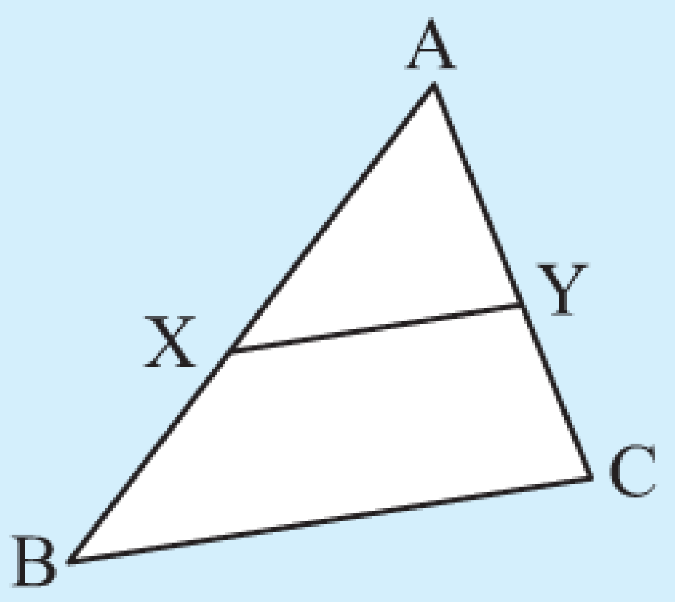
$$\therefore \quad \cancel{AX.AY} + AX.YC = \cancel{AX.AY} + AY.XB$$

$$\therefore \quad \frac{AX}{XB} = \frac{AY}{YC}$$




We write AX × AY as AX.AY

CONVERSE TO PARALLEL LINES WITHIN A TRIANGLE THEOREM



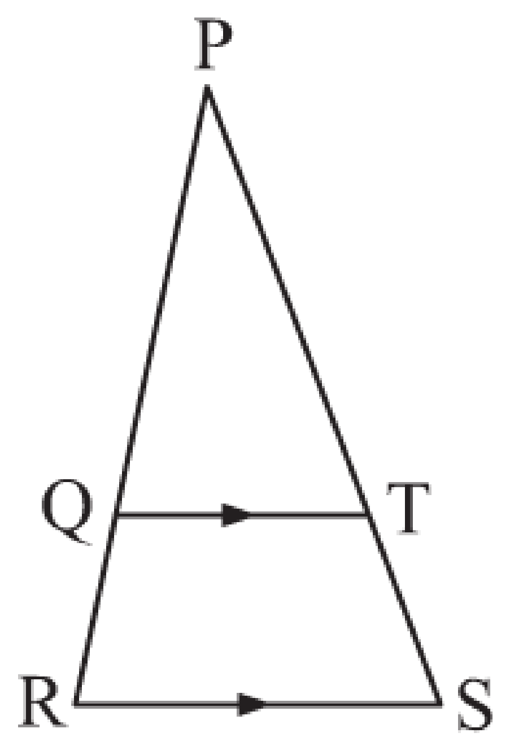
If $\dfrac{AX}{AY} = \dfrac{BX}{CY}$, then [XY] ∥ [BC].

EXERCISE 2A

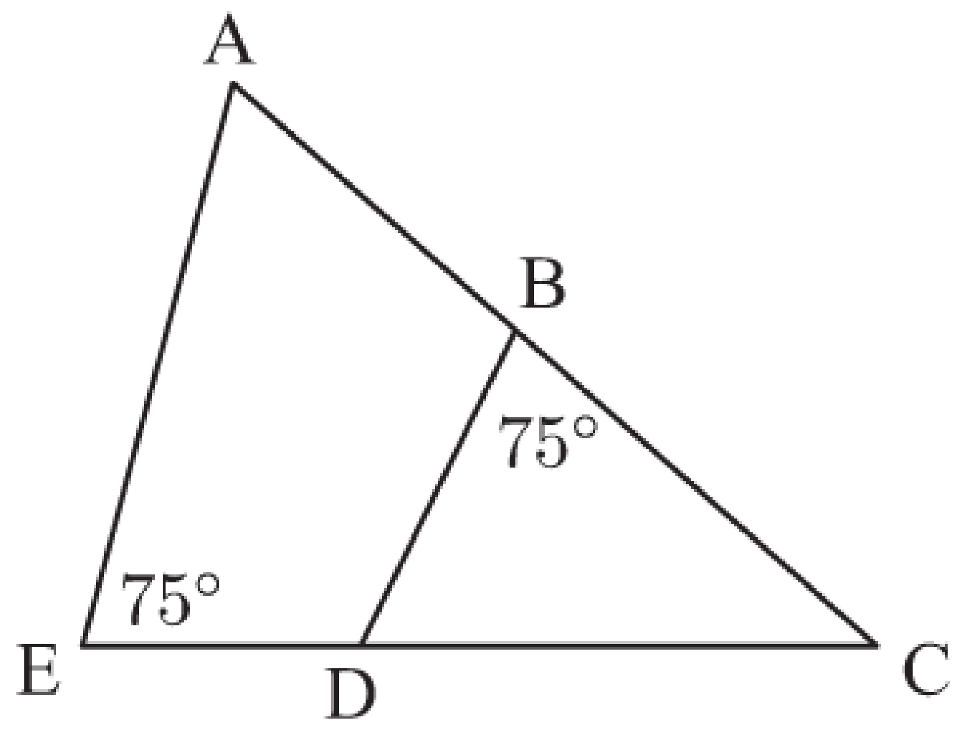
1 For each of the following figures:

- **i** Identify similar triangles and prove that they are similar.
- **ii** Write an equation connecting the lengths of corresponding sides.

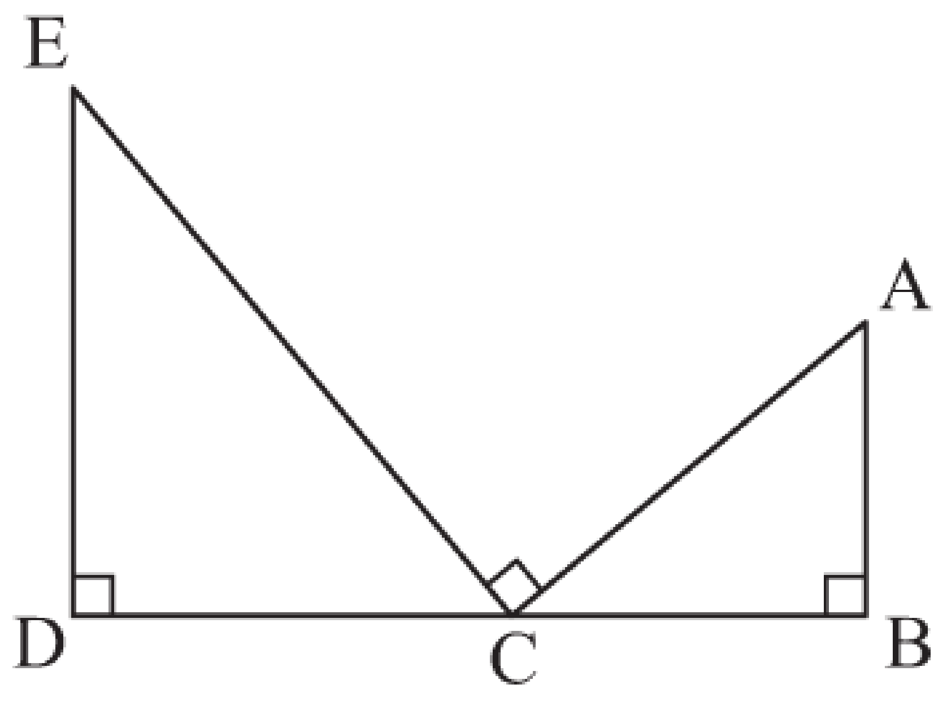
a



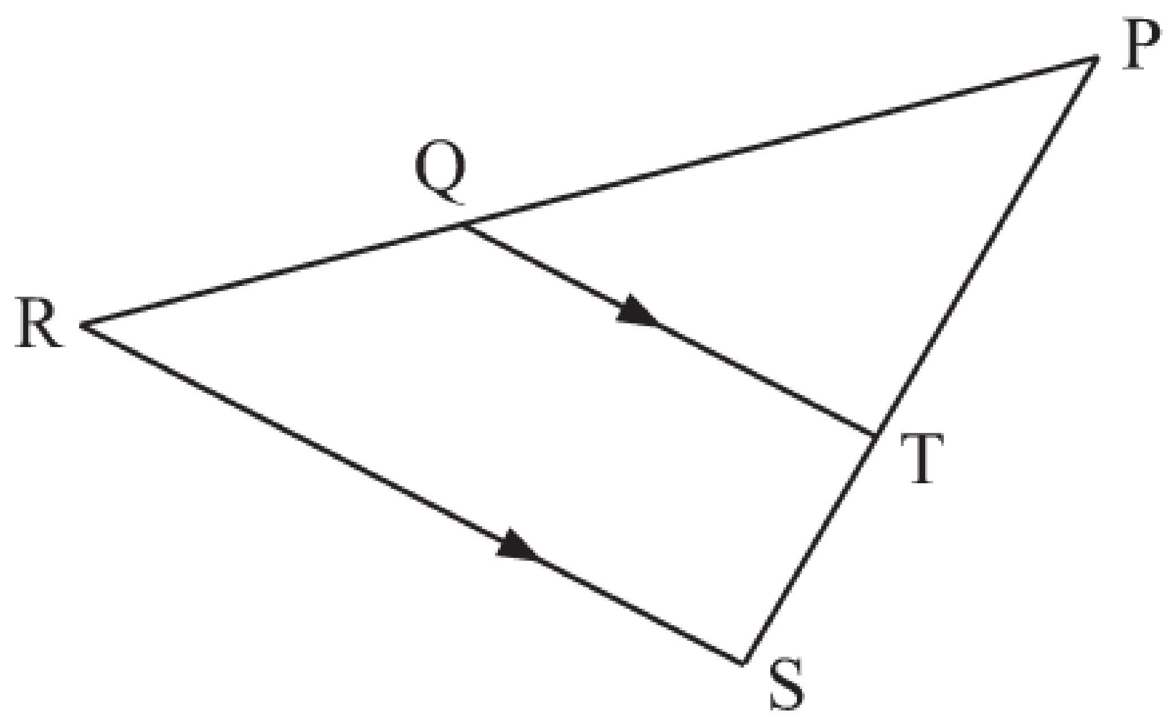
b



c



2 In the given figure, PQ = 5 cm, QR = 3 cm, and TS = 2 cm. Find the length of [PT].



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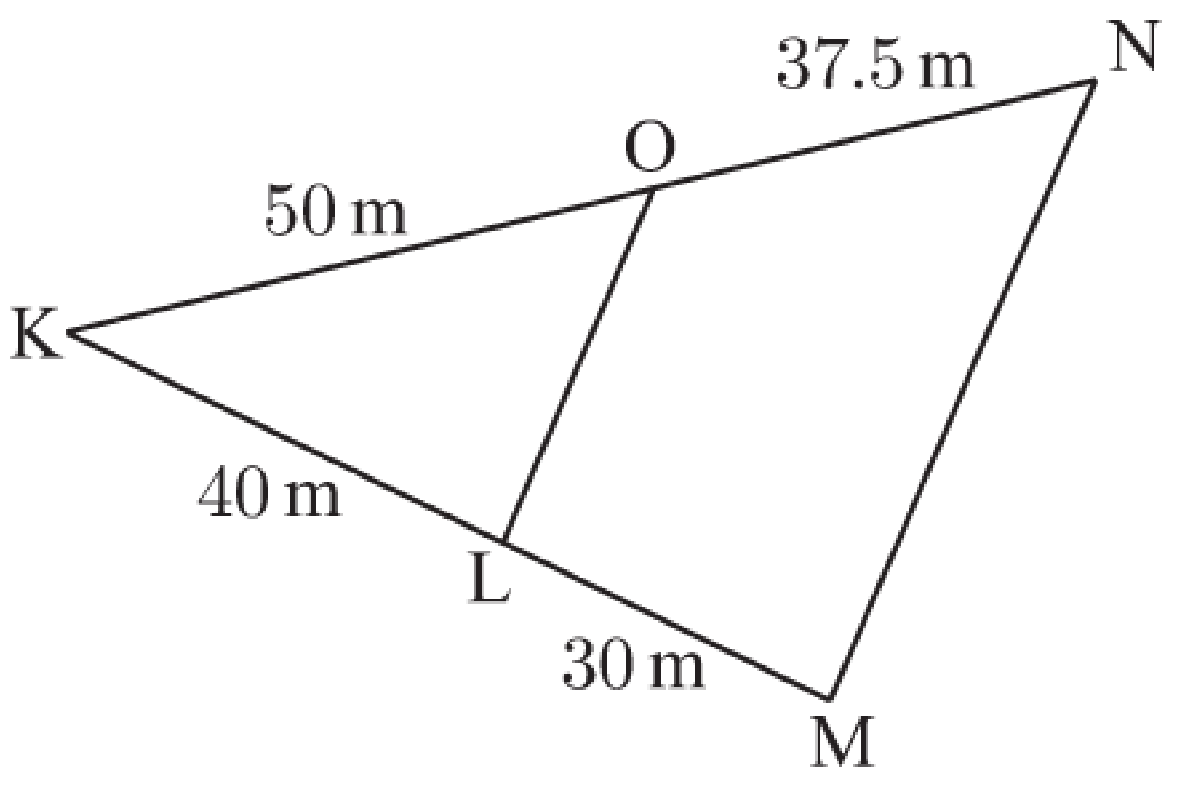
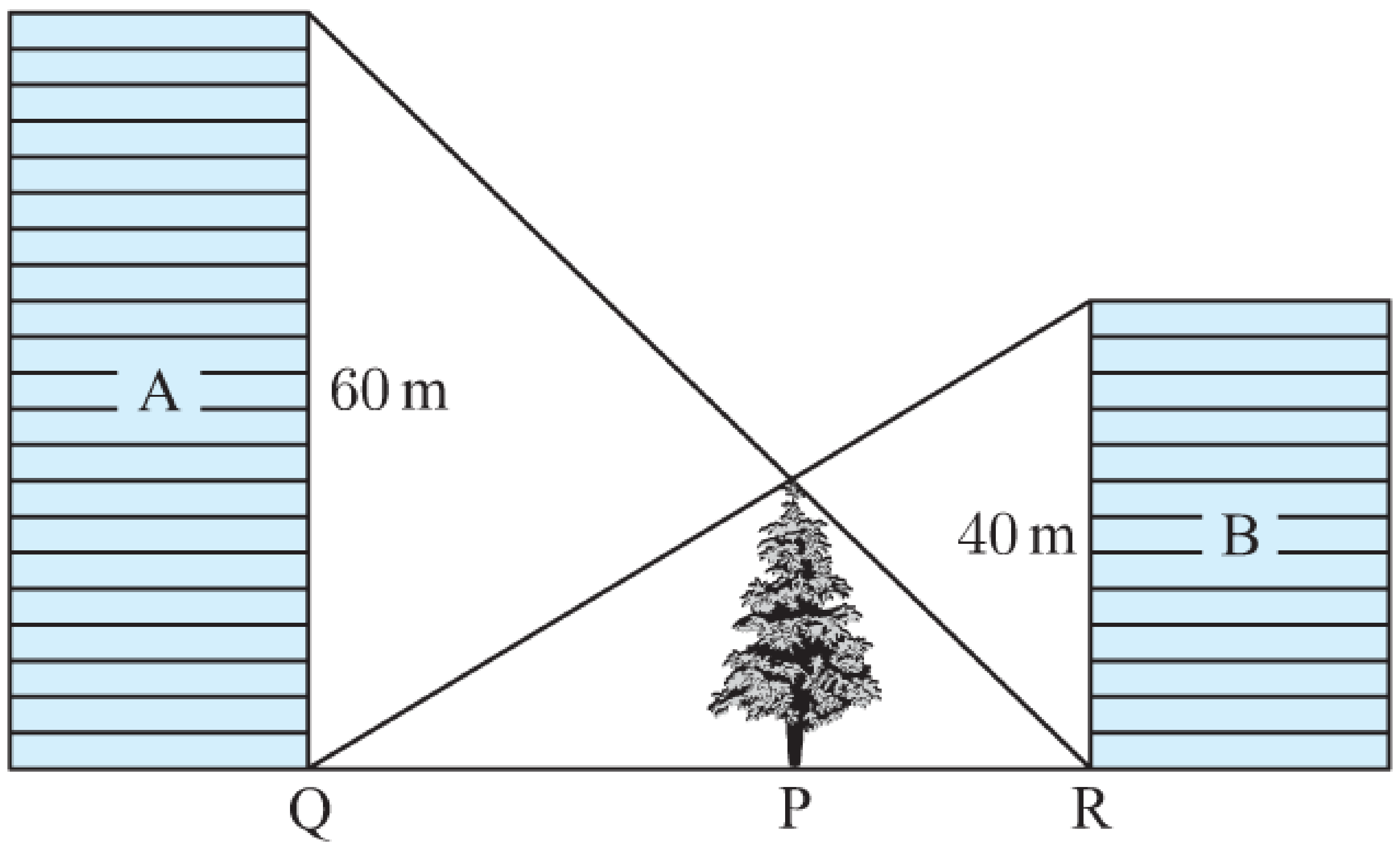


a Prove that [OL] is parallel to [NM].

b If OL = 32 m, find the length of [MN].

4 ABCD is a trapezium with [AB] parallel to [DC]. The diagonals of the trapezium meet at M. Prove that △ABM is similar to △CDM.

5 A pine tree grows between two buildings A and B. On one day it was observed that the top of A, the apex of the tree, and the foot of B line up, and at the same time the foot of A, the apex of the tree, and the top of B line up. Find the height of the tree.

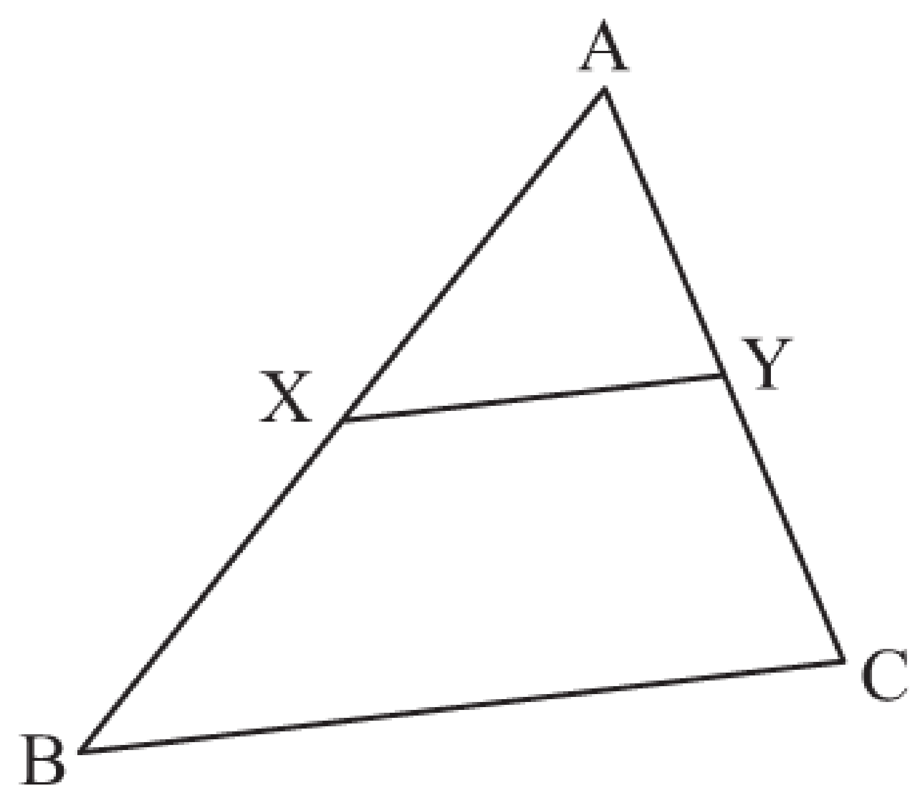


6 PQRS is a parallelogram and T lies on [PS]. [QT] produced meets [RS] produced at U. Prove that QT.PS = QU.PT.

7 ABC is an isosceles triangle with AB = AC. X lies on [AC] such that $CB^2 = CX.CA$. Prove that BX = BC.

8 Triangle ABC has altitudes [AP] and [BQ] where P lies on [BC], and Q lies on [AC]. H is the intersection of [AP] and [BQ].
Prove that AH.HP = BH.HQ.

9 Prove the converse to the *parallel lines within a triangle theorem*:



If $\frac{AX}{AY} = \frac{BX}{CY}$ then [XY] $\parallel$ [BC].

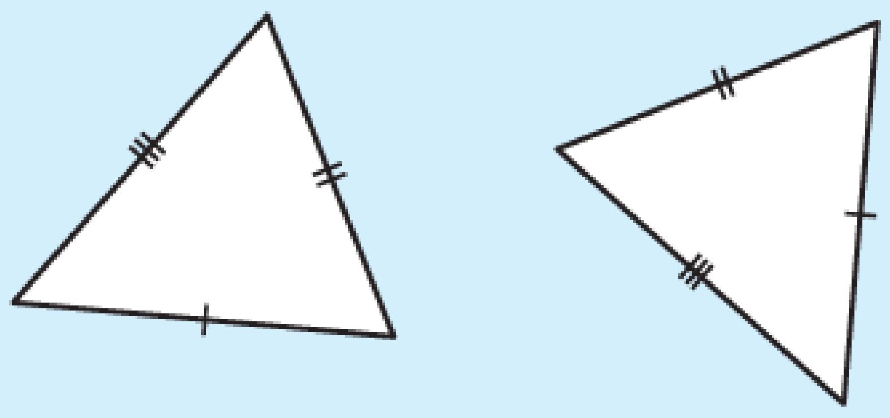
B CONGRUENT TRIANGLES

Two triangles are **congruent** if they are identical in every respect apart from position and orientation. They have the same shape and size.

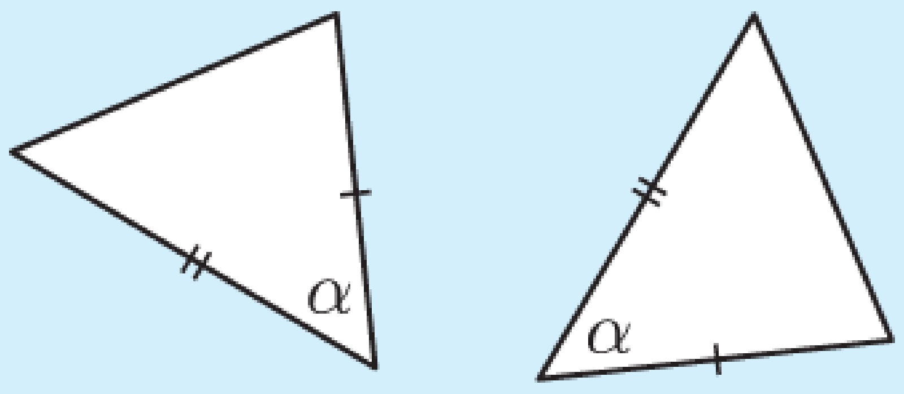
TESTS FOR THE CONGRUENCE OF TRIANGLES

Two triangles are congruent if one of the following is true:

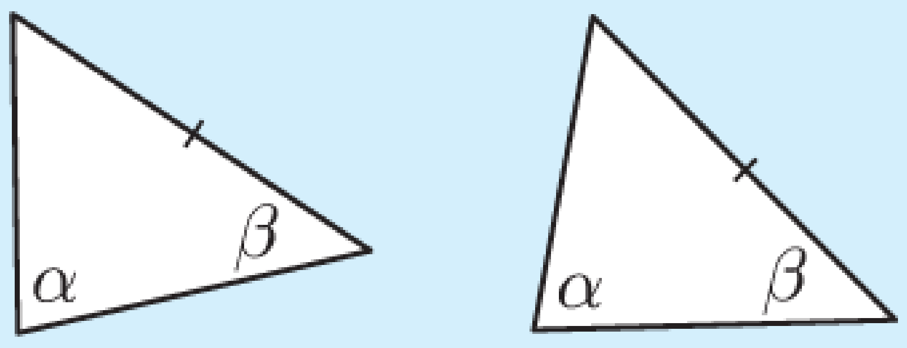
- corresponding sides have equal length (SSS)



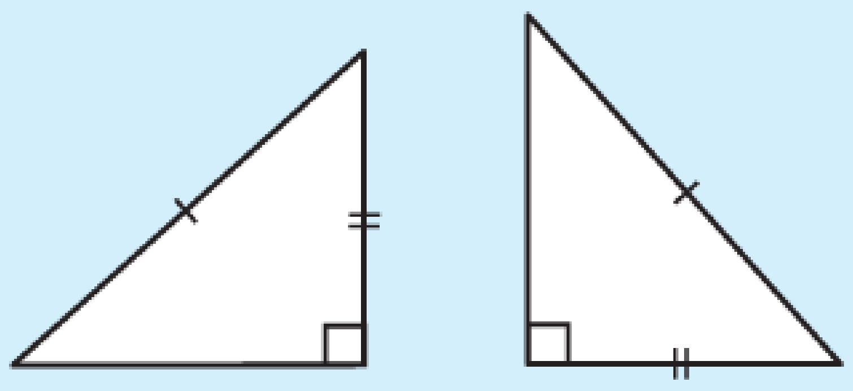
- two corresponding sides are equal in length and the included angles are equal (SAS)



- two corresponding angles are equal and any corresponding sides are equal in length (AAcorS)

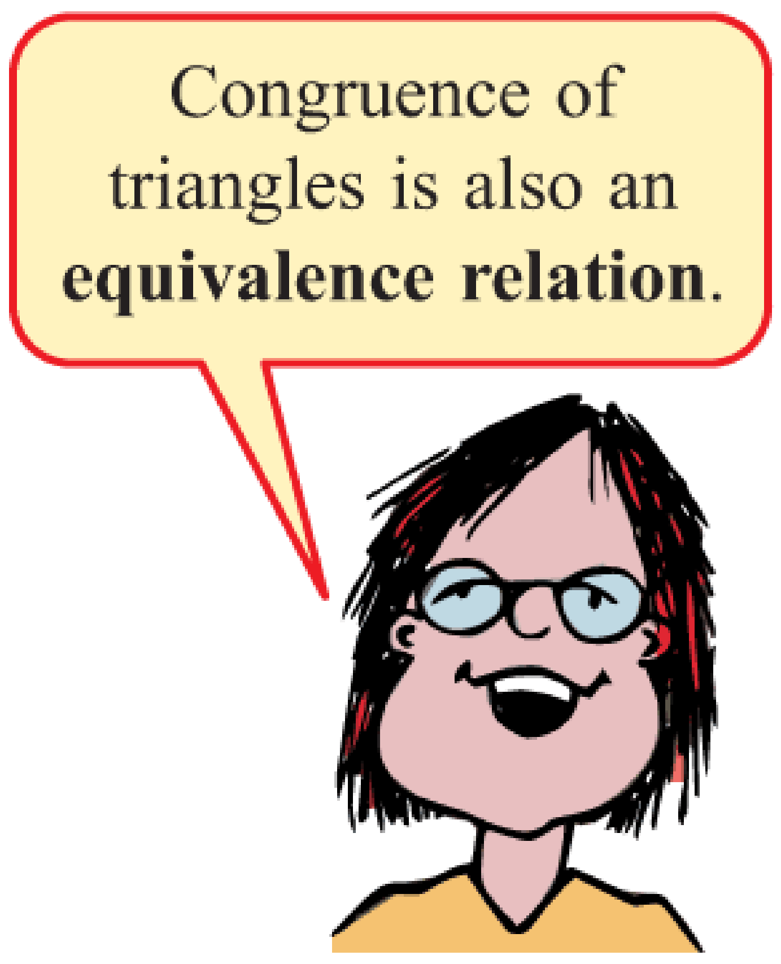


- each triangle is right angled, the hypotenuses are equal in length, and a pair of corresponding sides are equal (RHS).



These 4 tests for congruence are a consequence of the fact that two or more people will draw exactly the same triangle when given details of:

- the lengths of the sides of the triangle
- the lengths of two sides of the triangle and the angle between them
- two angle sizes and one side length
- a right angled triangle with known hypotenuse and one other side length.

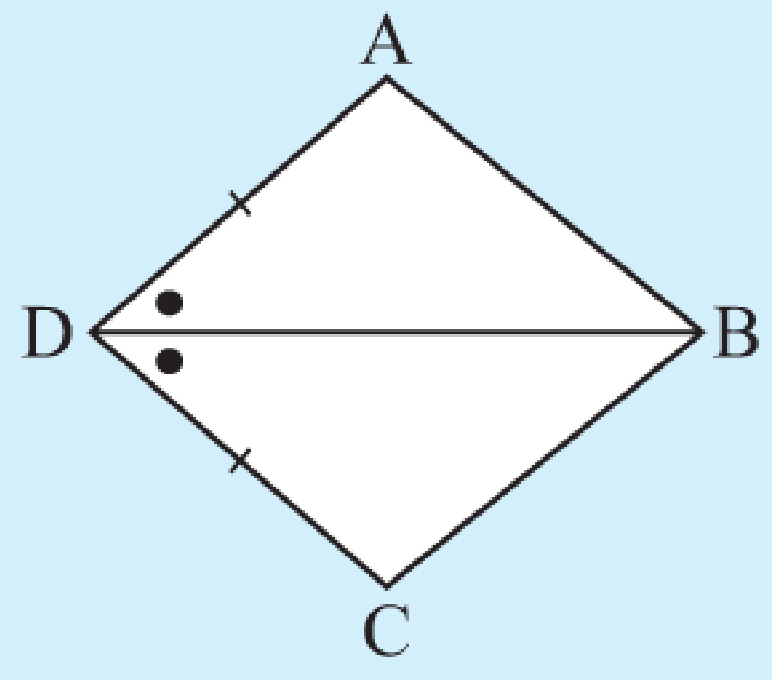


Congruence of triangles is also an **equivalence relation**.

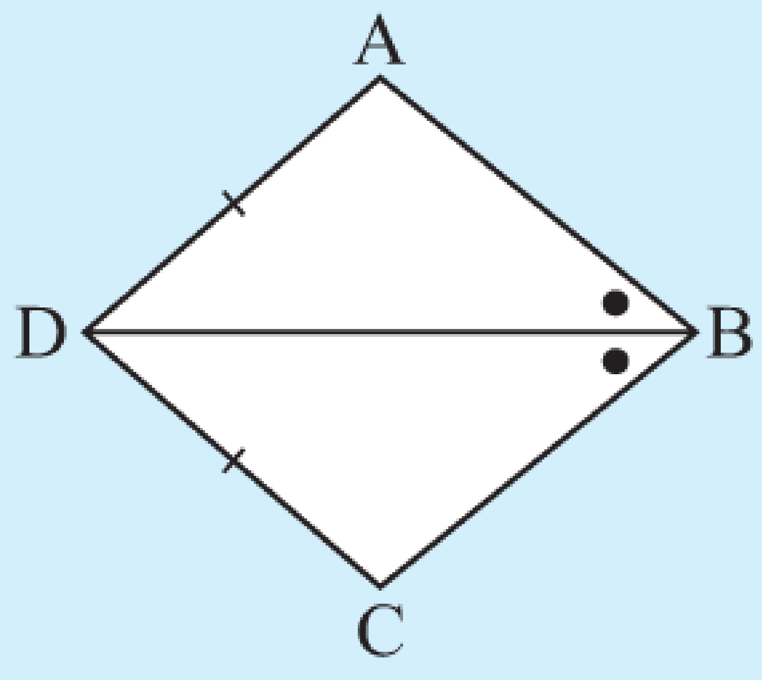
Example 2

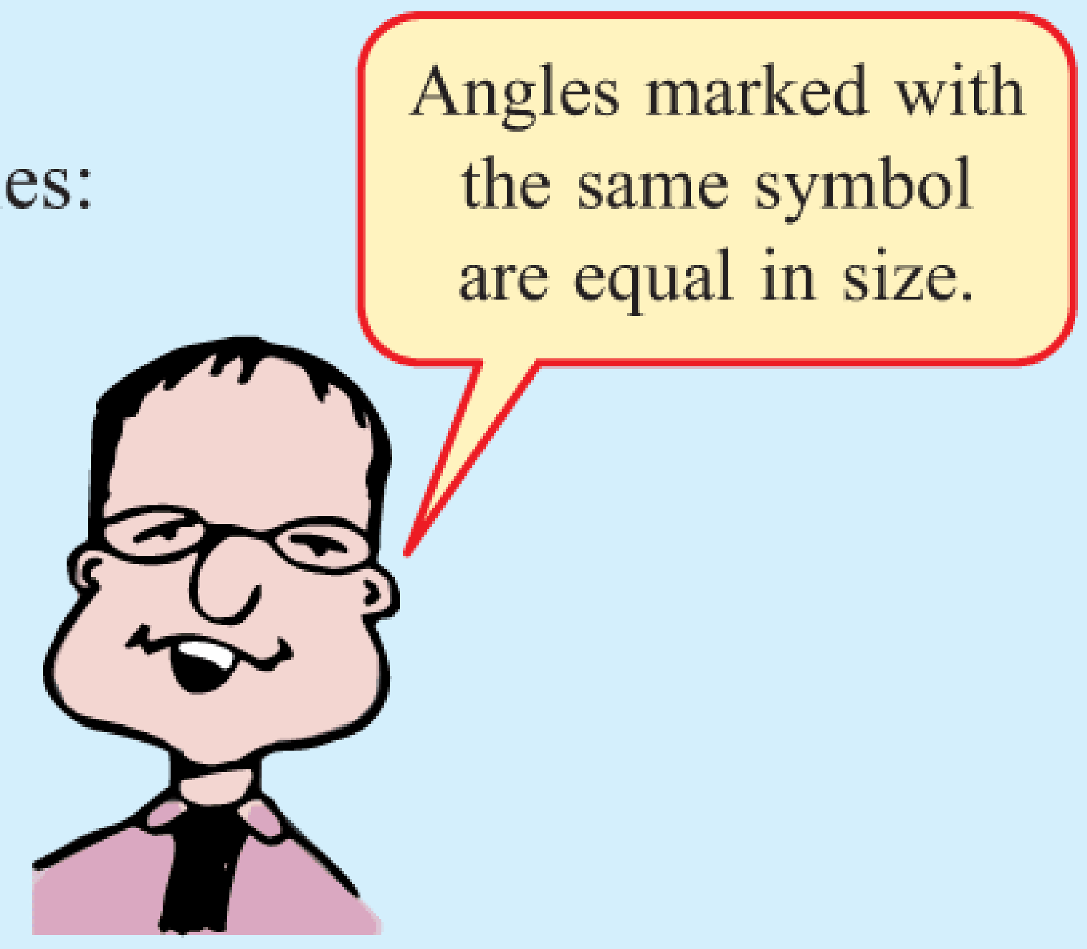
Determine whether each of these figures contains congruent triangles:

a



b





Angles marked with the same symbol are equal in size.

a We observe that:

- DA = DC
- [DB] is common
- $\widehat{ADB} = \widehat{CDB}$

$\therefore$ $\triangle$s ADB and CDB are congruent {SAS}.

b We observe that:

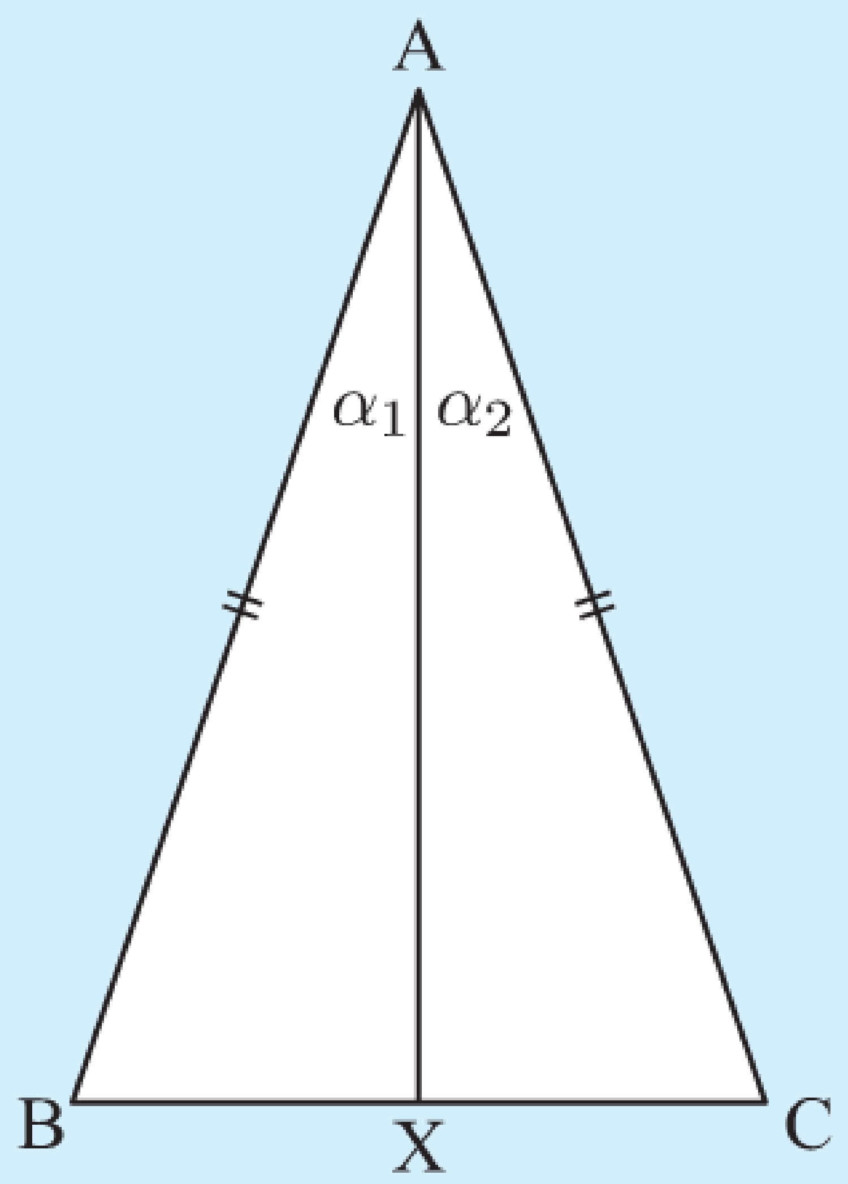
- DA = DC
- [DB] is common
- $\widehat{ABD} = \widehat{CBD}$.

None of the four tests apply, so we cannot establish congruence.

{The equal angles are *not included* angles for the equal sides.}

Example 3

Use congruence to prove that the bisector of the apex of an isosceles triangle bisects the base at right angles.



Consider an isosceles $\triangle$ABC where AB = AC and $\alpha_1 = \alpha_2$.

We observe that:

- $\alpha_1 = \alpha_2$
- AB = AC
- [AX] is common to both $\triangle$s

$\therefore$ $\triangle$s ABX and ACX are congruent {SAS}.

Consequently BX = CX and $\widehat{AXB} = \widehat{AXC}$.

But these angles add to 180° {angles on a line}

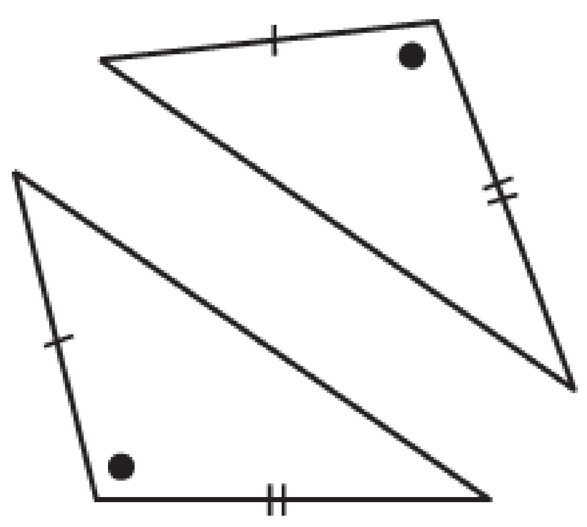
$\therefore$ each of them is a right angle

$\therefore$ [AX] bisects [BC] at right angles.

EXERCISE 2B

1 In this question you may **not** assume any properties of isosceles triangles. Determine whether each of these figures contains congruent triangles:

a



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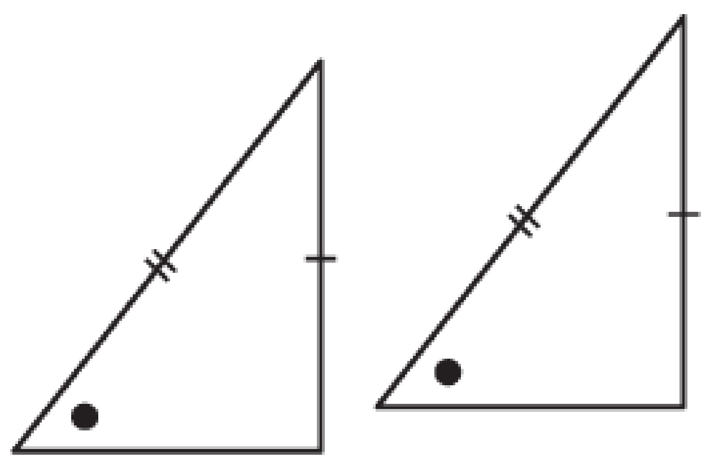
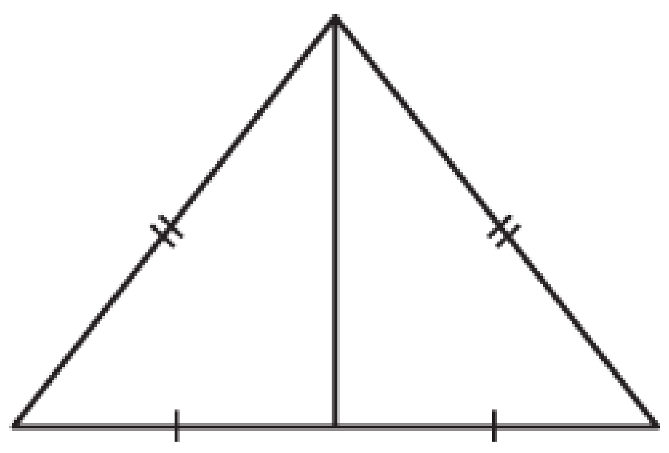


c



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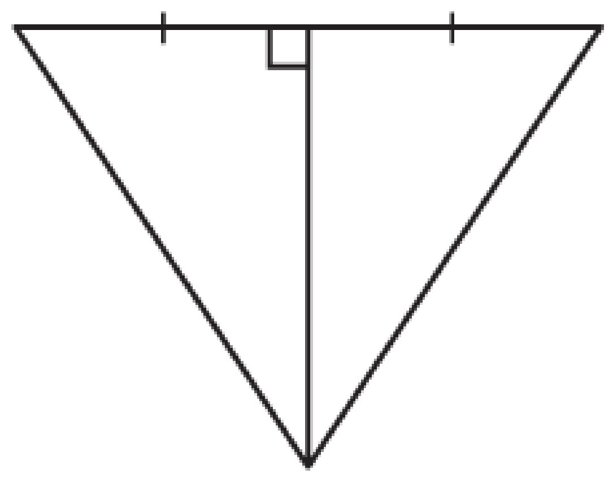

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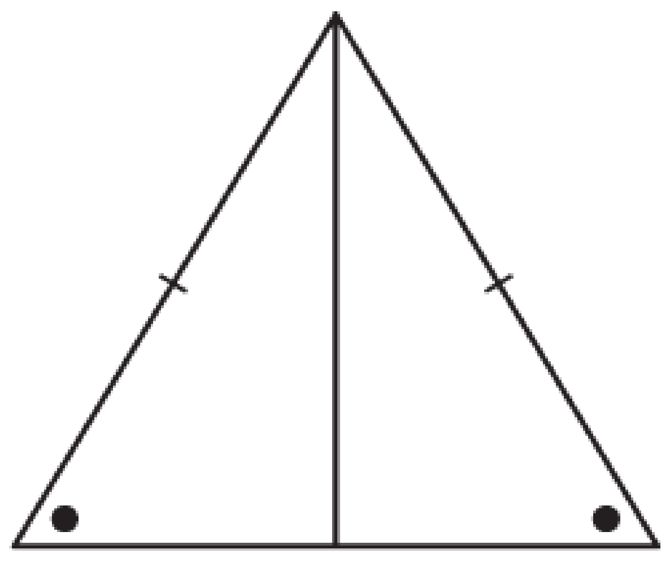

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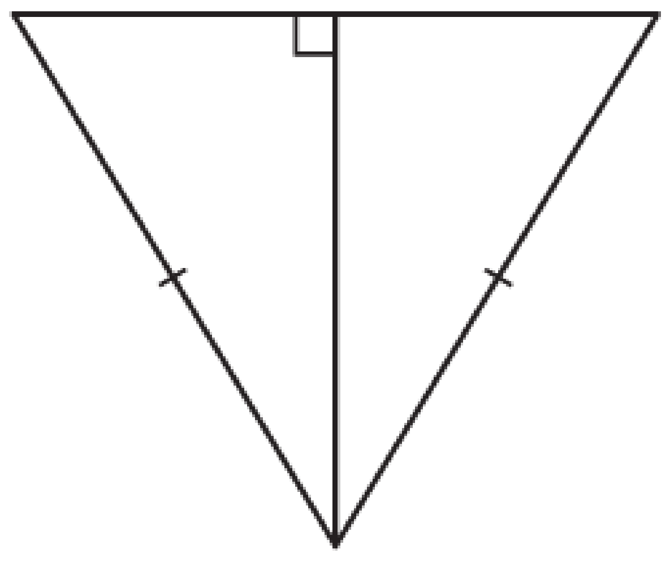

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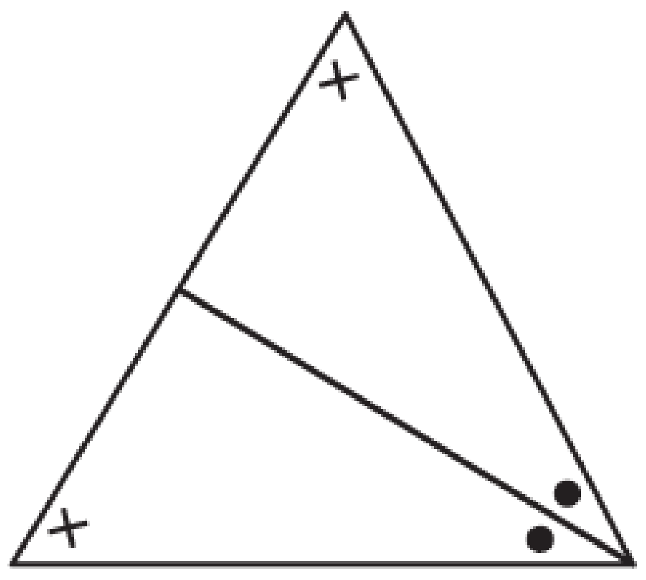

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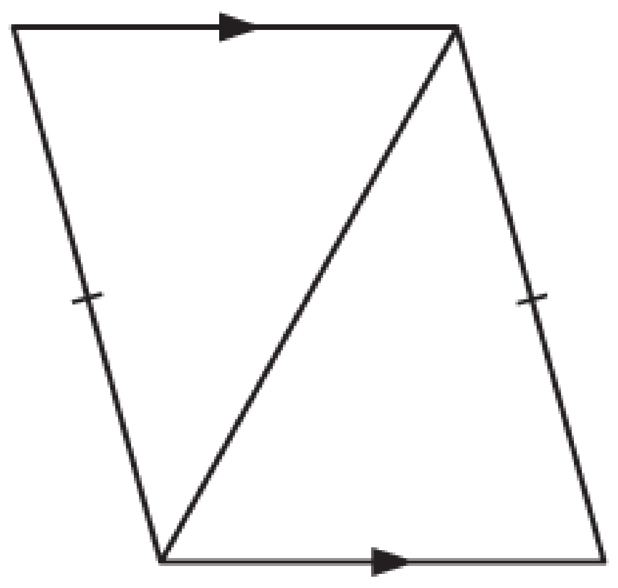
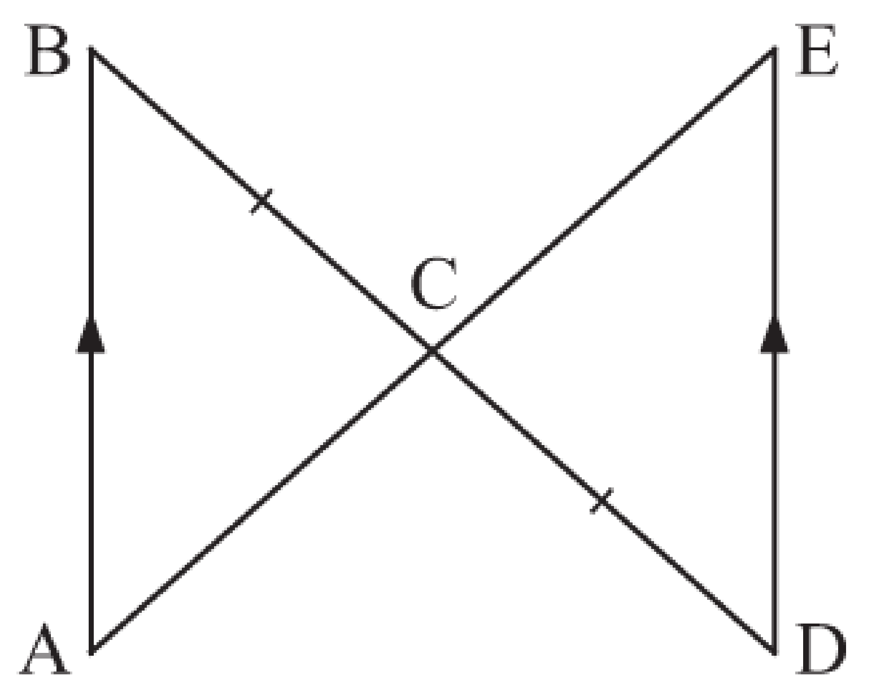


2 **a** Explain why triangles ABC and EDC are congruent.

b If $AC = 6$ cm and $B\widehat{A}C = 42°$, find:

- **i** the length of [CE]
- **ii** the size of $D\widehat{E}C$.



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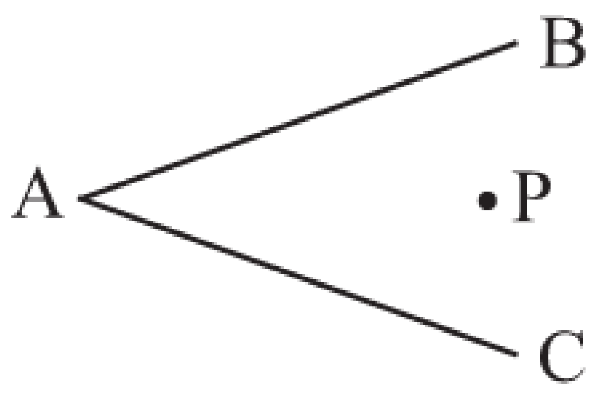
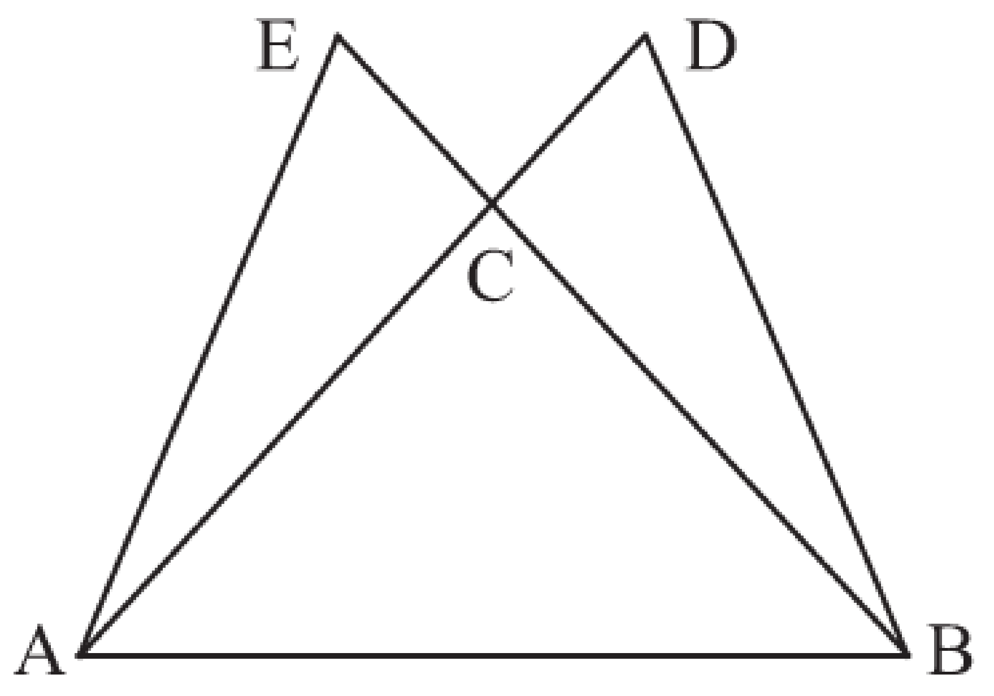


Point P is equidistant from both [AB] and [AC]. Use congruence to show that P lies on the bisector of $B\widehat{A}C$.

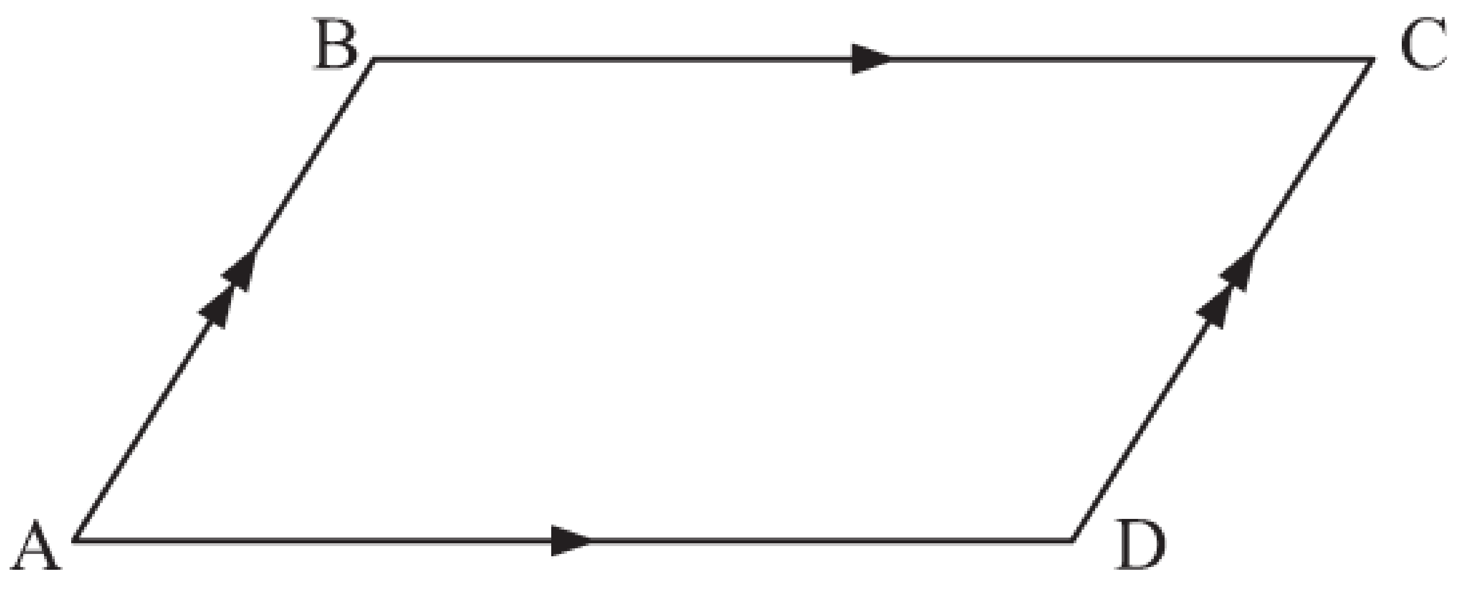
4 Triangle ABC is isosceles with $AC = BC$.
BC and AC are produced to E and D respectively so that $CE = CD$.

Prove that $AE = BD$.



5 You are given a parallelogram ABCD.
Use congruence of triangles to prove that:

- **a** opposite sides are equal in length
- **b** opposite angles are equal
- **c** the diagonals bisect each other.



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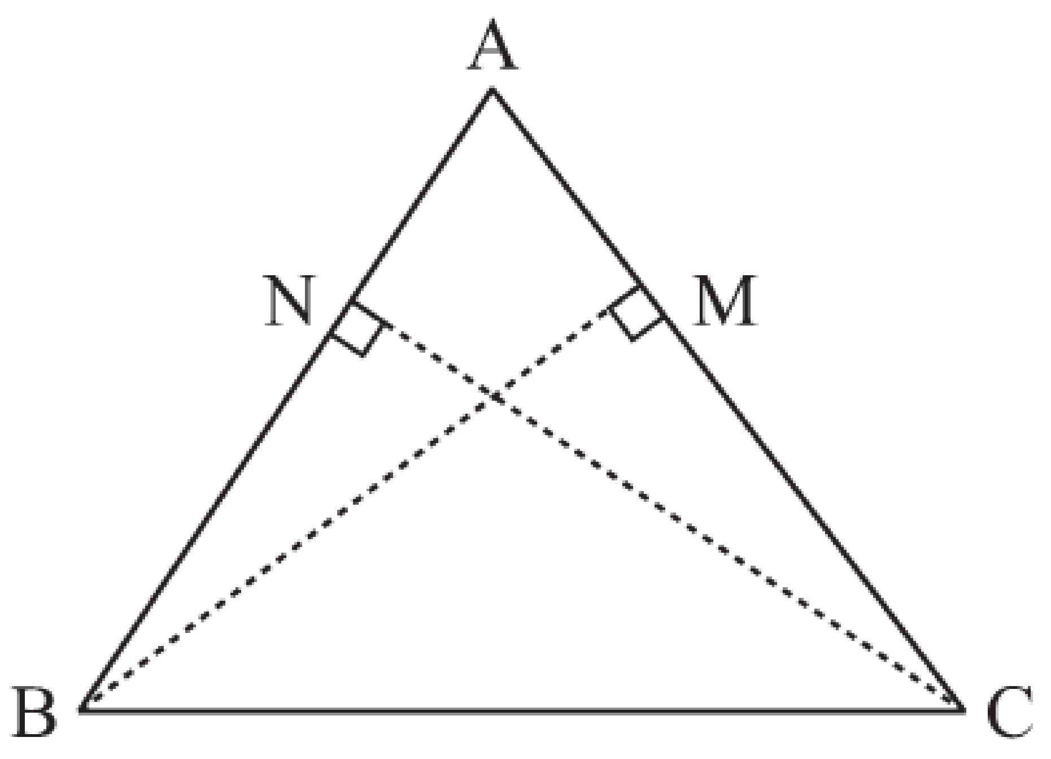
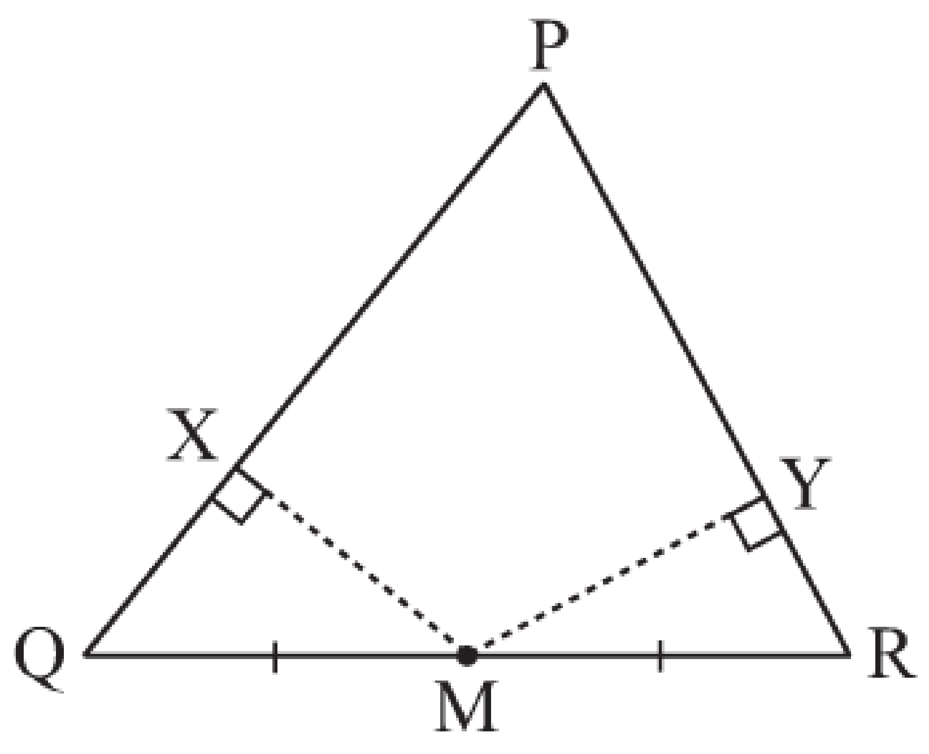


In $\triangle ABC$, [BM] is drawn perpendicular to [AC], and [CN] is drawn perpendicular to [AB].
If these perpendiculars are equal in length, prove that:

- **a** $\triangle$s BCM and CBN are congruent
- **b** $\triangle ABC$ is isosceles.

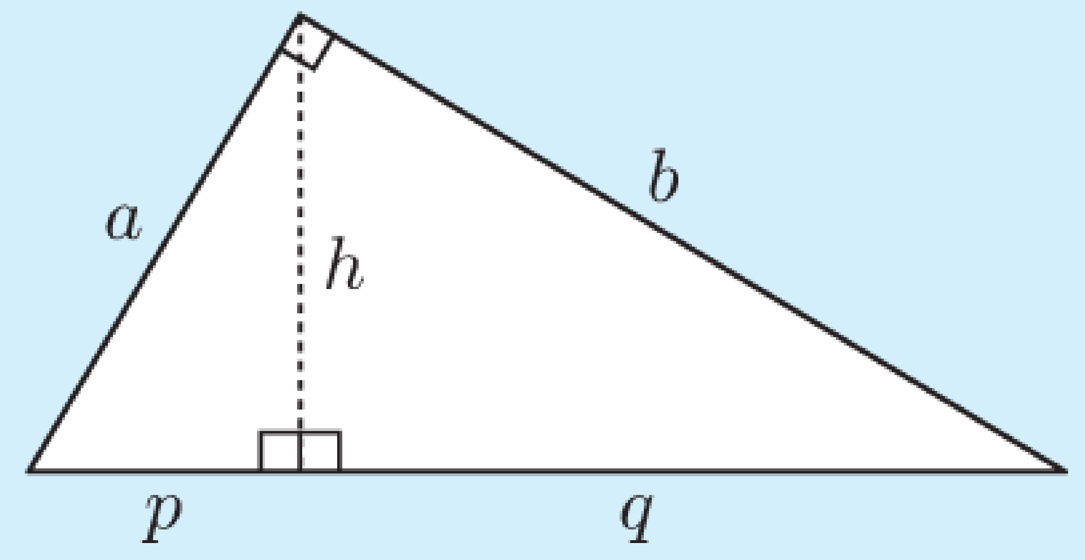
7 In $\triangle PQR$, M is the midpoint of [QR]. [MX] is drawn perpendicular to [PQ], and [MY] is drawn perpendicular to [PR]. If the perpendiculars are equal in length, prove that:

- **a** $\triangle MQX$ is congruent to $\triangle MRY$
- **b** $\triangle PQR$ is isosceles.



C PROPORTIONALITY IN RIGHT ANGLED TRIANGLES

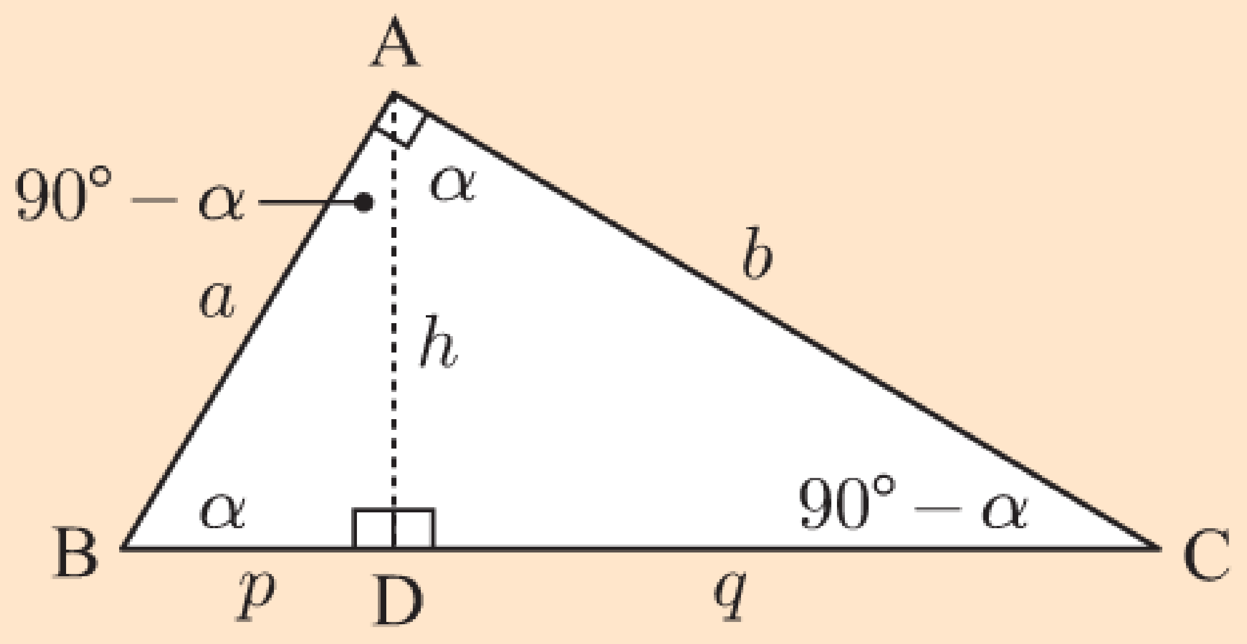
EUCLID'S THEOREM FOR PROPORTIONAL SEGMENTS IN A RIGHT ANGLED TRIANGLE



In the given figure:

- $h^2 = pq$
- $a^2 = p(p+q)$
- $b^2 = q(p+q)$

Proof:



In $\triangle$ABC, let $\widehat{ABC} = \alpha$.

$\therefore$ $\widehat{BAD} = 90° - \alpha$ {angles of a triangle}

$\therefore$ $\widehat{CAD} = \alpha$

Also, $\widehat{ACD} = 90° - \alpha$ {angles of a triangle}

$\triangle$s ABC, DBA, and DAC are equiangular and hence are similar.

Corresponding sides are in proportion, so:

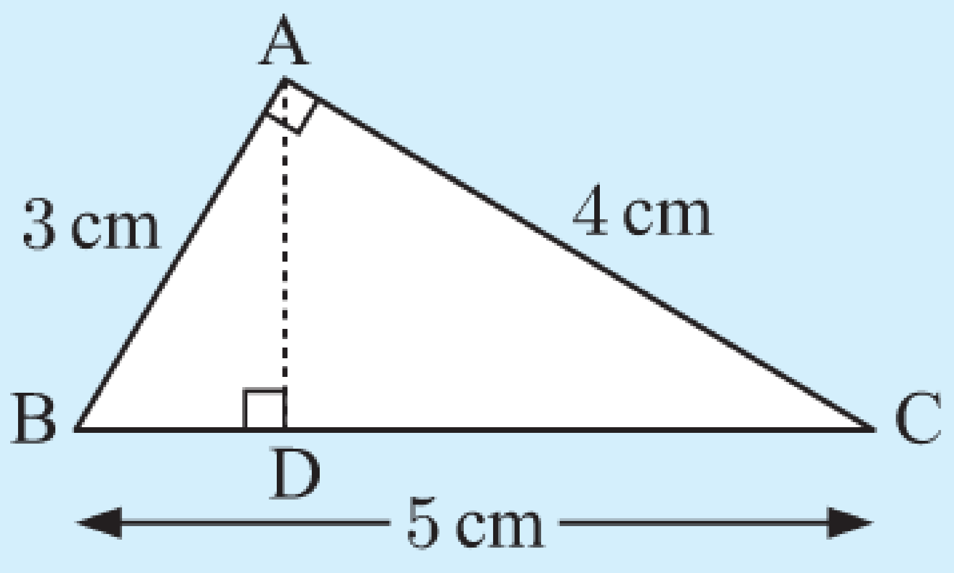
- $\dfrac{DB}{DA} = \dfrac{DA}{DC} \;\Rightarrow\; \dfrac{p}{h} = \dfrac{h}{q} \;\Rightarrow\; h^2 = pq$
- $\dfrac{BA}{BC} = \dfrac{DB}{AB} \;\Rightarrow\; \dfrac{a}{p+q} = \dfrac{p}{a}$

  $\Rightarrow\; a^2 = p(p+q)$
- $\dfrac{AC}{BC} = \dfrac{DC}{AC} \;\Rightarrow\; \dfrac{b}{p+q} = \dfrac{q}{b}$

  $\Rightarrow\; b^2 = q(p+q)$

Example 4

Find BD and AD in:



From Euclid's theorem, $BA^2 = BD.BC$

$\therefore$ $3^2 = BD \times 5$

$\therefore$ BD $= 1.8$ cm

Also $AD^2 = BD \times DC$

$\therefore$ AD $= \sqrt{1.8 \times 3.2} = 2.4$ cm

Notice in the previous **Example** that we can obtain the same result by calculating the area of $\triangle$ABC in two different ways:

$$\tfrac{1}{2} \times 5 \times \text{AD} = \tfrac{1}{2} \times 3 \times 4$$
$$\Rightarrow \quad \text{AD} = \tfrac{12}{5} = 2.4 \text{ cm}$$

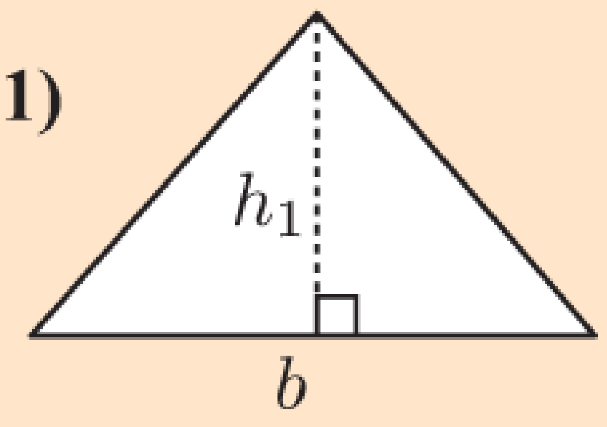
AREA COMPARISON THEOREM

Theorem: Areas of triangles are proportional to:

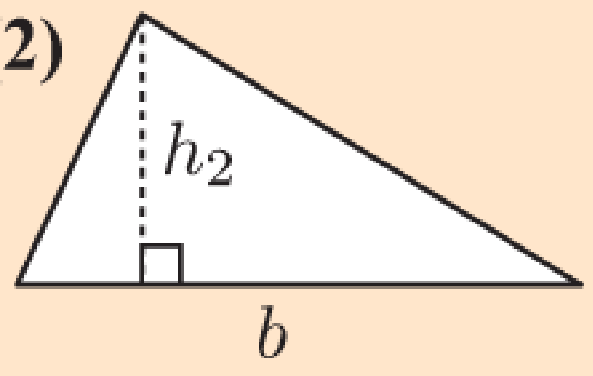
- altitudes if bases are equal
- bases if altitudes are equal
- squares of corresponding sides if the triangles are similar.

Proof:

- **(1)**



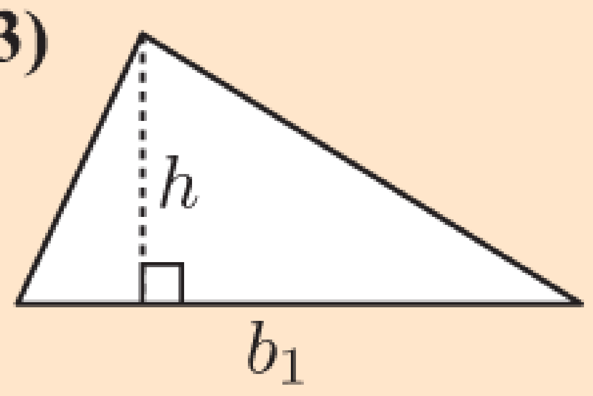
(2)



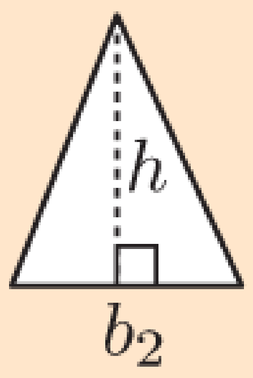
$$\frac{\text{Area of } \textbf{(1)}}{\text{Area of } \textbf{(2)}} = \frac{\frac{1}{2}bh_1}{\frac{1}{2}bh_2} = \frac{h_1}{h_2}$$

proves the first part.

- **(3)**



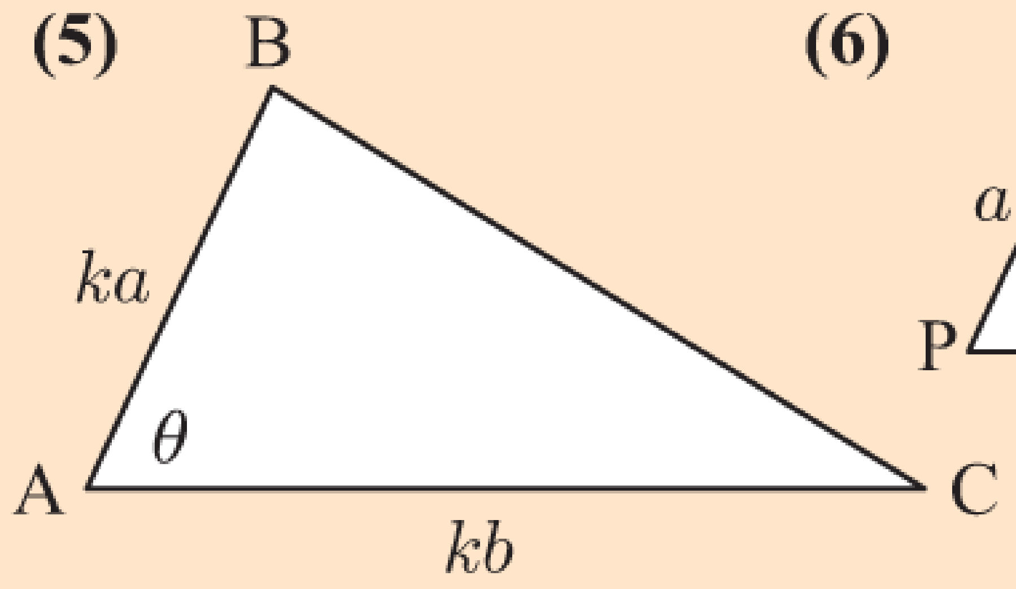
(4)



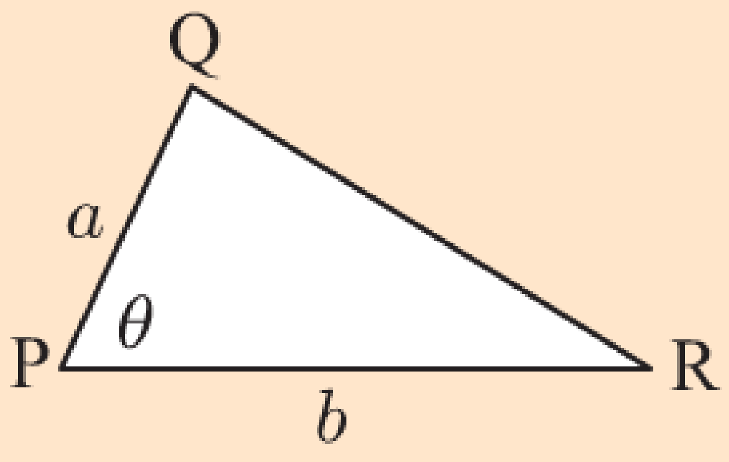
$$\frac{\text{Area of } \textbf{(3)}}{\text{Area of } \textbf{(4)}} = \frac{\frac{1}{2}b_1h}{\frac{1}{2}b_2h} = \frac{b_1}{b_2}$$

proves the second part.

- **(5)**



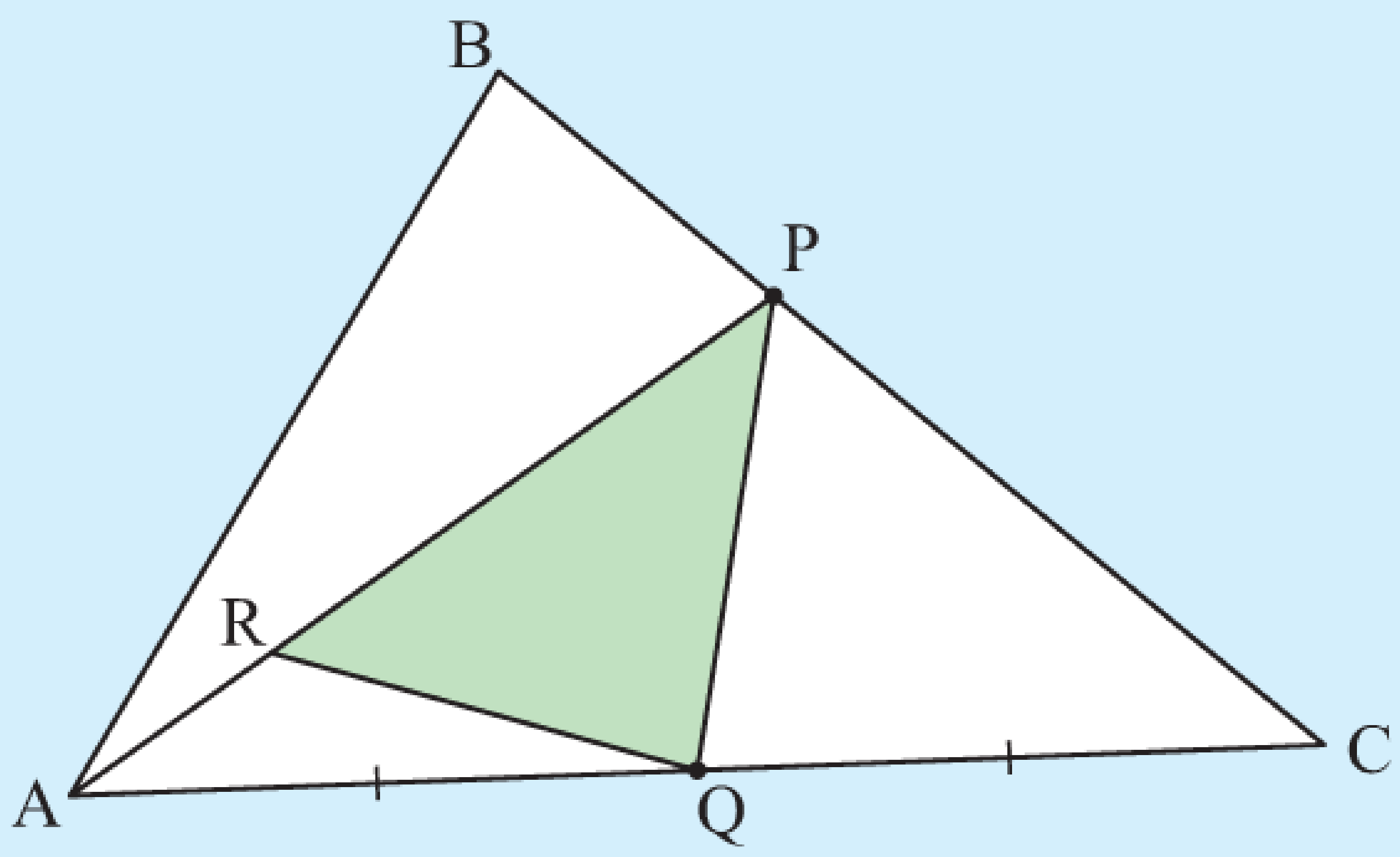
(6)



$$\frac{\text{Area of } \textbf{(5)}}{\text{Area of } \textbf{(6)}} = \frac{\frac{1}{2}(ka)(kb)\sin\theta}{\frac{1}{2}ab\sin\theta}$$
$$= k^2$$
$$= \frac{(ka)^2}{a^2}$$
$$= \frac{\text{AB}^2}{\text{PQ}^2}$$

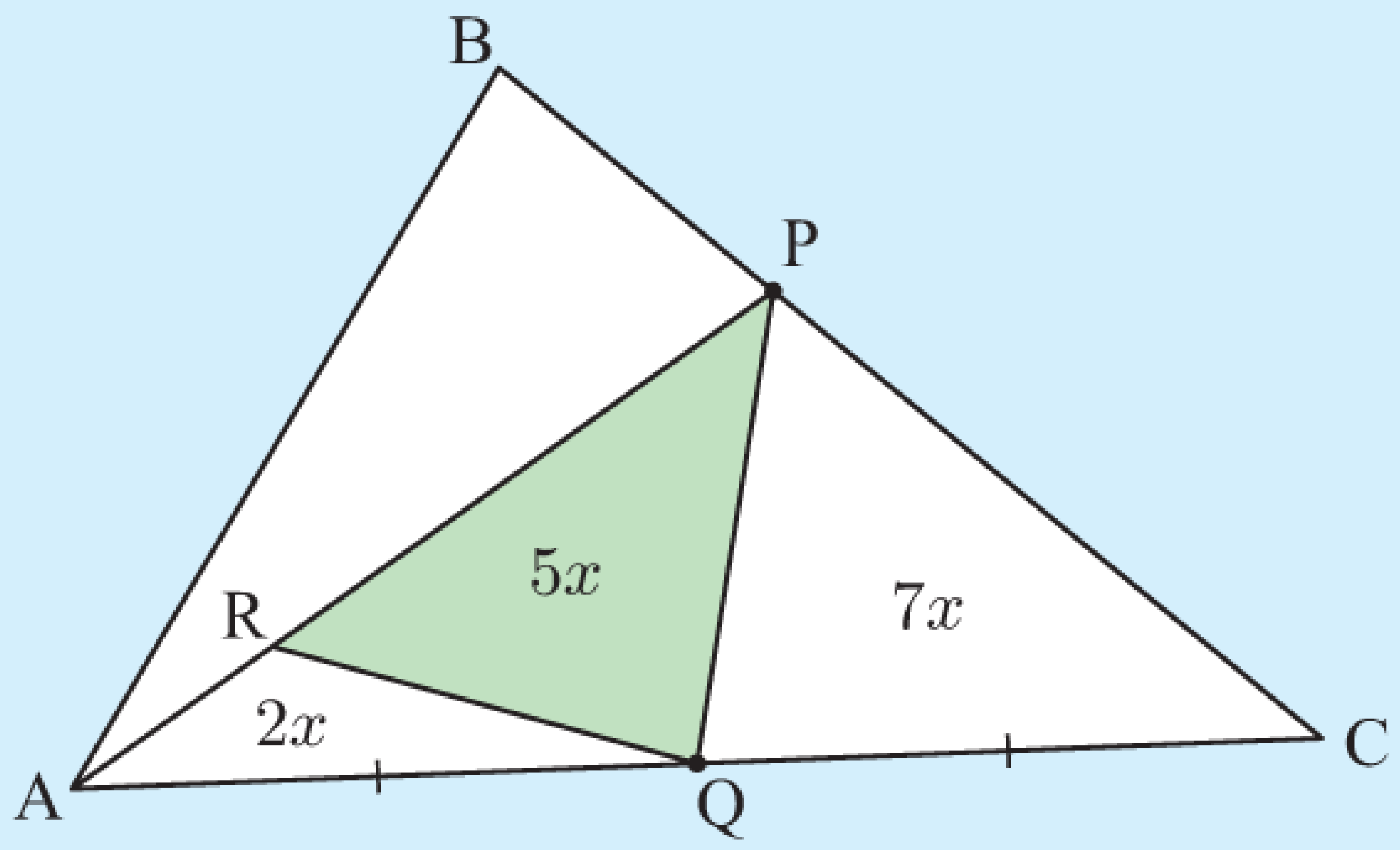
proves the third part.

Example 5



In $\triangle$ABC, P divides [BC] in the ratio 1 : 2, Q is the midpoint of [AC], and R divides [AP] in the ratio 2 : 5.

By area, what fraction is $\triangle$PQR of $\triangle$ABC?


B
P
R
5x
7x
2x
A
Q
C

Let $\triangle$ARQ have area $2x$.

$\therefore$ $\triangle$RQP has area $5x$,

and $\triangle$PQC has area $2x + 5x = 7x$.

$\therefore$ $\triangle$APC has area $14x$.

But $\dfrac{\text{area of }\triangle\text{ABP}}{\text{area of }\triangle\text{APC}} = \frac{1}{2}$

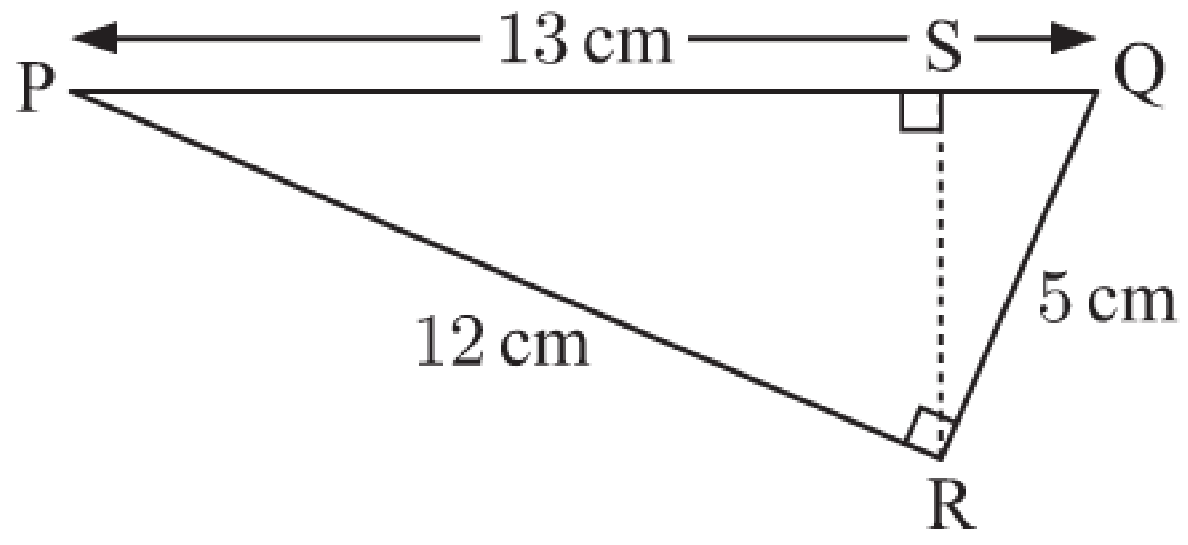
$\therefore$ area of $\triangle$ABP $= \frac{1}{2}(14x) = 7x$

Thus $\triangle$ABC has area $7x + 14x = 21x$

$$\therefore \frac{\text{area of }\triangle\text{PQR}}{\text{area of }\triangle\text{ABC}} = \frac{5x}{21x} = \frac{5}{21}$$

EXERCISE 2C

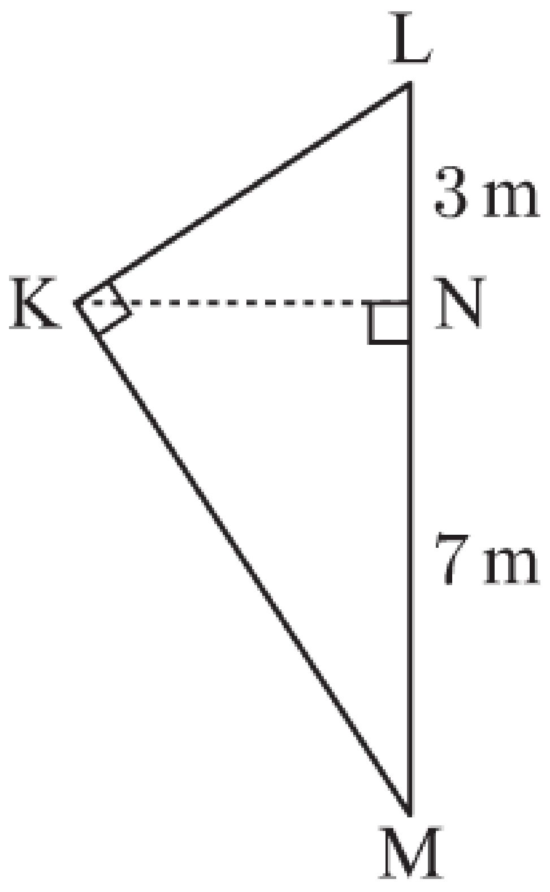
1


P
13 cm
S
Q
12 cm
5 cm
R

Find the length of:

a [QS] **b** [RS]

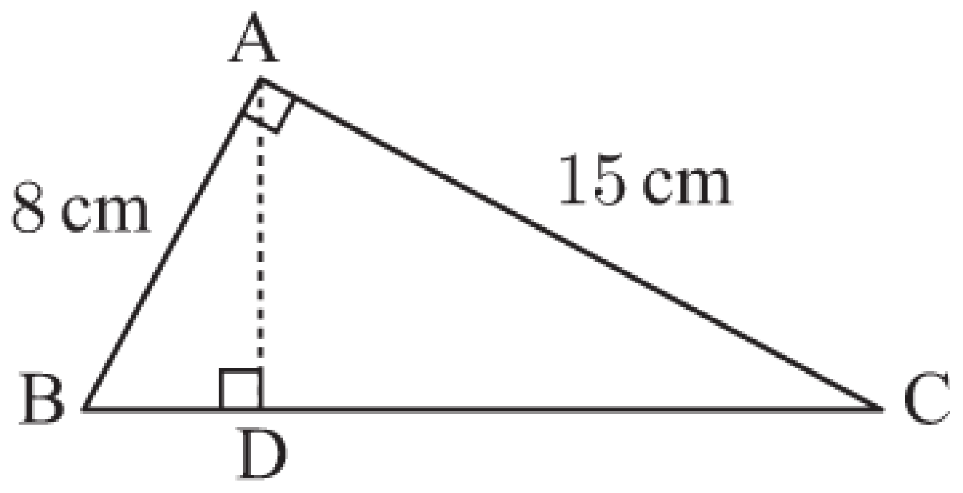
2



Find the length of:

a [KN] **b** [KL] **c** [KM]

3


A
8 cm
15 cm
B
D
C

Find the length of:

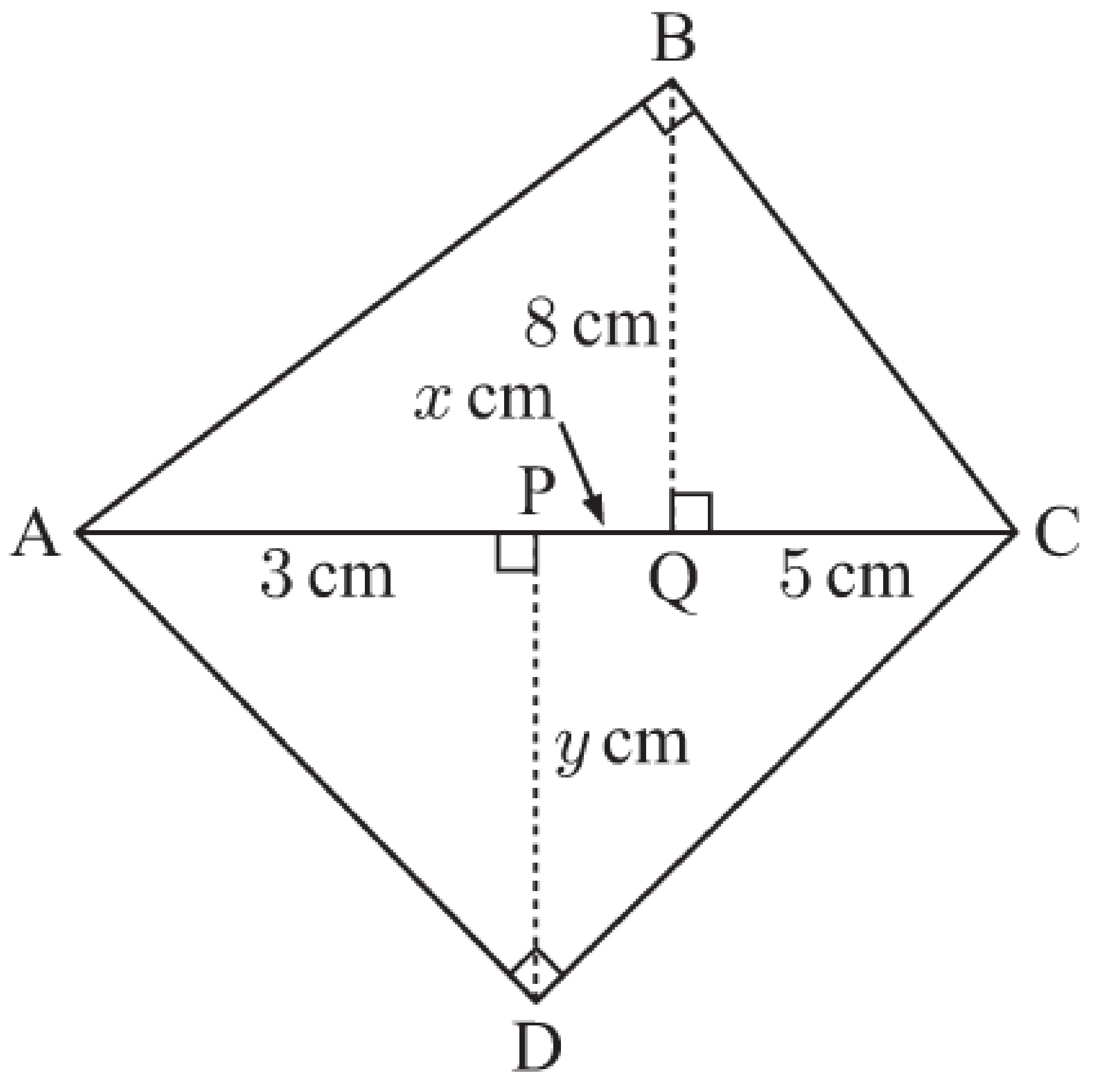
a [BC] **b** [DC] **c** [AD]

4 In the diagram alongside, the quadrilateral ABCD has not been drawn to scale.

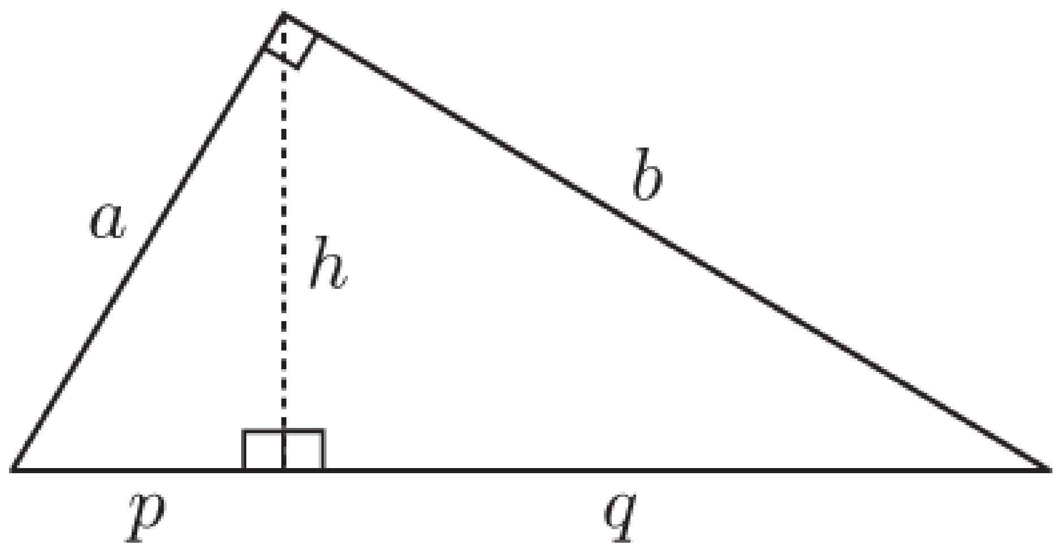
a Find the value of x.

b Find the value of y.

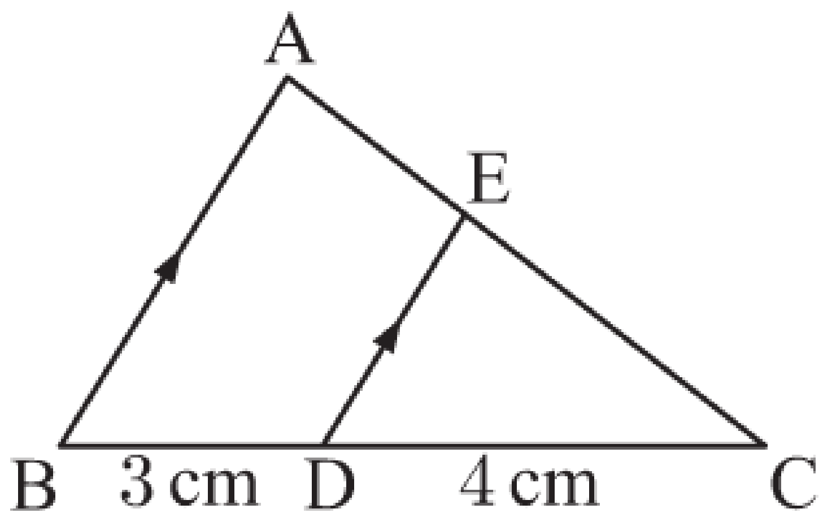
c Find the perimeter of ABCD using Euclid's theorem only.


B
8 cm
x cm
P
A
3 cm
Q
5 cm
C
y cm
D

5 Our proof of Euclid's theorem for proportional segments in a right angled triangle used similar triangles only. Use Euclid's theorem to prove Pythagoras' theorem $a^2 + b^2 = (p+q)^2$.



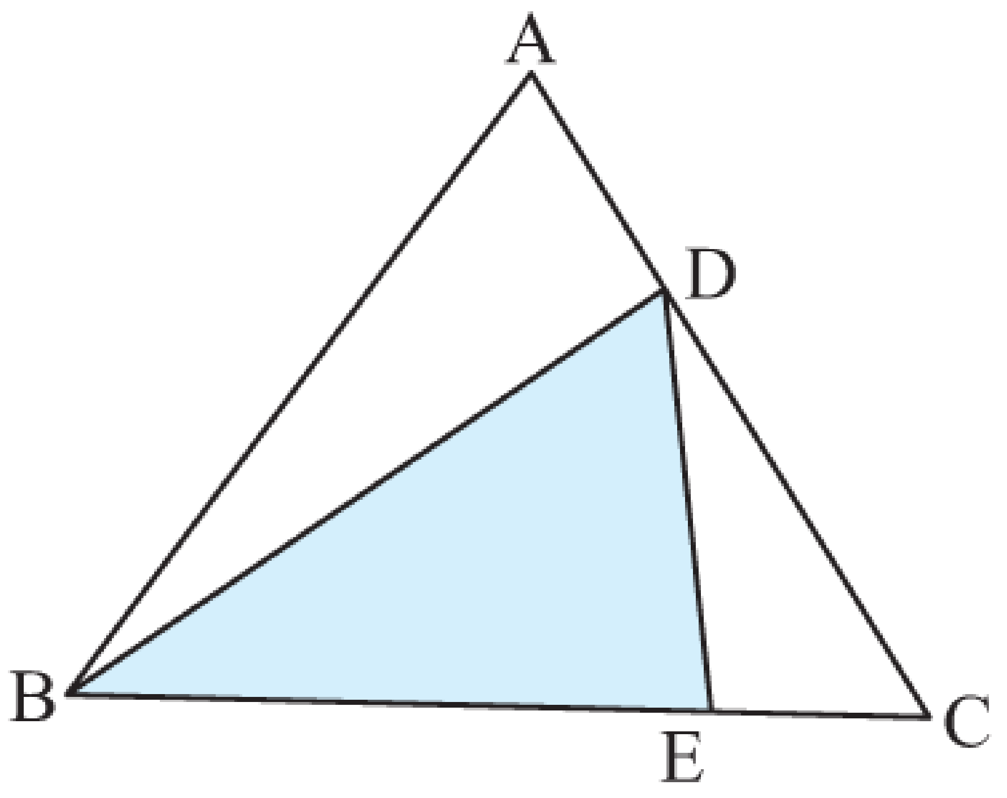
6



a Find the ratio of:
  i area $\triangle$DEC : area $\triangle$ABC
  ii area $\triangle$DEC : area ABDE.

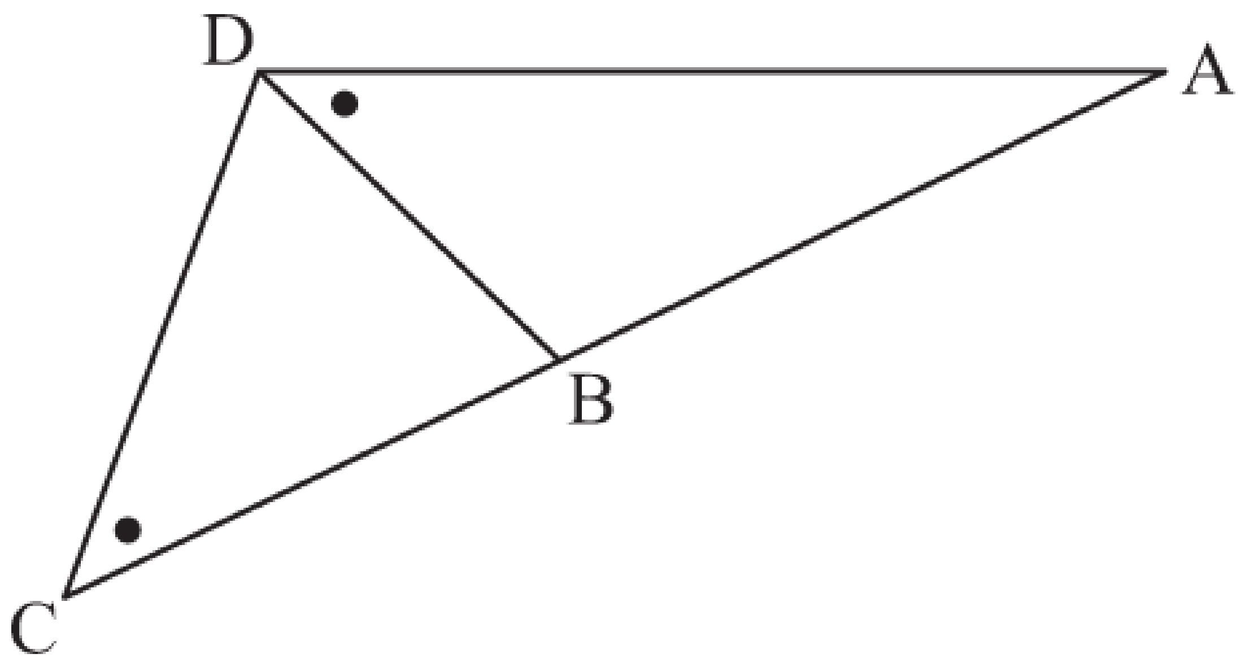
b If the area of ABDE $= 6$ cm^2, find the area of $\triangle$ABC.

7



D divides [AC] in the ratio $1:2$.
E divides [BC] in the ratio $3:1$.
By area, what fraction of $\triangle$ABC is $\triangle$BDE?

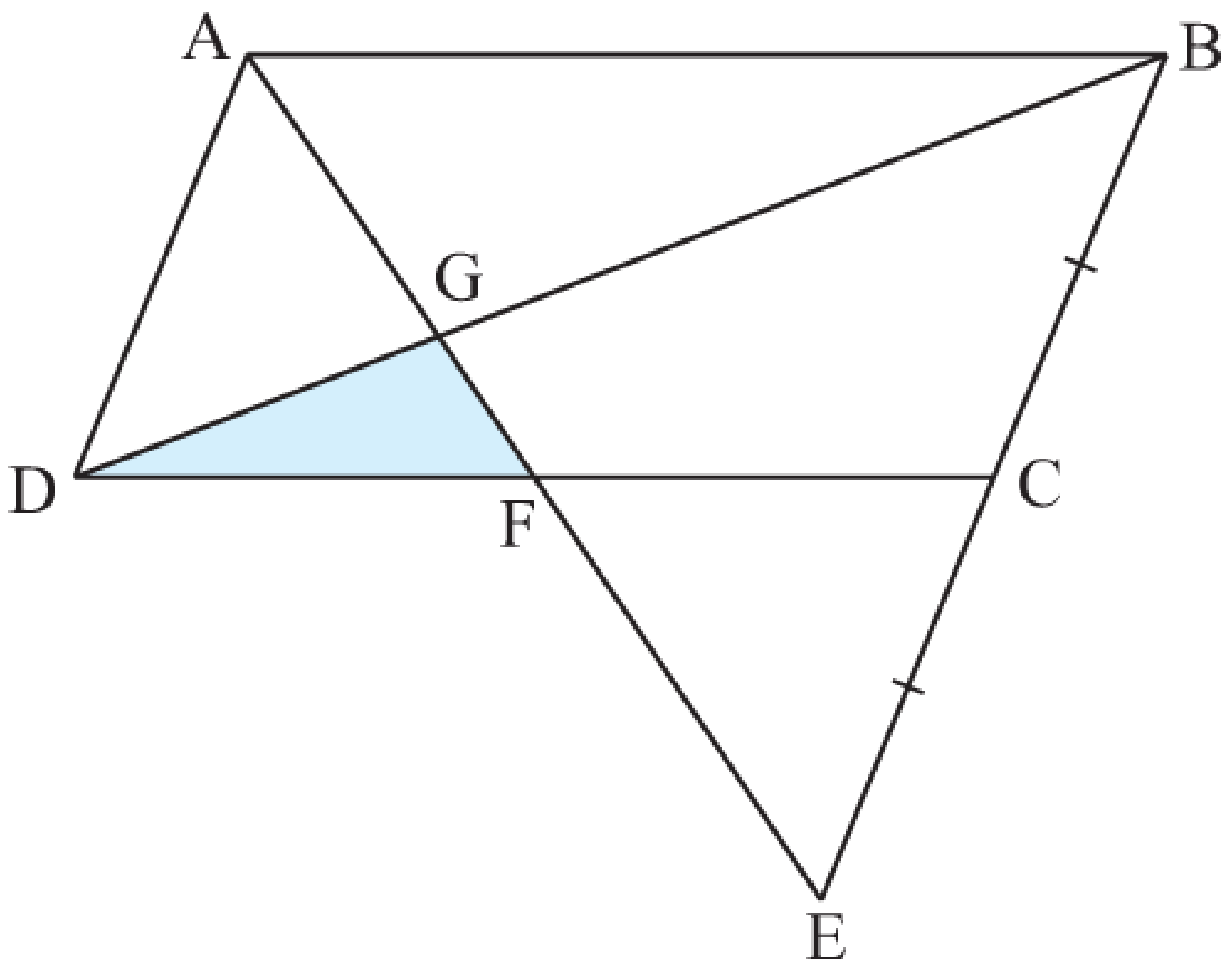
8



In the given figure, $\widehat{ADB} = \widehat{ACD}$.

Prove that $\dfrac{AD^2}{AC^2} = \dfrac{AB}{AC}$.

9



ABCD is a parallelogram.
[BC] is produced to E such that $BC = CE$.
F is the point of intersection of [AE] and [DC].
G is the point of intersection of [BD] and [AE].
What fraction of the parallelogram is occupied by $\triangle$DGF?

D CIRCLE GEOMETRY

CIRCLE TERMINOLOGY

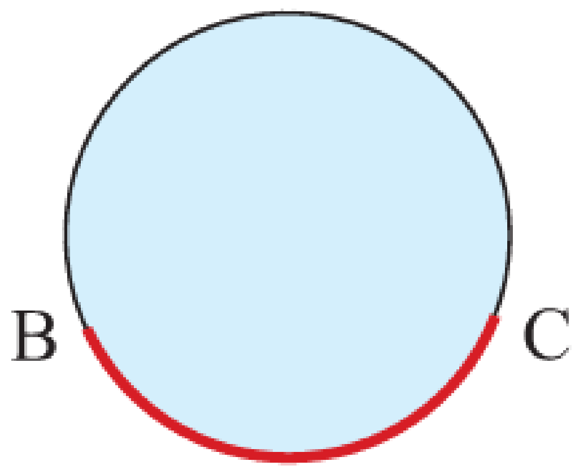
If an arc is less than half the circle, it is called a **minor arc**. If an arc is greater than half the circle, it is called a **major arc**.

A chord divides the interior of a circle into two regions called **segments**. The larger region is called a **major segment** and the smaller region is called a **minor segment**.

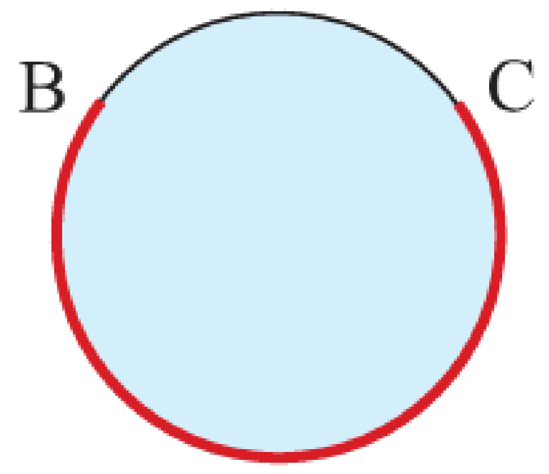
Consider minor arc BC.

We can say that the arc BC **subtends** the angle BAC at A which lies on the circle.

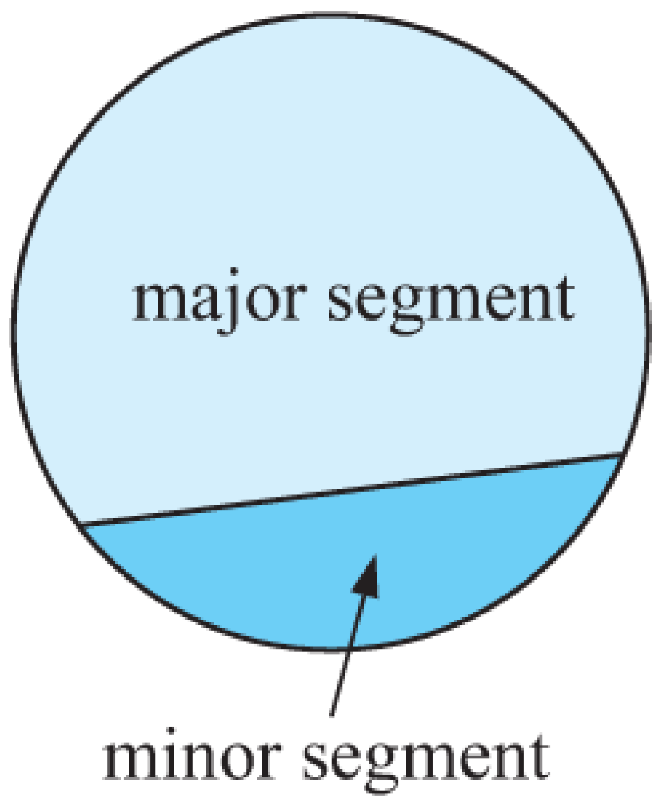
We also say that the arc BC subtends an angle at the centre of the circle, which is angle BOC.

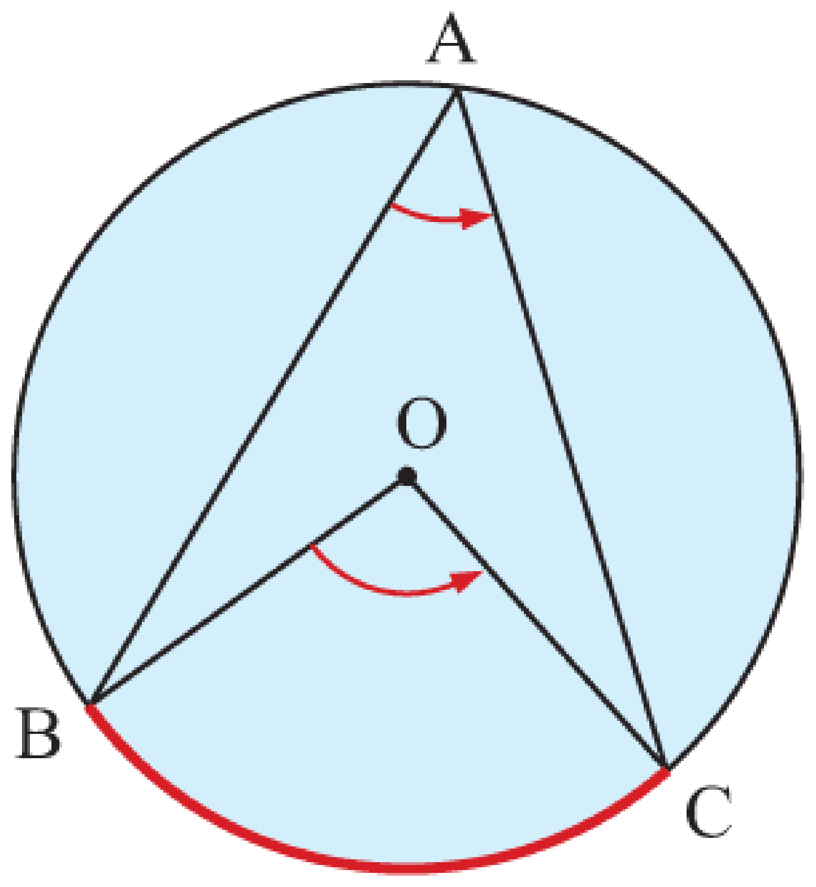


minor arc BC

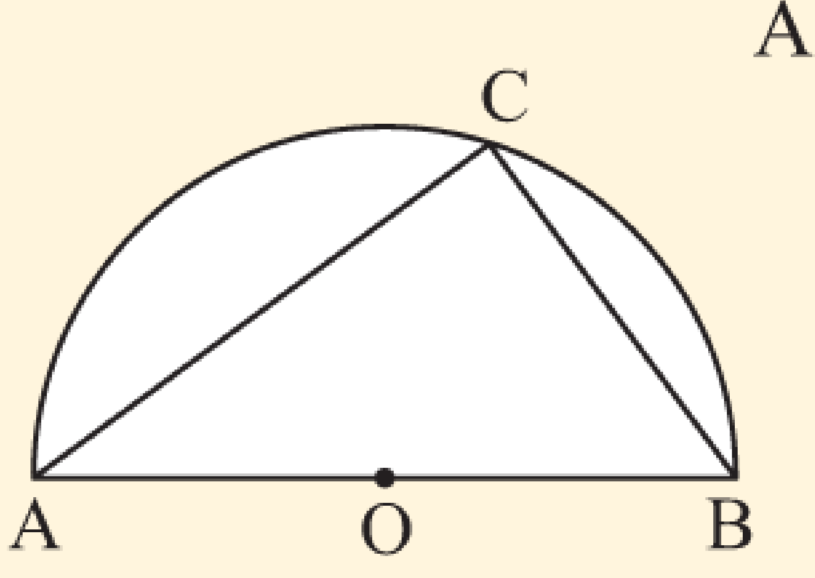


major arc BC



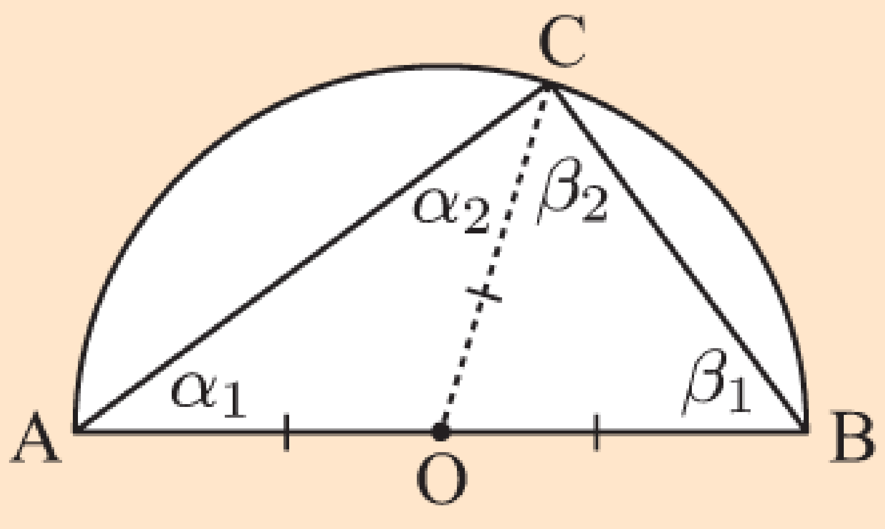


CIRCLE THEOREMS

Name of theorem	*Statement*	*Diagram*
Angle in a semi-circle	The angle in a semi-circle is a right angle.	$A\widehat{C}B = 90°$ 

GEOMETRY PACKAGE

Proof:



Since OA = OB = OC, triangles OAC and OBC are isosceles.

$\therefore\ \alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {isosceles triangle}

Now in triangle ABC,

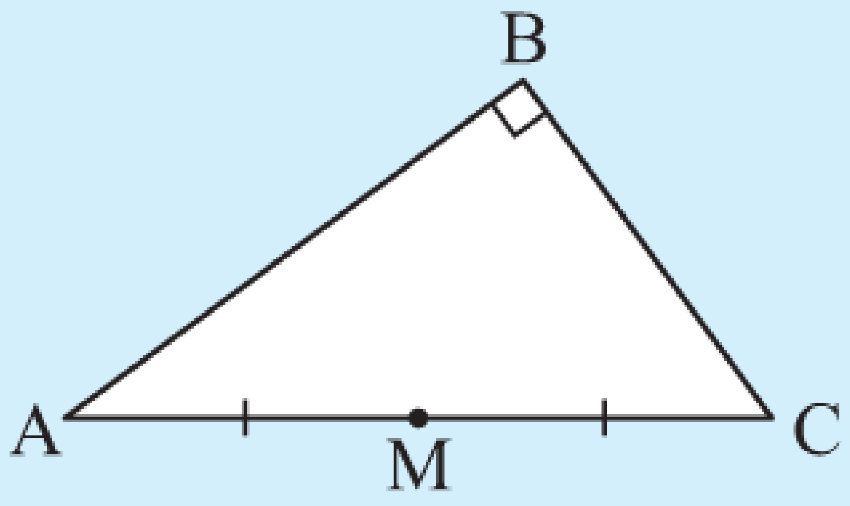
$\alpha_1 + \beta_1 + (\alpha_2 + \beta_2) = 180°$ {angles of a triangle}

$\therefore\ 2\alpha + 2\beta = 180°$

$\therefore\ \alpha + \beta = 90°$

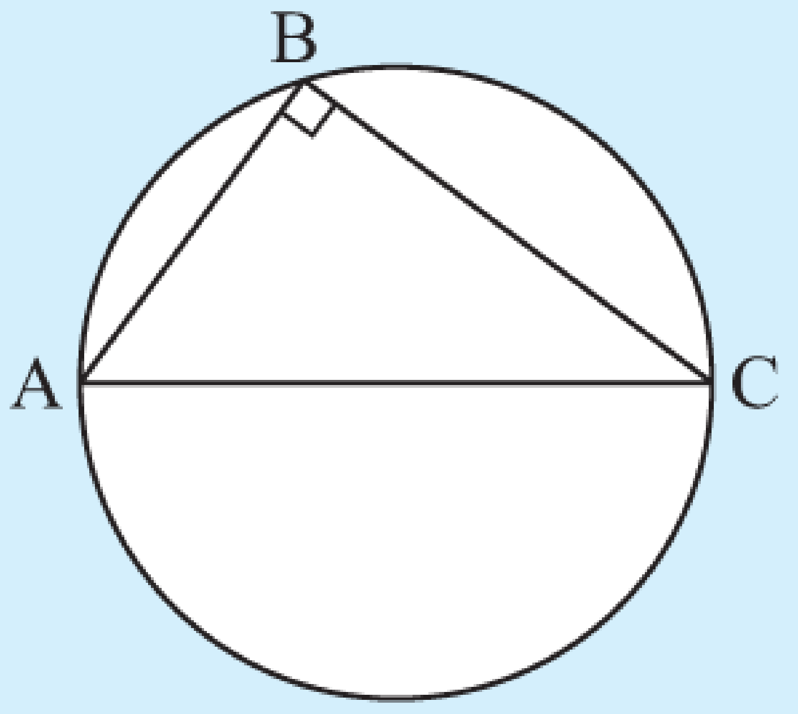
$\therefore\ A\widehat{C}B$ is a right angle.

Converse 1:



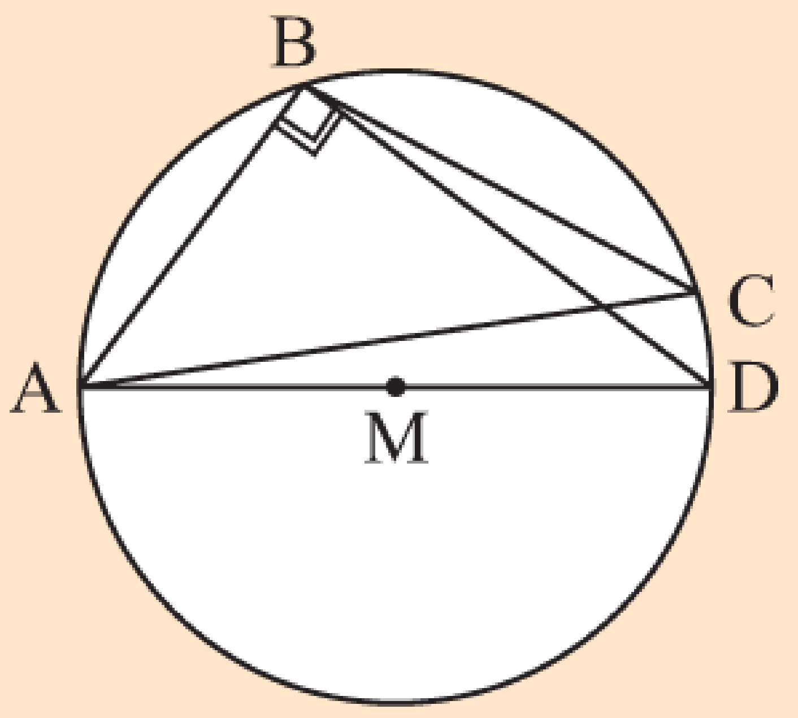
If M is the midpoint of the hypotenuse of a right angled triangle, then a circle can be drawn through A, B, and C with M its centre on diameter [AC].

Converse 2:



If A, B, and C lie on a circle and $A\widehat{B}C$ is a right angle, then [AC] is a diameter of the circle.

Proof of converse 2:



Let [AD] be a diameter of the circle with centre M. Join [BD].

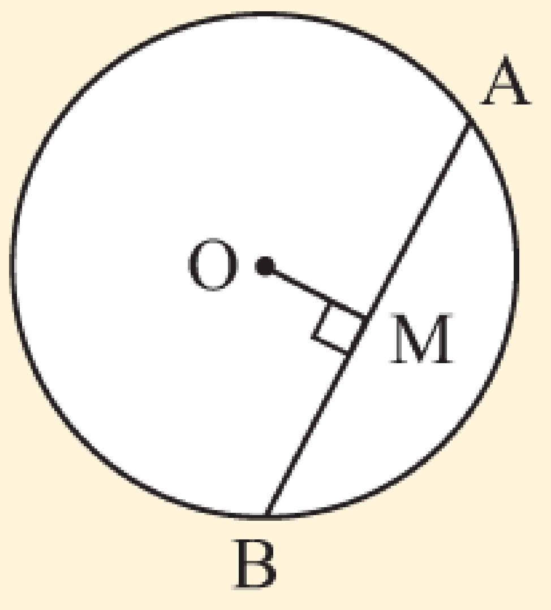
$\therefore$ $A\widehat{B}D$ is a right angle {angle in a semi-circle}

Now C also lies on the circle such that $A\widehat{B}C$ is a right angle.

$\therefore$ $C\widehat{B}D = 90^\circ - 90^\circ = 0$

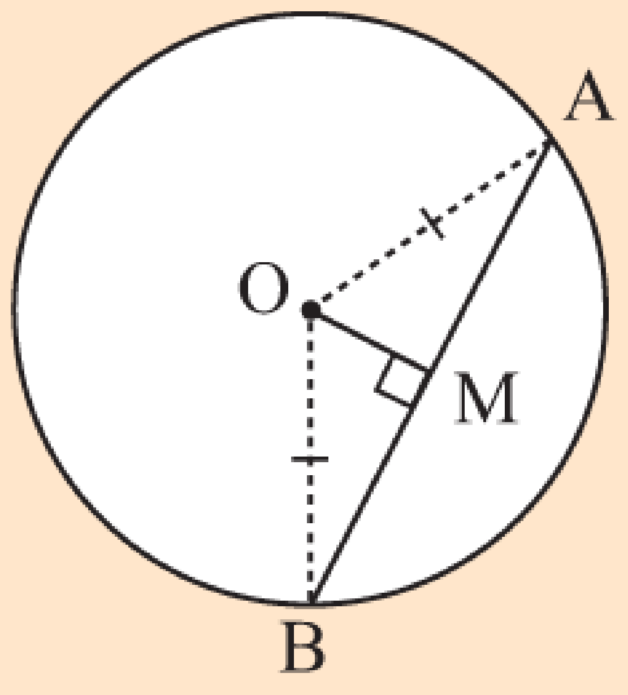
$\therefore$ C and D are coincident

$\therefore$ [AC] is a diameter of the circle.

Name of theorem	*Statement*	*Diagram*
Chord of a circle	The perpendicular from the centre of a circle to a chord, bisects the chord.	AM = BM 

GEOMETRY PACKAGE

Proof:



OA = OB {equal radii}

$\therefore$ triangle OAB is isosceles

$\therefore$ AM = MB {isosceles triangle}

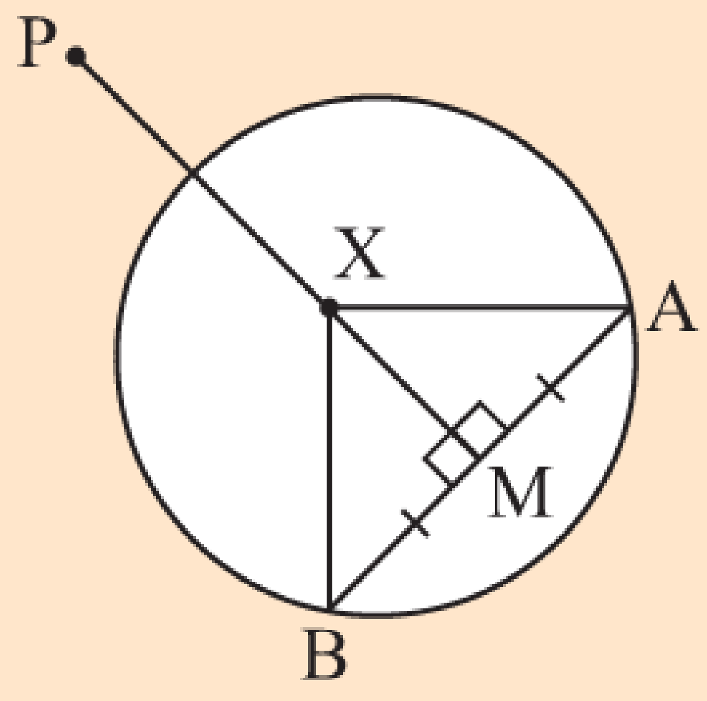
Converse 1:

The line from the centre of a circle to the midpoint of a chord, is perpendicular to the chord.

Converse 2:

The perpendicular bisector of a chord of a circle, passes through the circle's centre.

Proof of converse 2:



Let X be any point on the perpendicular bisector of [AB].

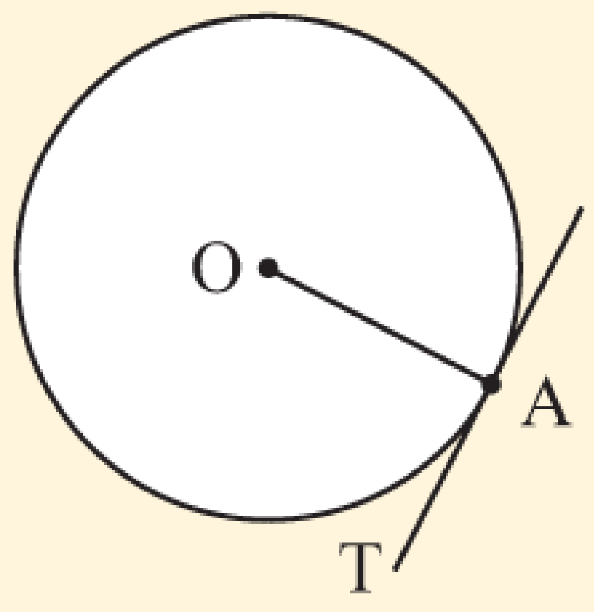
$\triangle$s XAM and XBM are congruent {SAS}

$\therefore$ XA = XB

Now choose X so that XA = XB = r, where r is the radius of the circle.

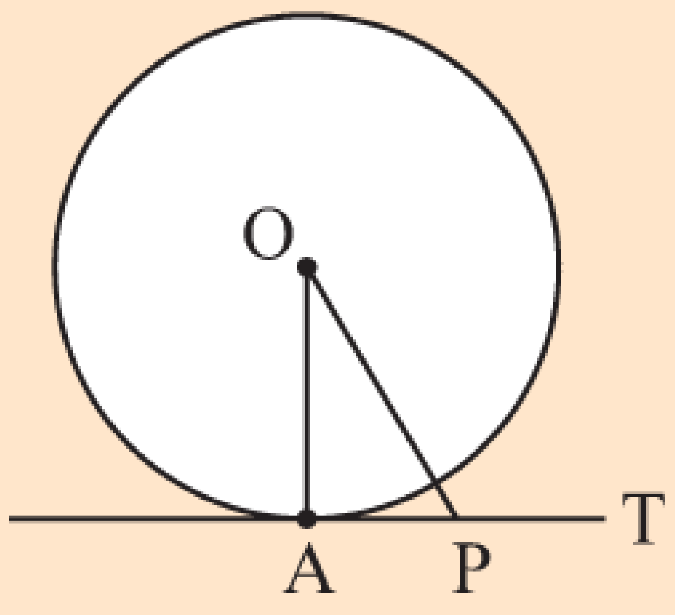
$\therefore$ X is necessarily the circle's centre.
{distance r from both A and B, and lies within the circle}

$\therefore$ the perpendicular bisector of the chord passes through the circle's centre.

Name of theorem	*Statement*	*Diagram*
Radius-tangent	The tangent to a circle is perpendicular to the radius at the point of contact.	$O\widehat{A}T = 90°$ 

GEOMETRY PACKAGE

Proof:



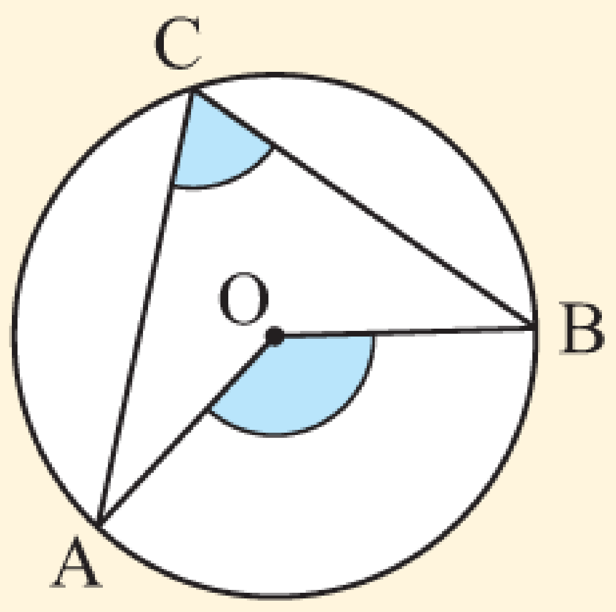
Consider a circle with centre O, and a tangent to the circle with point of contact A.

Suppose P is any point on the tangent and P is not at A.

$\therefore$ P lies outside the circle.

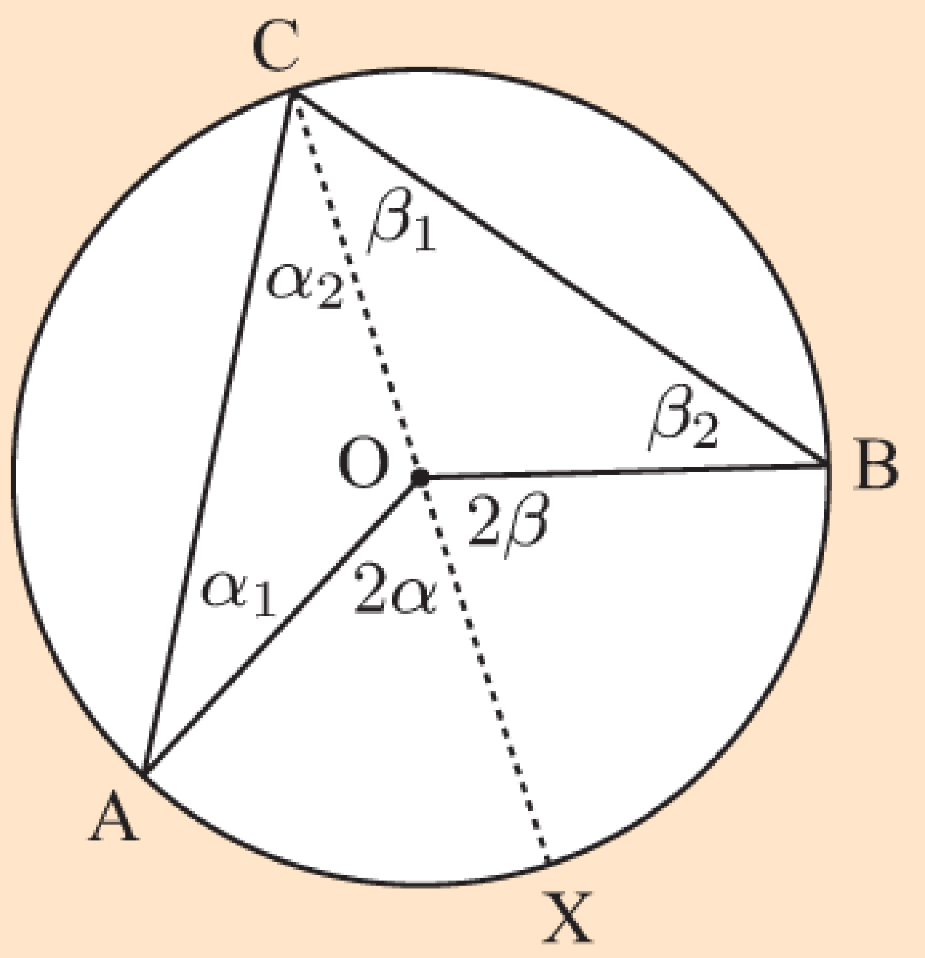
$\therefore$ OA is the shortest distance from O to the tangent.

$\therefore$ [OA] is perpendicular to the tangent.

Name of theorem	*Statement*	*Diagram*
Angle at the centre	The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.	$A\widehat{O}B = 2A\widehat{C}B$ 

GEOMETRY PACKAGE

Proof:


C
β_1
α_2
β_2
B
O
2β
α_1
2α
A
X

$OA = OC = OB$ {equal radii}

$\therefore$ triangles AOC and OBC are isosceles

$\therefore$ $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {isosceles triangle}

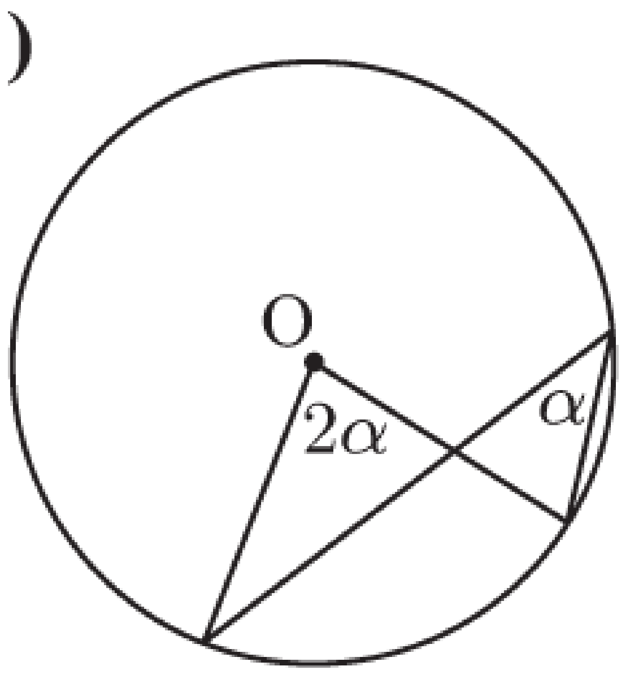
But $A\widehat{O}X = 2\alpha$ and $B\widehat{O}X = 2\beta$ {exterior angle of a triangle}

$$\therefore\ A\widehat{O}B = 2\alpha + 2\beta$$
$$= 2 \times A\widehat{C}B$$

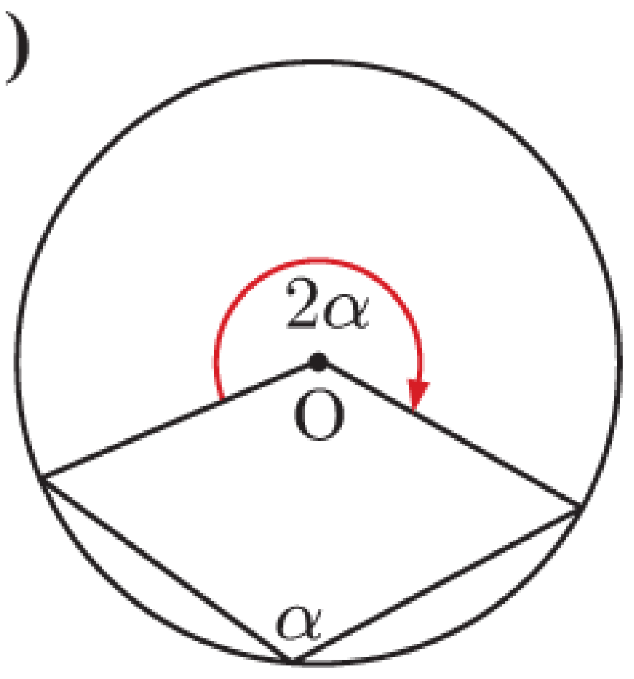
The following diagrams show other cases of **the angle at the centre theorem**. These cases can be easily shown using the **geometry package**.

GEOMETRY PACKAGE

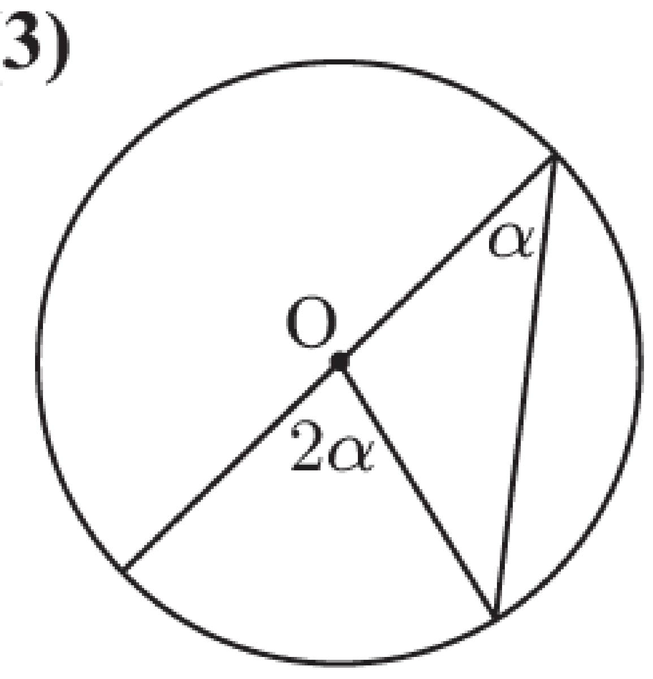
(1)


O
2α
α

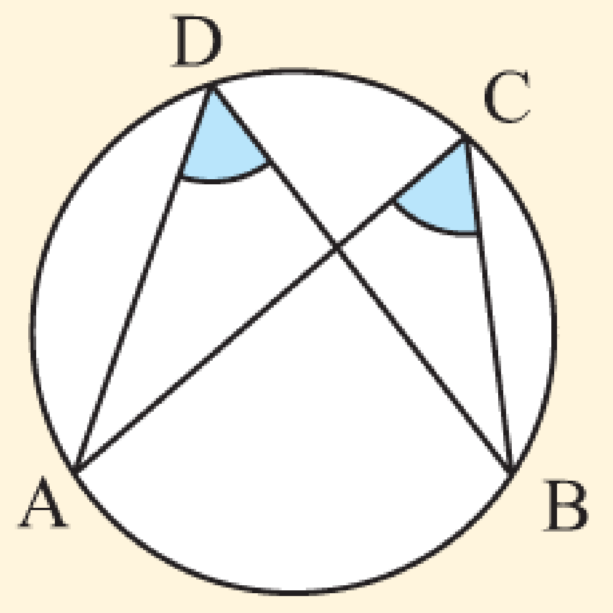
(2)


2α
O
α

(3)

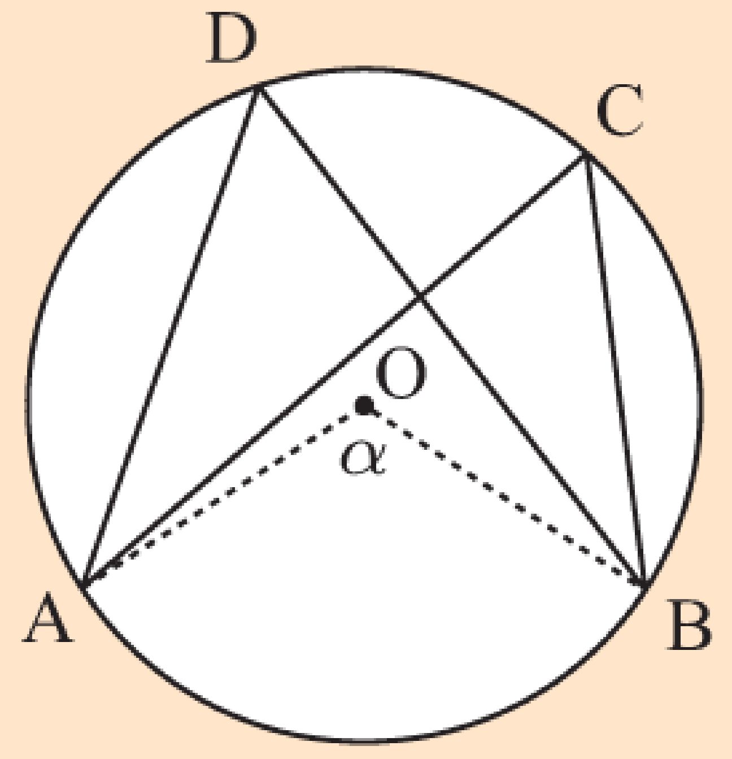

α
O
2α

In case **(2)**, letting $2\alpha = 180°$ we have another proof of the angle in a semi-circle theorem. So, the angle in a semi-circle could be considered as a corollary of the angle at the centre theorem.

Corollary	*Statement*	*Diagram*
Angles subtended by the same arc	Angles subtended by an arc on the circle are equal in size.	$A\widehat{D}B = A\widehat{C}B$ 

GEOMETRY PACKAGE

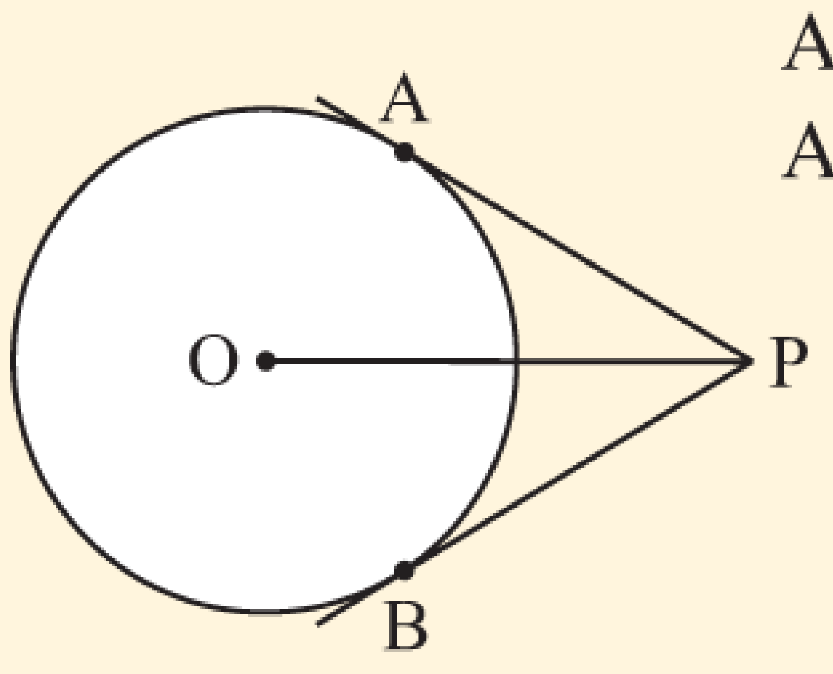
Proof:


D
C
O
α
A
B

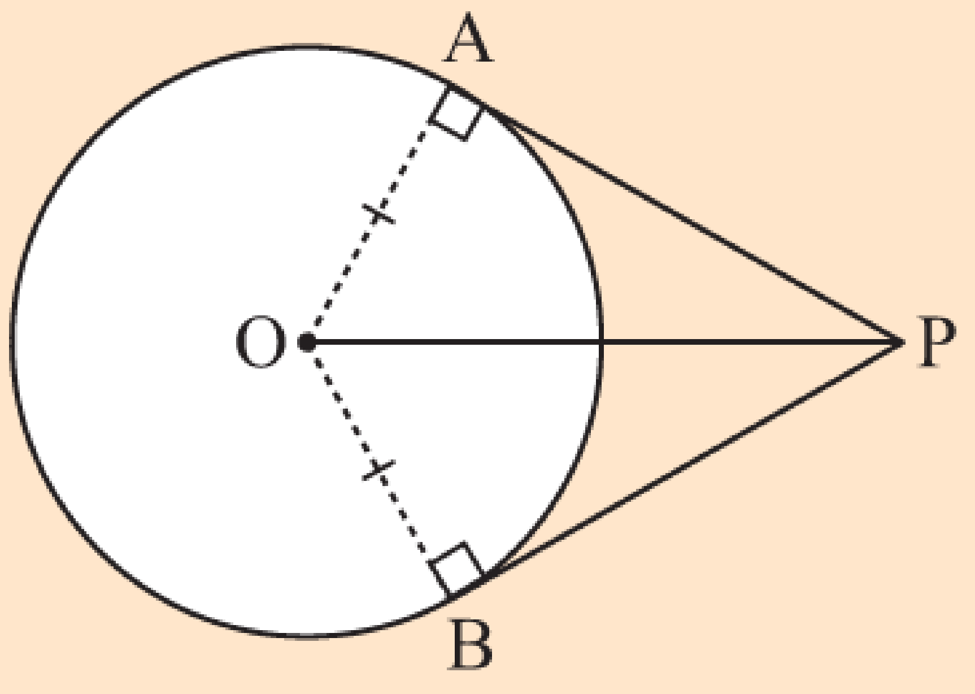
$A\widehat{D}B = \frac{1}{2}\alpha$ {angle at the centre}

and $A\widehat{C}B = \frac{1}{2}\alpha$ {angle at the centre}

$\therefore$ $A\widehat{D}B = A\widehat{C}B$

Name of theorem	*Statement*	*Diagram*
Tangents from an external point	Tangents from an external point are equal in length, and the line joining the point to the centre bisects the angle at the point.	 $AP = BP$ $A\widehat{P}O = B\widehat{P}O$

Proof:

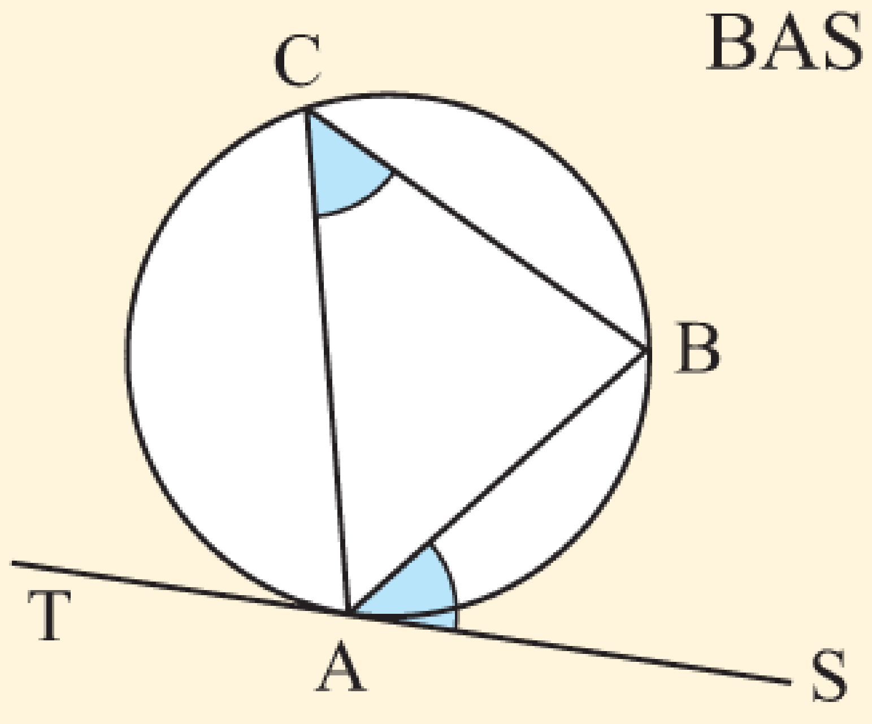


We observe that:

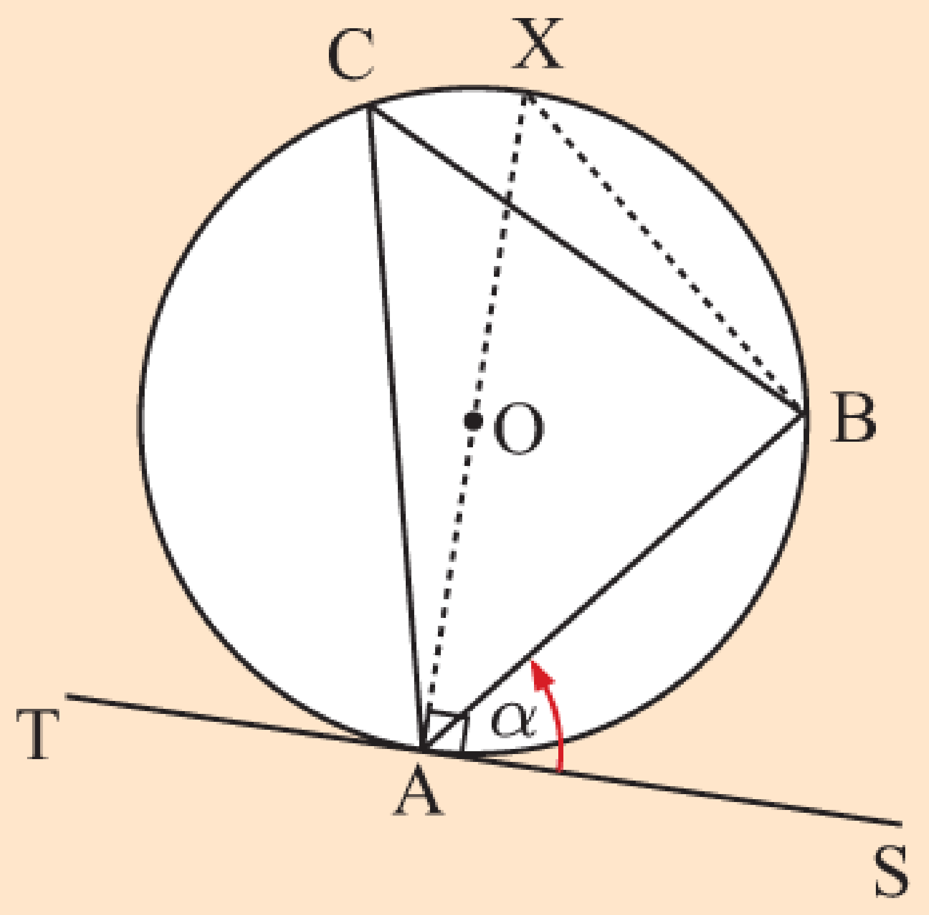
- $O\widehat{A}P = O\widehat{B}P = 90^\circ$ {radius-tangent}
- $OA = OB$ {equal radii}
- OP is common to both

$\therefore$ $\triangle$s OAP and OBP are congruent (RHS).

Consequently, $AP = BP$ and $A\widehat{P}O = B\widehat{P}O$.

Name of theorem	*Statement*	*Diagram*
Angle between tangent and chord	The angle between a tangent and a chord at the point of contact, is equal to the angle subtended by the chord in the alternate segment.	 $B\widehat{A}S = B\widehat{C}A$

Proof:



We draw AOX and BX.

$X\widehat{A}S = 90^\circ$ {radius-tangent}

$A\widehat{B}X = 90^\circ$ {angle in a semi-circle}

Let $B\widehat{A}S = \alpha$

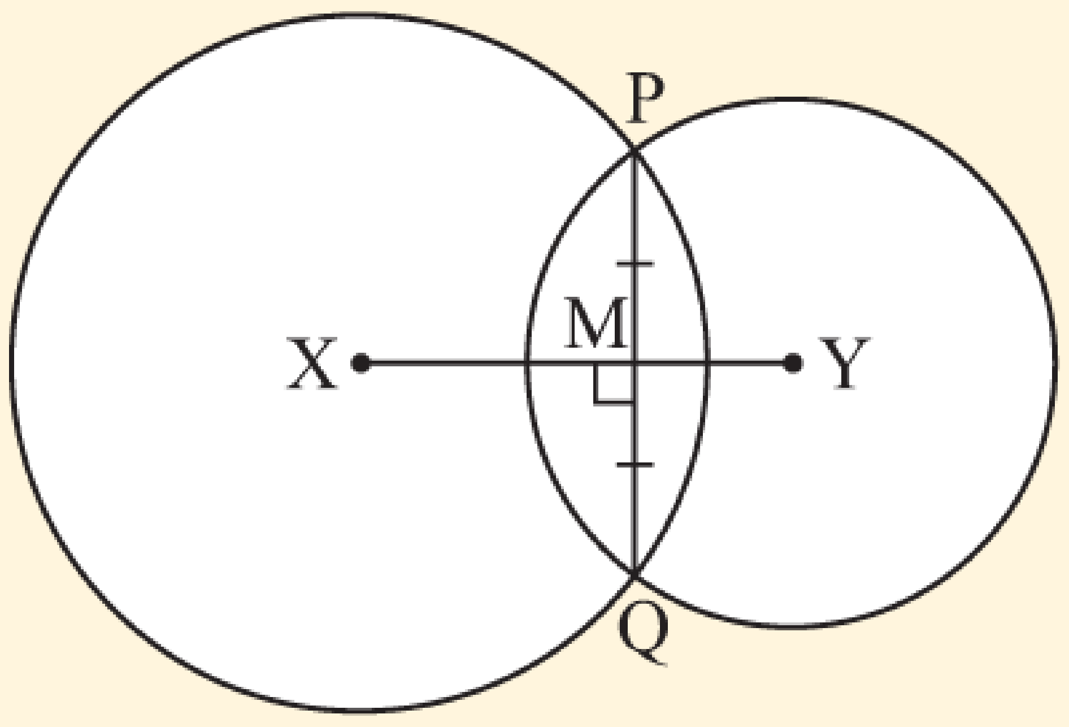
$\therefore$ $B\widehat{A}X = 90^\circ - \alpha$

So, in $\triangle ABX$,

$B\widehat{X}A = 180^\circ - 90^\circ - (90^\circ - \alpha) = \alpha$

But $B\widehat{X}A = B\widehat{C}A$ {angles subtended by the same arc}

$\therefore$ $B\widehat{C}A = B\widehat{A}S = \alpha$

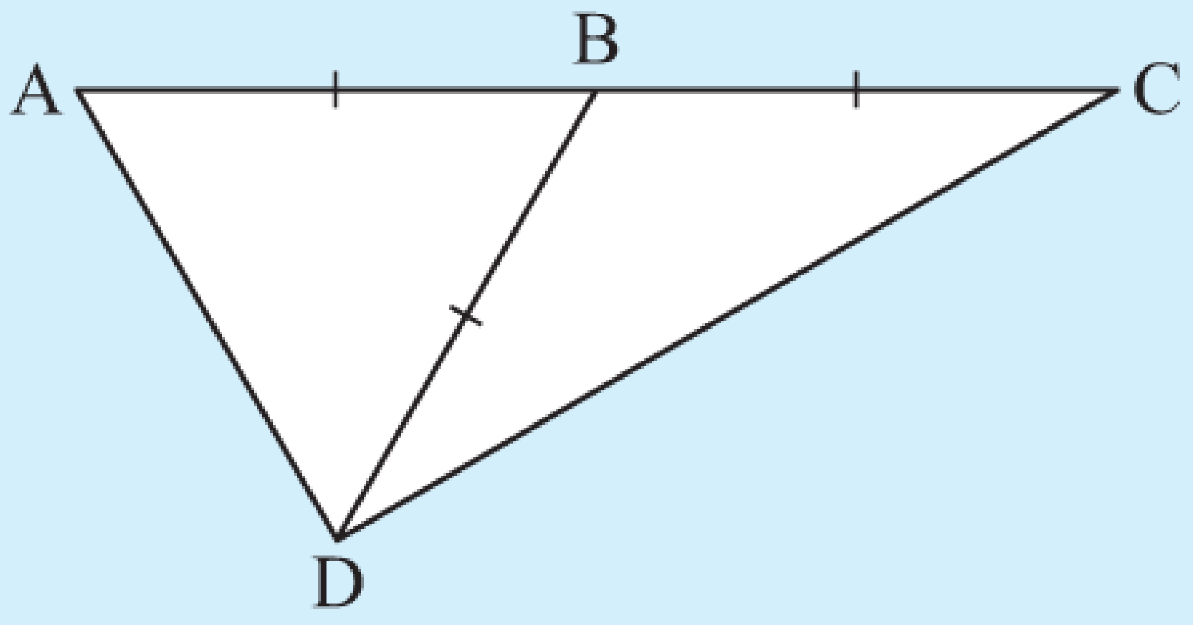
Name of theorem	*Statement*	*Diagram*
Intersecting circles	The line joining the centres of two intersecting circles bisects the common chord at right angles.	 [XY] $\perp$ [PQ] MP = MQ

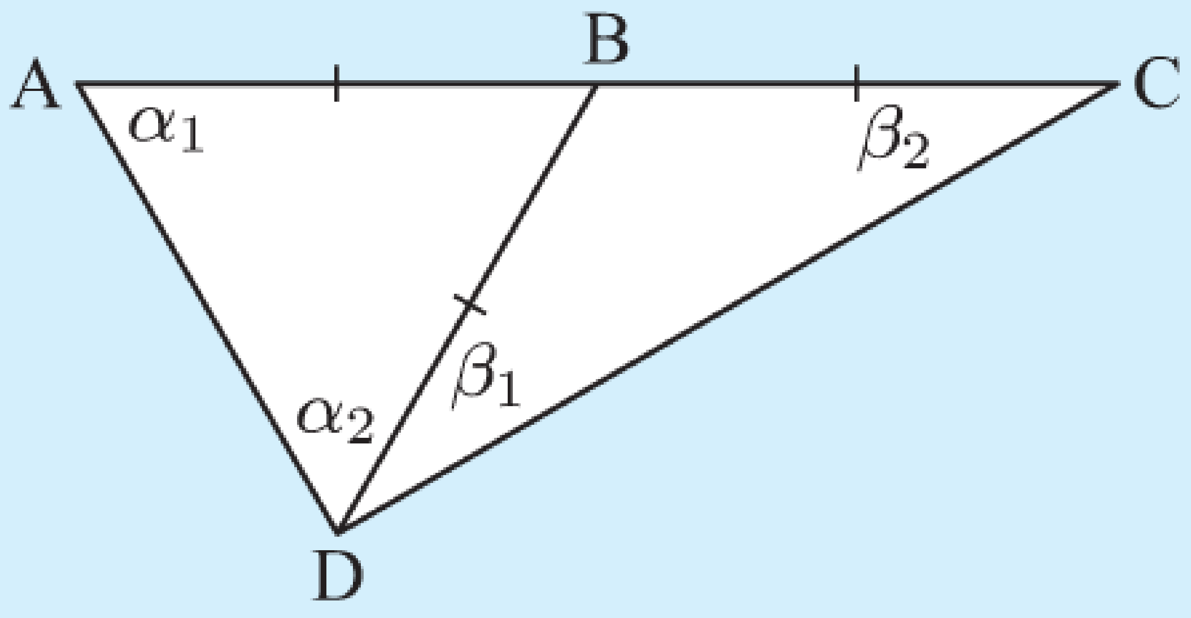
GEOMETRY PACKAGE

USING CIRCLE THEOREMS

Example 6

Show that $A\widehat{D}C$ is a right angle:





Since AB = BD, $\triangle$ABD is isosceles.

$\therefore$ $\alpha_1 = \alpha_2$ {isosceles triangle}

Likewise, $\beta_1 = \beta_2$ in isosceles triangle BCD.

Thus in triangle ADC,

$$\alpha + (\alpha + \beta) + \beta = 180^\circ$$

{angles of a triangle}

$$\therefore \quad 2\alpha + 2\beta = 180^\circ$$

$$\therefore \quad \alpha + \beta = 90^\circ$$

$\therefore$ $A\widehat{D}C$ is a right angle.

Alternatively:

Since BA = BC = BD, a circle with centre B can be drawn through A, D, and C.

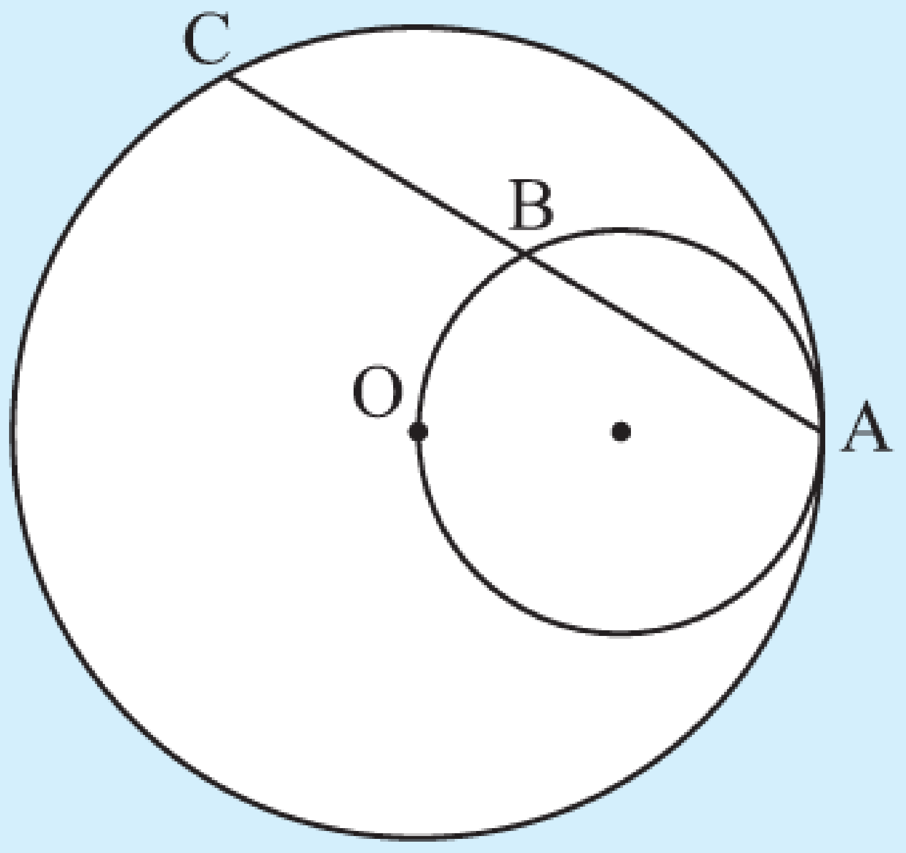
[AC] is a diameter.

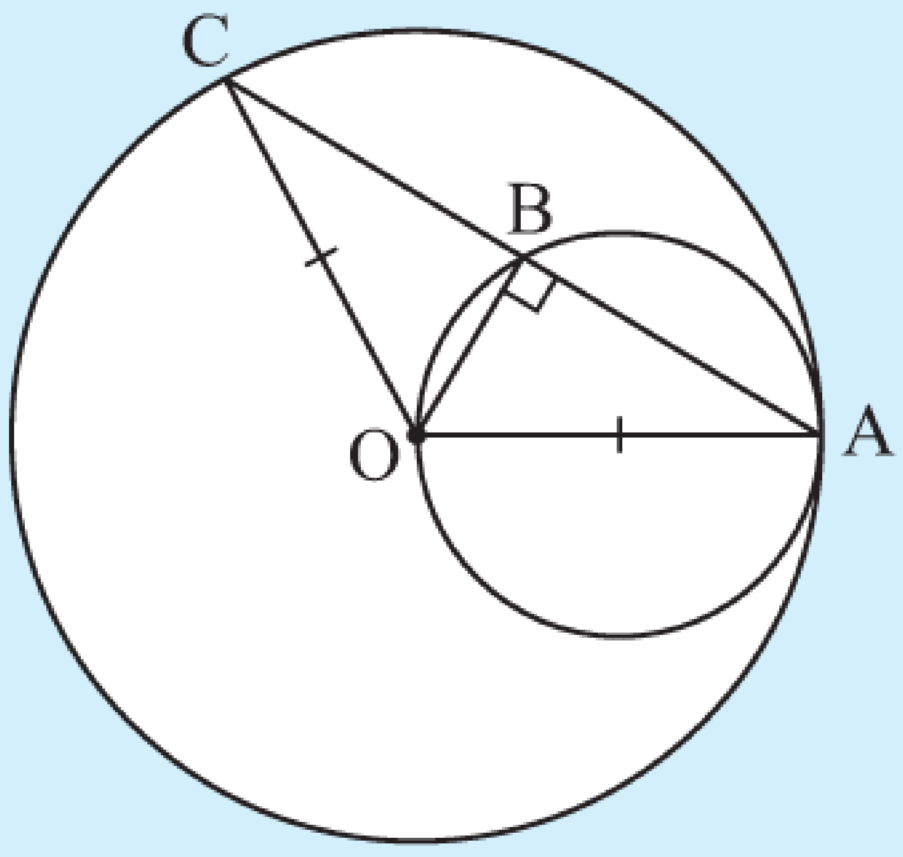
$\therefore$ $A\widehat{D}C$ is a right angle. {angle in a semi-circle}

Example 7

Given a circle with centre O, and a point A on the circle, a smaller circle with diameter [OA] is drawn. [AC] is any line drawn from A to the larger circle, cutting the smaller circle at B.

Prove that the smaller circle will always bisect [AC].





Join [OA], [OC], and [OB].

Now $O\widehat{B}A$ is a right angle. {angle in a semi-circle}

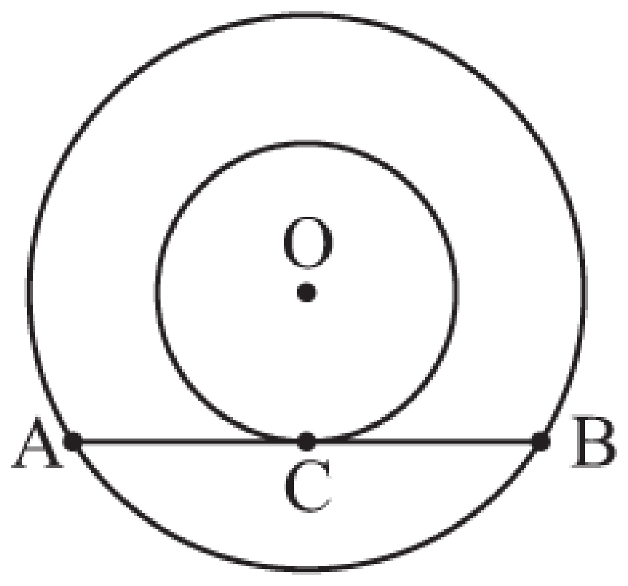
Thus [OB] is the perpendicular from the centre of the circle to the chord [AC].

$\therefore$ [OB] bisects [AC]. {chord of a circle theorem}

Thus B always bisects [AC].

EXERCISE 2D

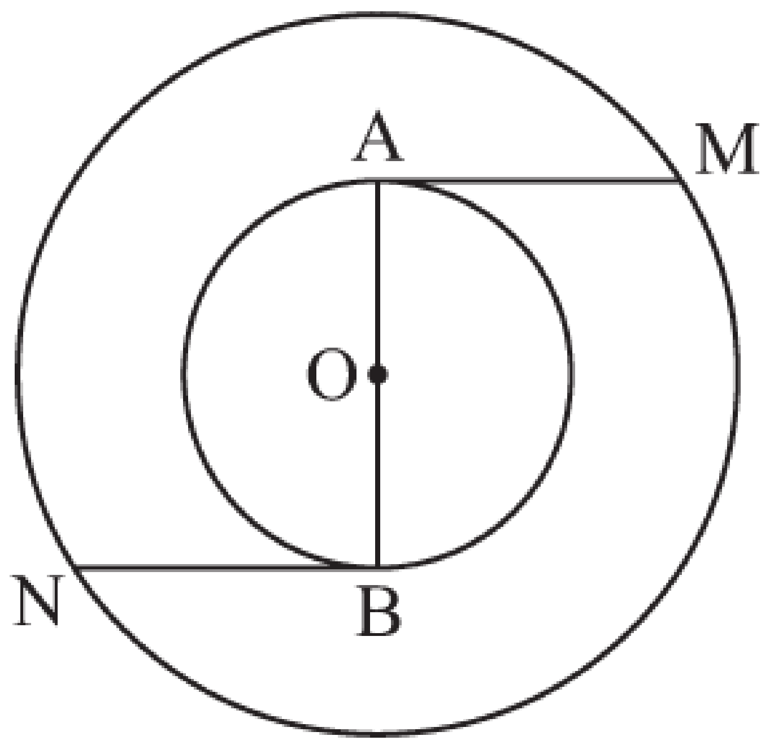
1 O is the centre of two concentric circles. [AB] is a tangent to the smaller circle at C. A and B are both on the larger circle. Prove that AC = BC.



2 Triangle PQR is inscribed in a circle. The angle bisector of $Q\widehat{P}R$ meets [QR] at S, and the circle at T. Prove that PQ.PR = PS.PT.

If PQR is *inscribed* in a circle, a circle is drawn through its three vertices.

3



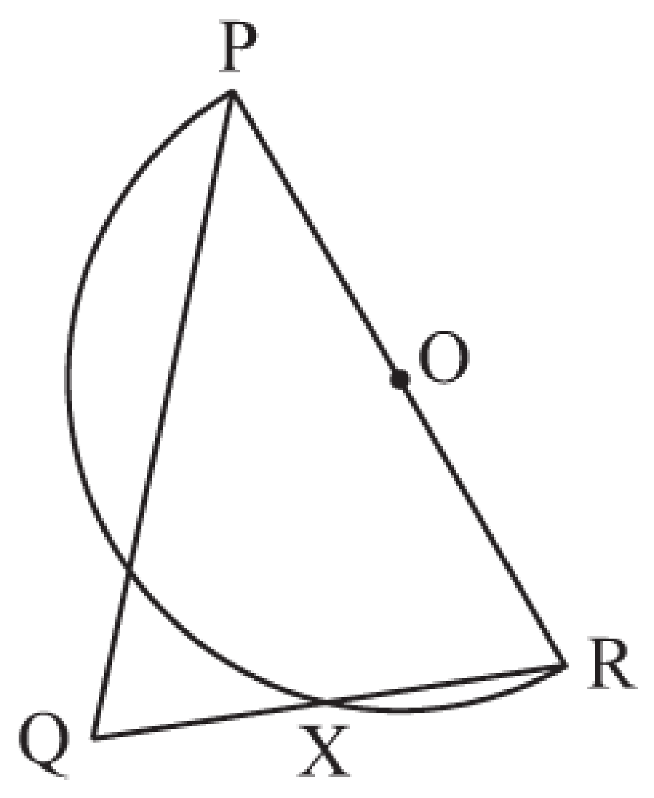
O is the centre of two concentric circles.
[AB] is a diameter of the smaller circle.
Tangents at A and B are drawn to cut the larger circle at M and N respectively.
Prove that AM = BN.

4 The tangent at P to a circle meets the chord [QR] produced at the point S. Prove that triangles SPQ and SRP are similar.

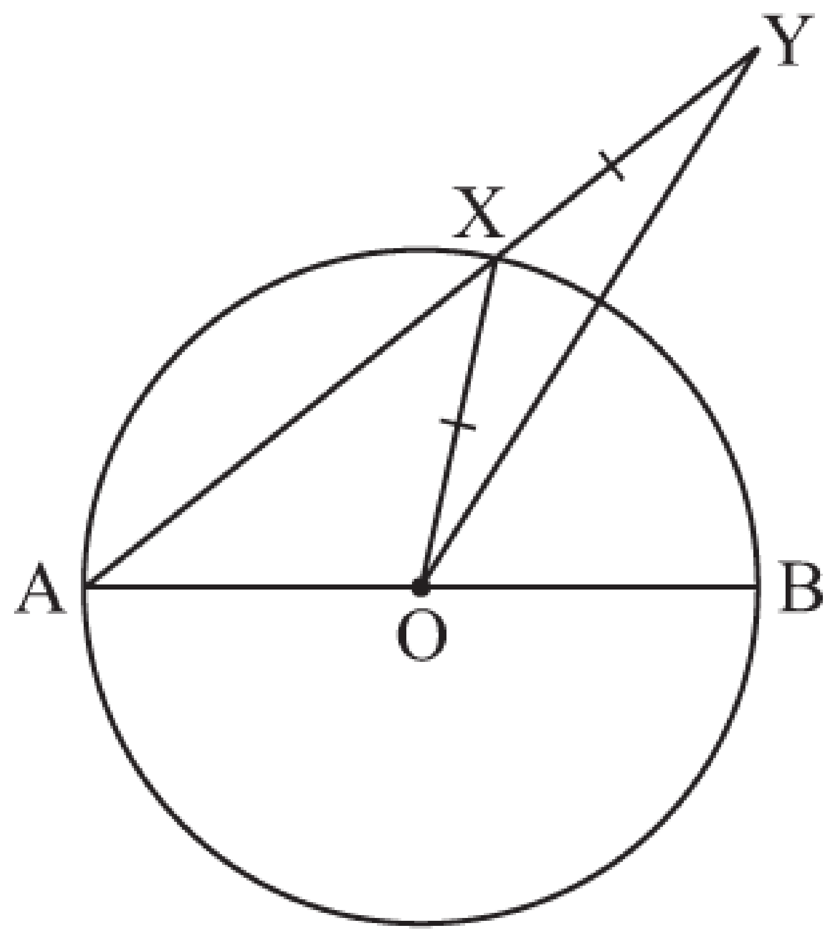
5 P, Q, R, and S are distinct points on a circle, and are in cyclic order. The diagonals of PQRS meet at A. Prove that triangles PQA and SRA are similar.

6 Prove the 'intersecting circles' theorem.

7 Triangle PQR is isosceles with $PQ = PR$. A semi-circle with diameter [PR] is drawn which cuts [QR] at X. Prove that X is the midpoint of [QR].



8 [AB] is a diameter of a circle with centre O. X is a point on the circle, and [AX] is produced to Y such that $OX = XY$.
Prove that $\widehat{YOB}$ is three times the size of $\widehat{XOY}$.



9 Triangle PQR is isosceles with $PQ = QR$. PQR is inscribed in a circle. [XP] is a tangent to the circle. Prove that [QP] bisects angle XPR.

10 [AB] is a diameter of a circle with centre O. [CD] is a chord parallel to [AB]. Prove that [BC] bisects the angle DCO, regardless of where [CD] is located.

11 [PQ] and [RS] are two perpendicular chords of a circle with centre O. Prove that $\widehat{POS}$ and $\widehat{QOR}$ are supplementary.

12 The bisector of $\widehat{YXZ}$ of $\triangle XYZ$ meets [YZ] at W. When a circle is drawn through X, it touches [YZ] at W, and cuts [XY] and [XZ] at P and Q respectively. Prove that $\widehat{YWP} = \widehat{ZWQ}$.

13 A, B, and C are three points on a circle. The bisector of $\widehat{CAB}$ cuts [BC] at P, and the circle at Q. Prove that $\widehat{APC} = \widehat{ABQ}$.

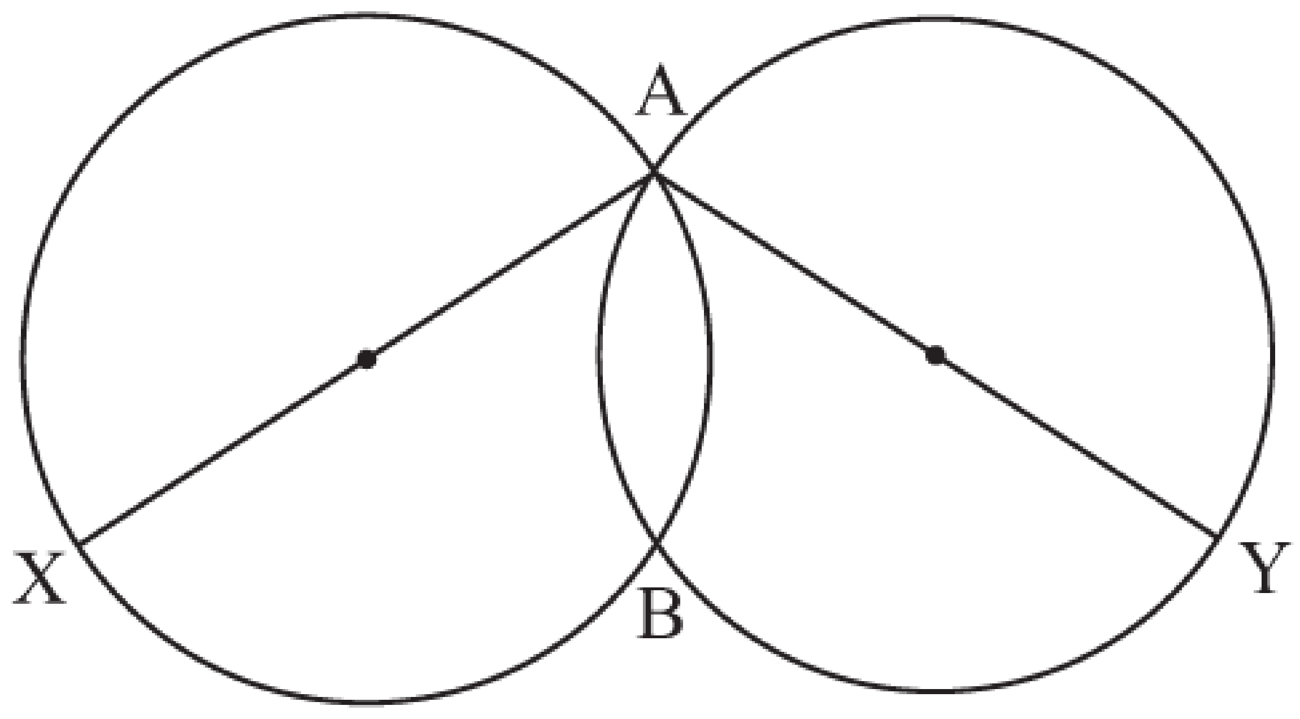
14 [AB] and [DC] are parallel chords of a circle. [AC] and [BD] intersect at E. Prove that:

a triangles ABE and CDE are isosceles

b $AC = BD$.

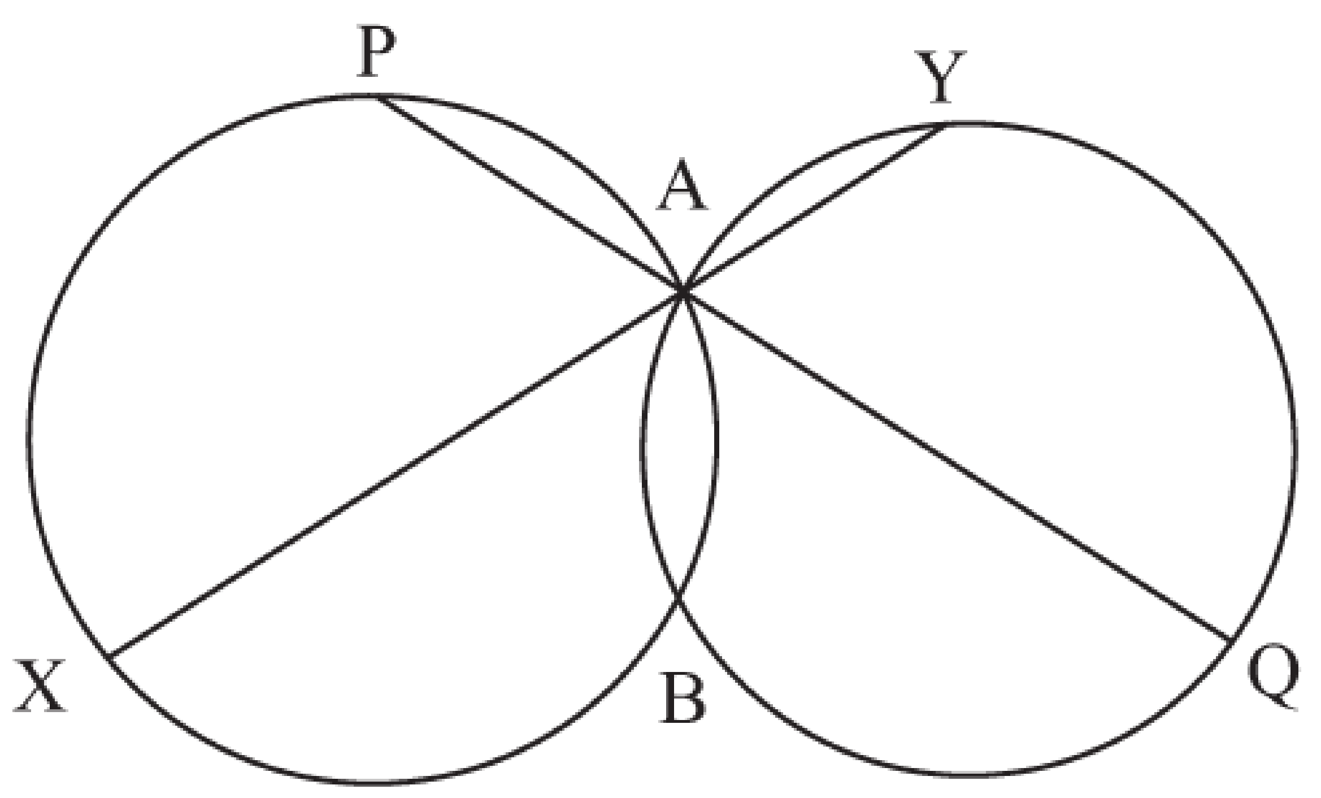
15 P is any point on a circle. [QR] is a chord of the circle parallel to the tangent at P. Prove that triangle PQR is isosceles.

16 Two circles intersect at A and B.
[AX] and [AY] are diameters, as shown.
Prove that X, B, and Y are collinear.



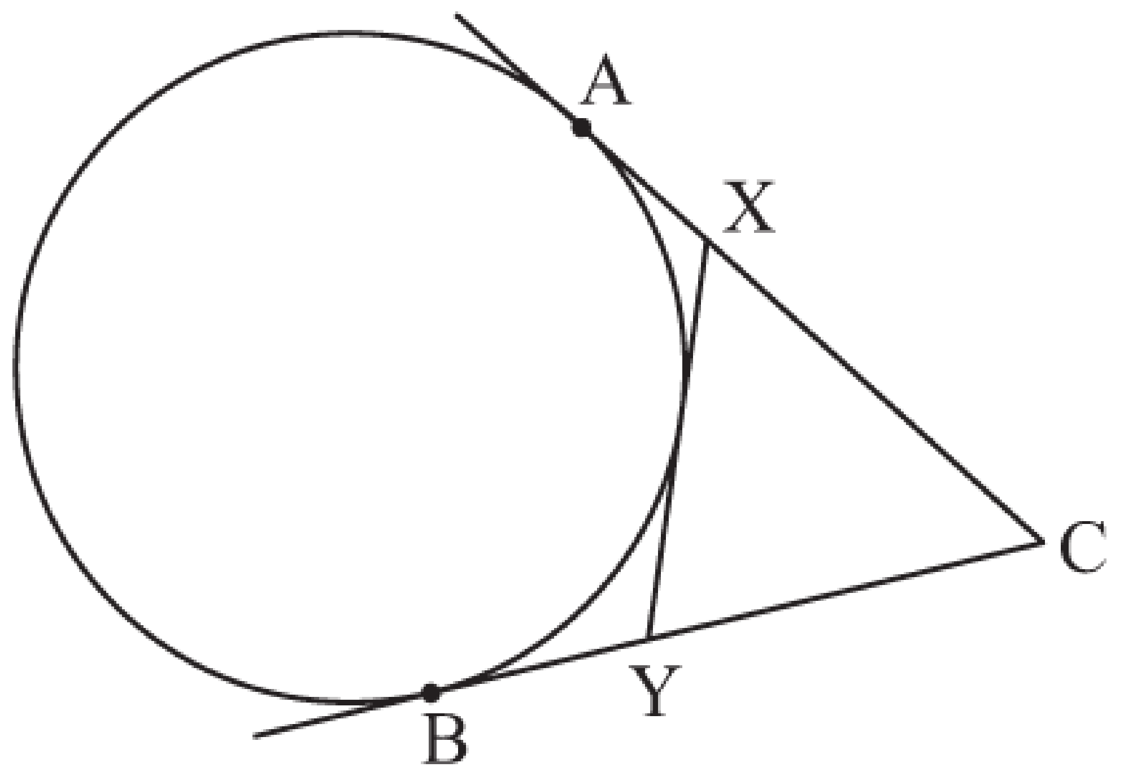
17 Two circles intersect at A and B. Straight lines [PQ] and [XY] are drawn through A to meet the circles as shown.

Show that $\widehat{XBP} = \widehat{YBQ}$.

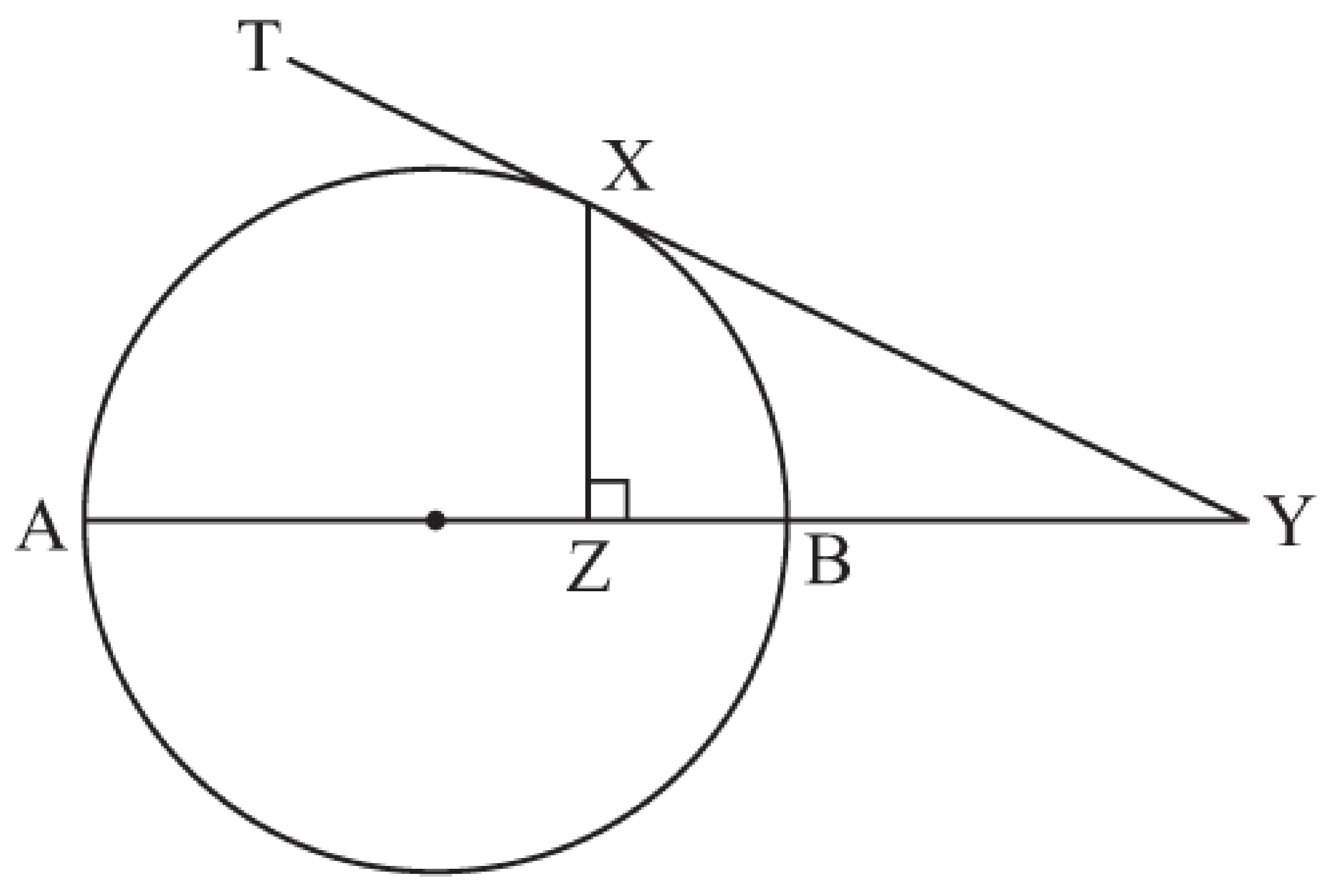


18 Triangle PQR is inscribed in a circle with [PR] as a diameter. The perpendicular from P to the tangent at Q, meets the tangent at S. Prove that [PQ] bisects angle SPR.

19 Tangents are drawn from a fixed point C to a fixed circle, meeting it at A and B. [XY] is a moving tangent which meets [AC] at X, and [BC] at Y. Prove that triangle XYC has constant perimeter.



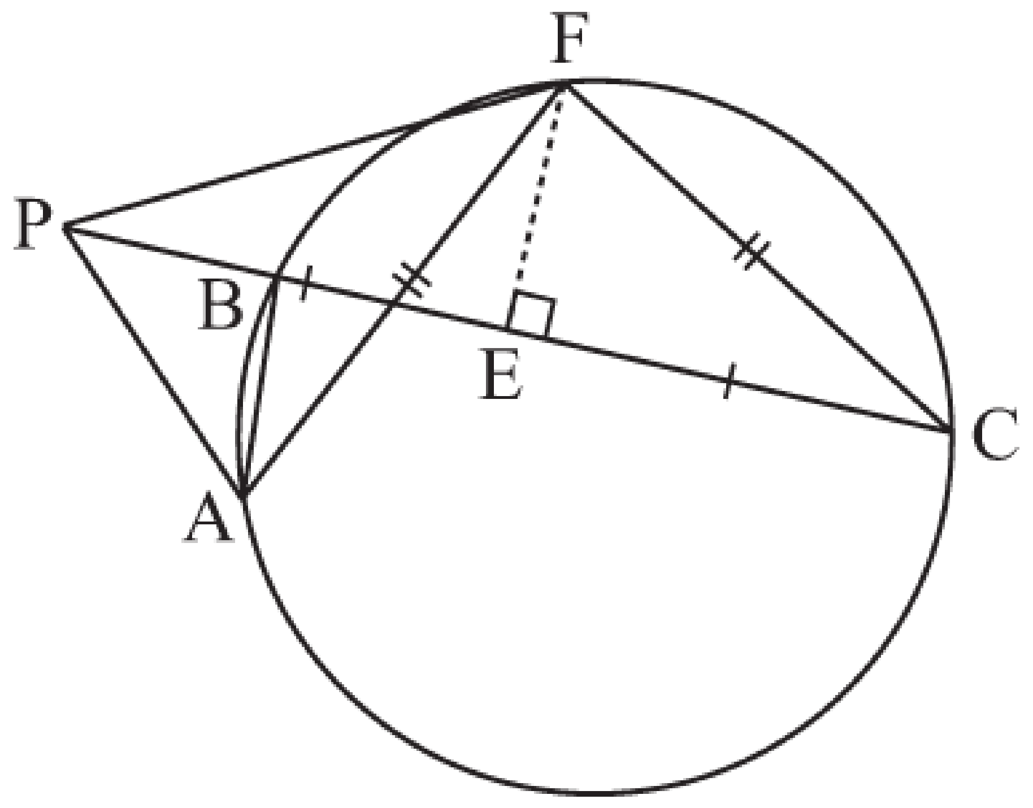
20 [AB] is a diameter of a circle. The tangent at X cuts the diameter produced at Y. [XZ] is perpendicular to [AY] at Z on [AY]. Prove that [XB] and [XA] are the bisectors of $\widehat{ZXY}$ and $\widehat{ZXT}$ respectively.



21 In the given figure, $AF = FC$ and $PE = EC$.

a Prove that triangle FPA is isosceles.

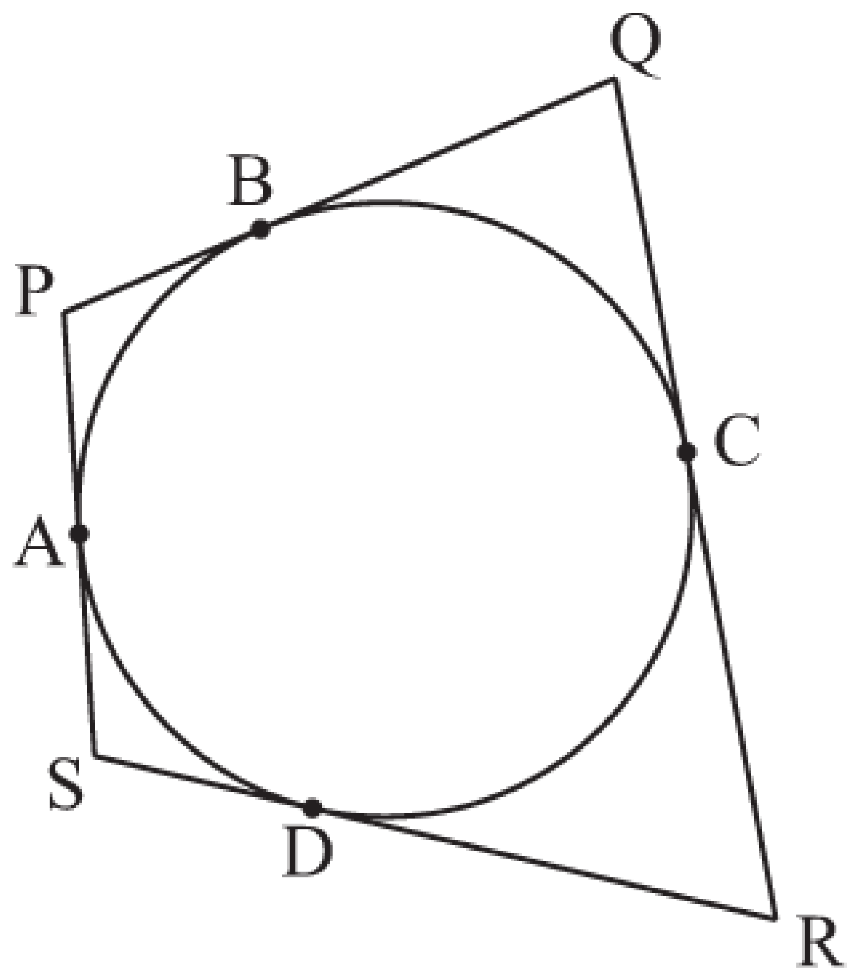
b Prove that $AB + BE = EC$.



22 Tangents from the external points P, Q, R, and S form a quadrilateral. This is called a **circumscribed polygon**.

What can be deduced about the opposite sides of the circumscribed quadrilateral?

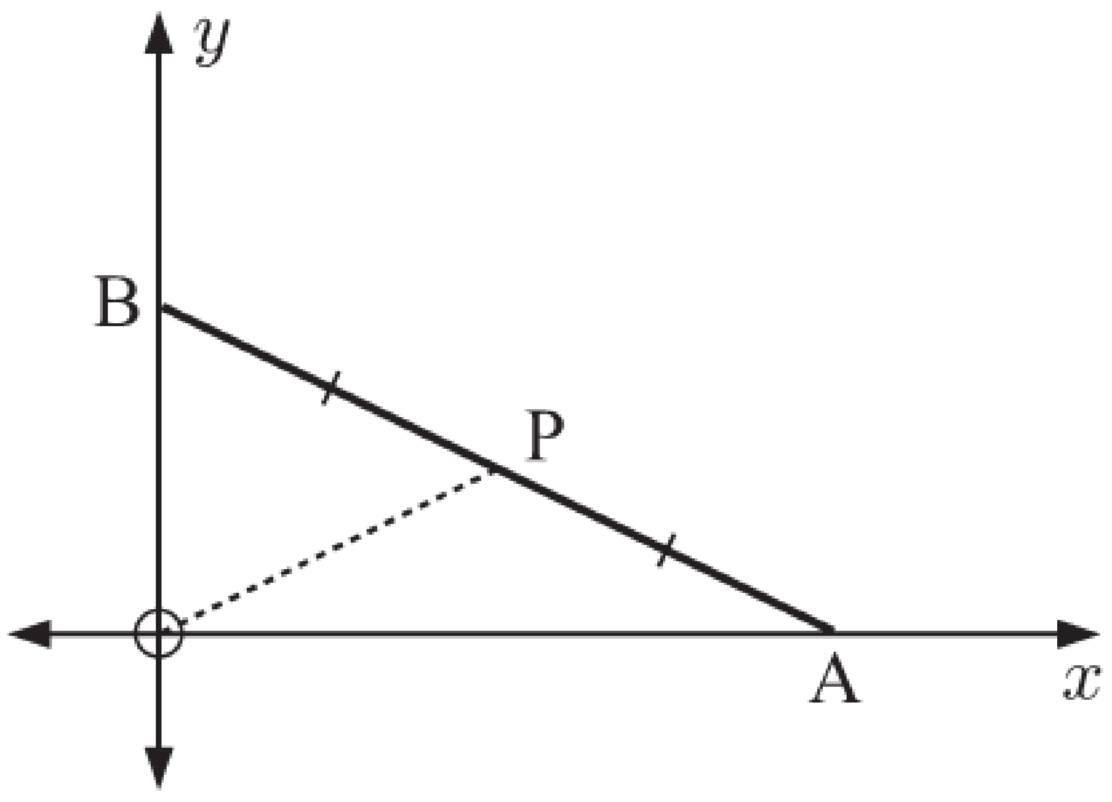
Prove your conjecture.



23 [POQ] is a diameter of a circle with centre O, and R is any other point on the circle. The tangent at R meets the tangents at P and Q at S and T respectively. Show that $S\widehat{O}T$ is a right angle.

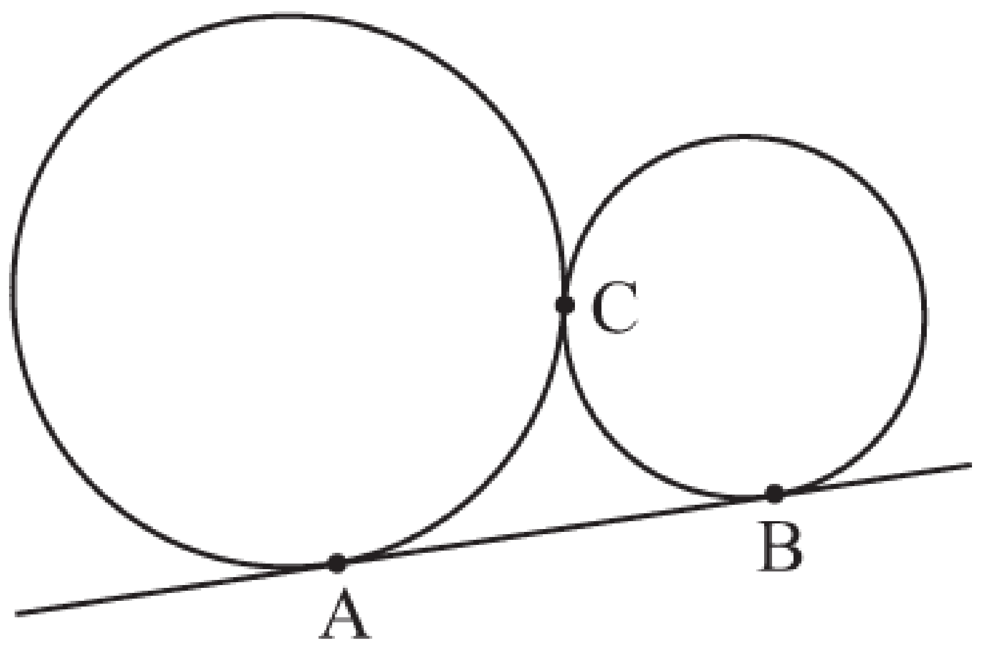
24 [PQ] and [PR] are tangents from an external point P to a circle with centre O.
[PS] is perpendicular to [PQ] and meets [OR] produced at S.
[QR] produced meets [PS] produced at T. Show that triangle STR is isosceles.

25 A solid thin bar [AB] moves so that A remains on the x-axis and B remains on the y-axis. There is a small light source at P, the midpoint of [AB]. Without using coordinate geometry methods, prove that as A and B move to all possible positions, the light traces out a circle.



DEMO

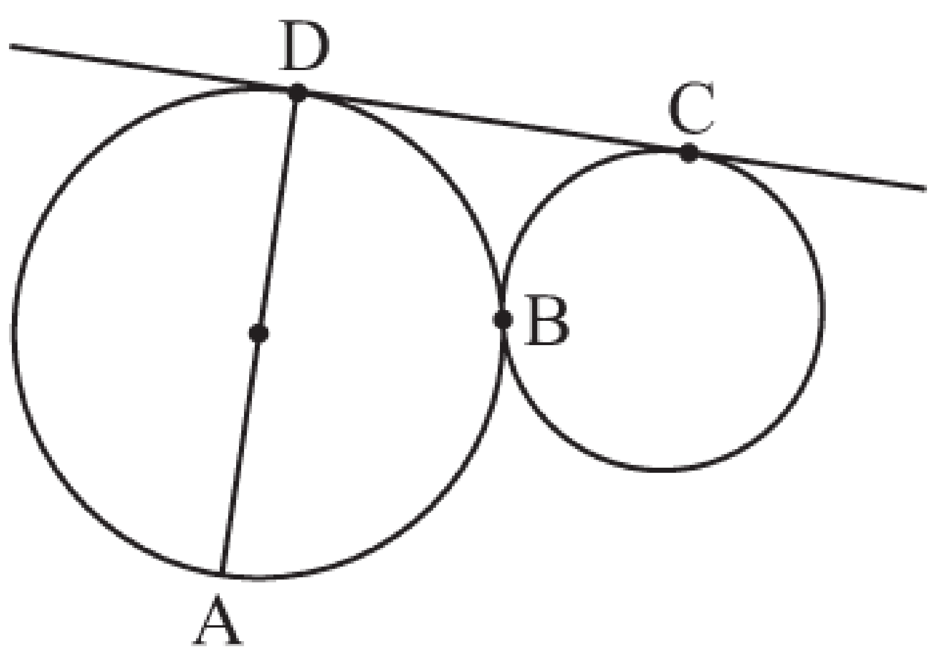
26



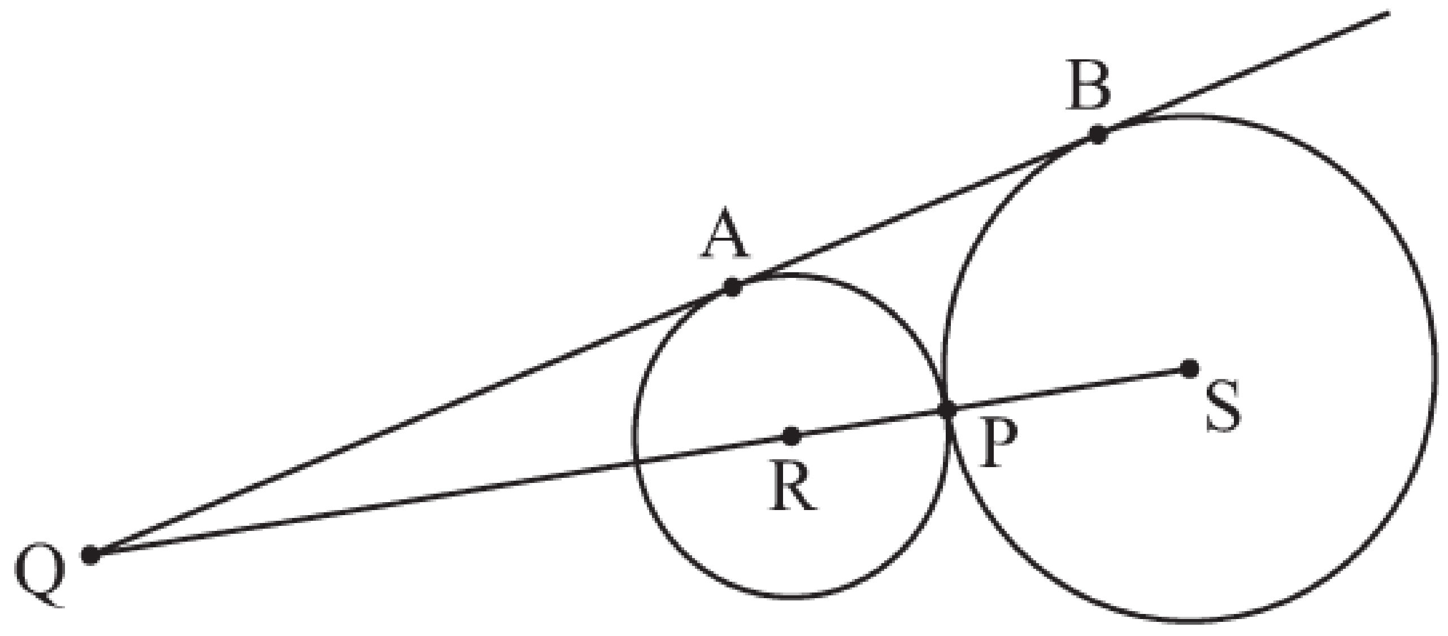
(AB) is a common tangent to two circles. Prove that:

- **a** the tangent through the point of contact C bisects [AB]
- **b** $A\widehat{C}B$ is a right angle.

27 Two circles touch externally at B. (CD) is a common tangent touching the circles at D and C.
[DA] is a diameter.
Prove that A, B, and C are collinear.

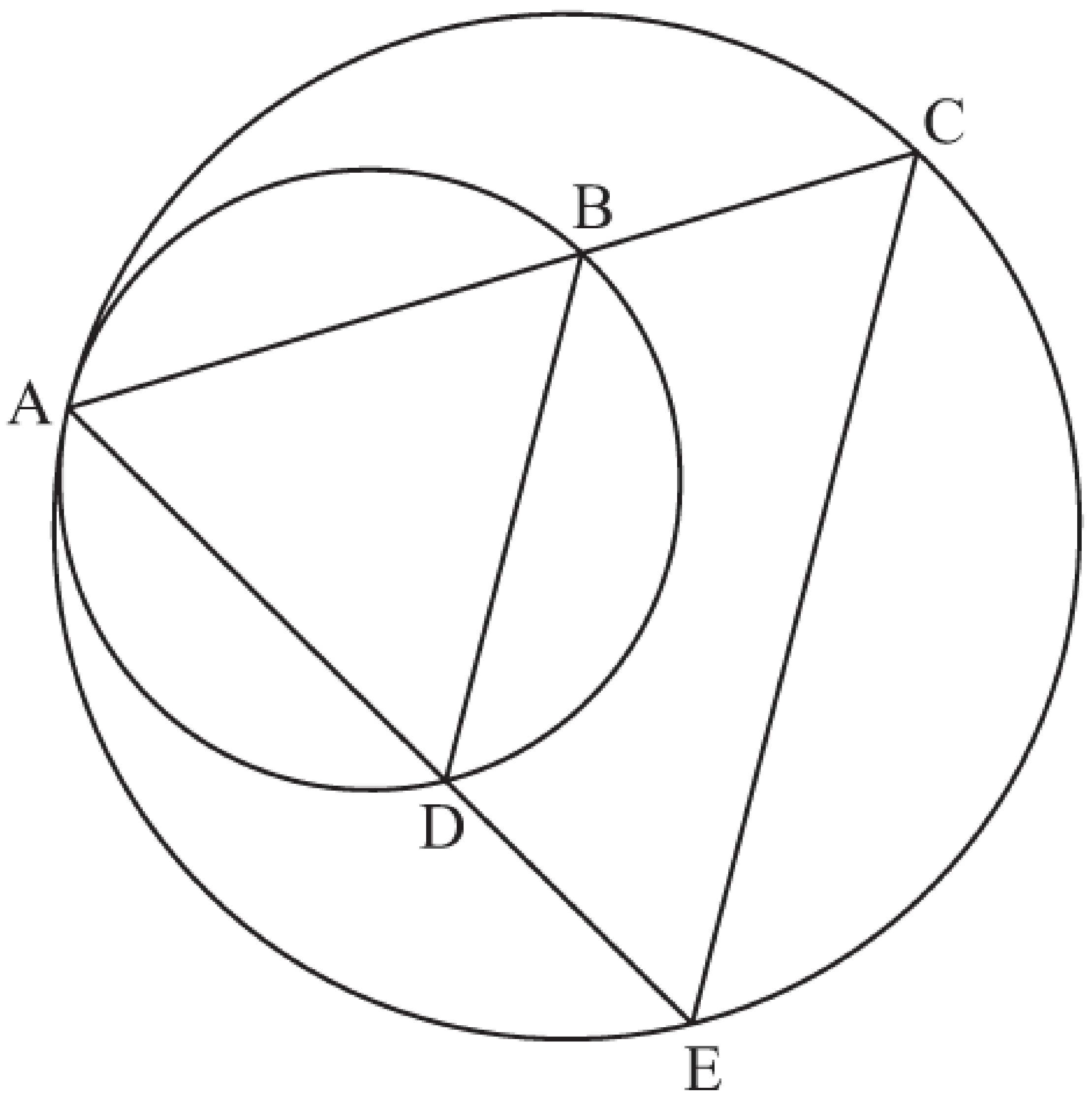


28



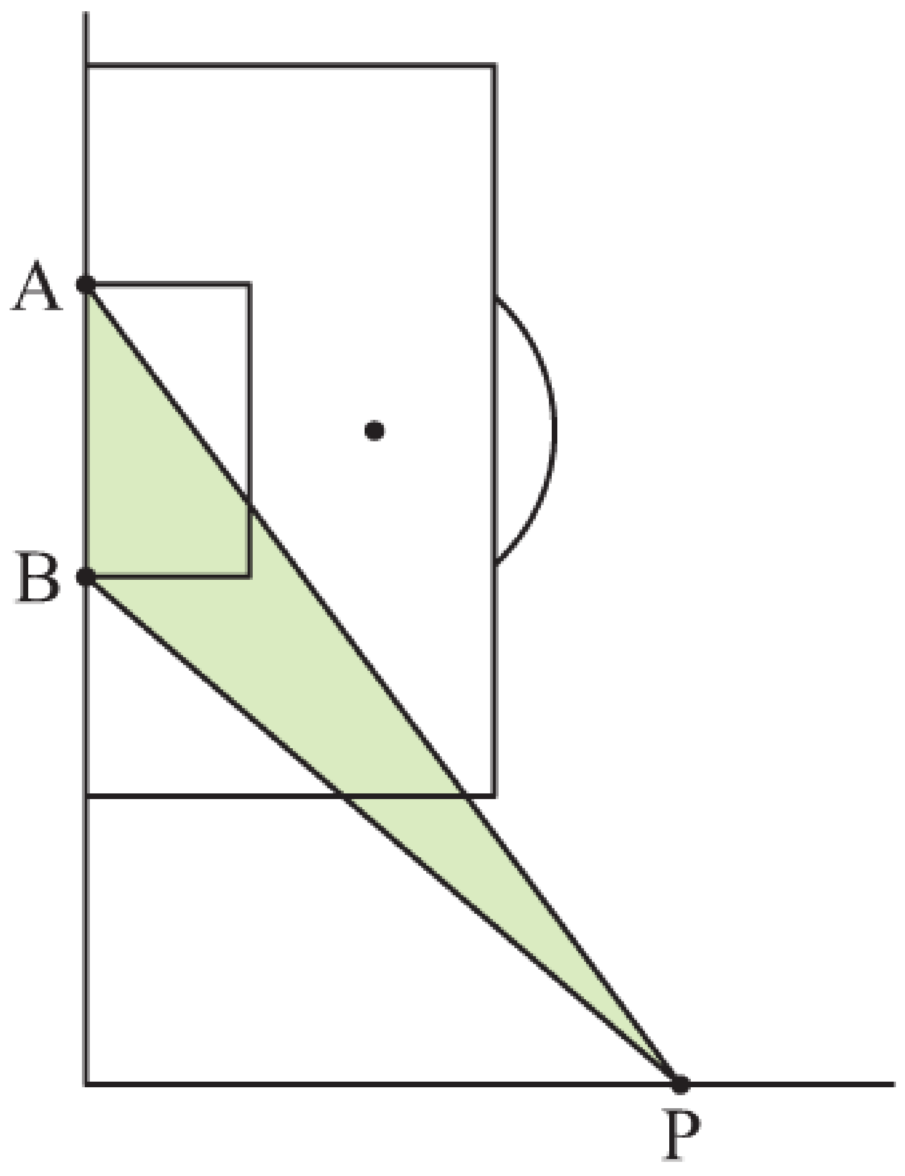
For the given figure, prove that $QP^2 = QA.QB$.

29 Two circles touch internally at point A. Chord [AC] of the larger circle cuts the smaller circle at B, and chord [AE] cuts the smaller circle at D.
Prove that [BD] is parallel to [CE].



30 Two circles touch internally at point P. The tangent to the inner circle at Q meets the outer circle at R and S. Prove that [QP] bisects $R\widehat{P}S$.

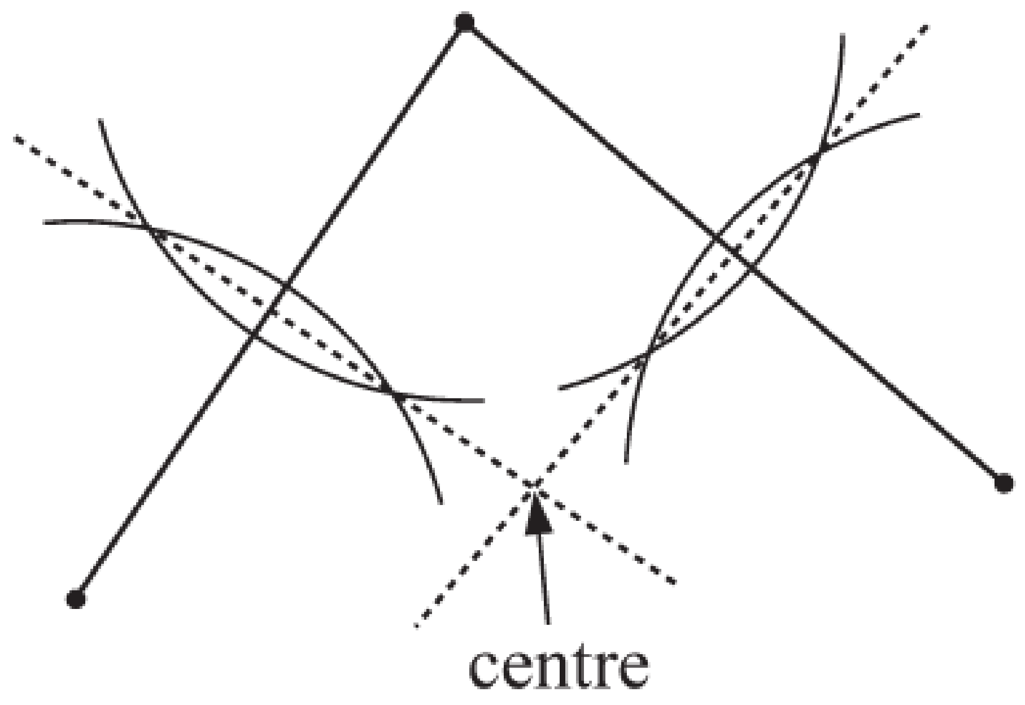
31 A and B are the goalposts on a football field. A photographer wants to find the point P on the boundary line such that his viewing angle of the goal, $A\widehat{P}B$, is maximised. Prove that P should be chosen so the boundary line is a tangent to the circle through A, B, and P.



E CONCYCLIC POINTS, CYCLIC QUADRILATERALS

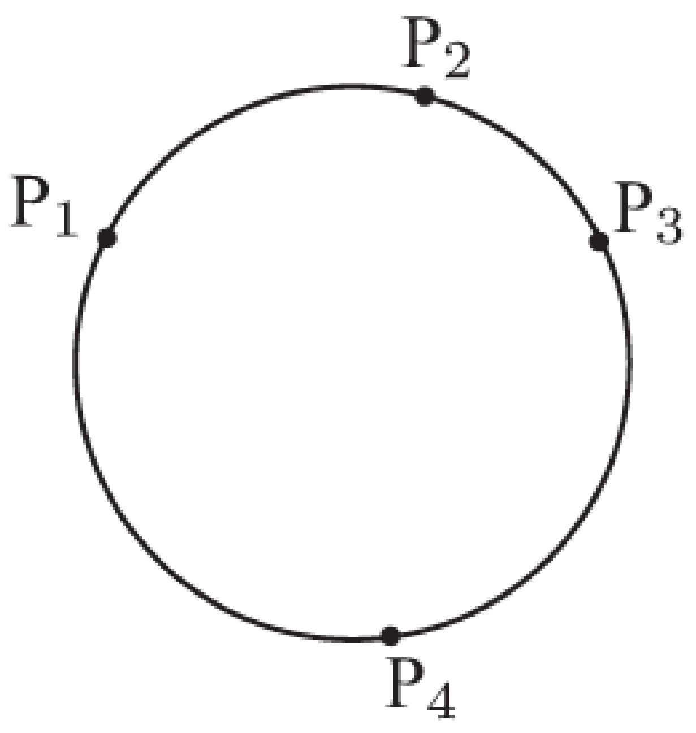
A circle can always be drawn through any three non-collinear points.

To find the circle's centre, we draw the perpendicular bisectors of the line segments joining the two pairs of points. The centre is the intersection of these two lines.

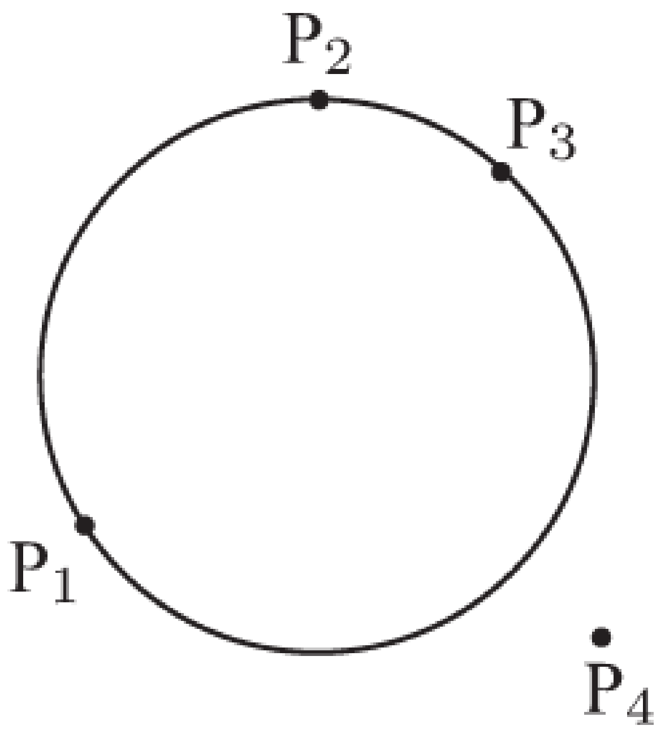


It may or may not be possible to draw a circle through any four given points in a plane.

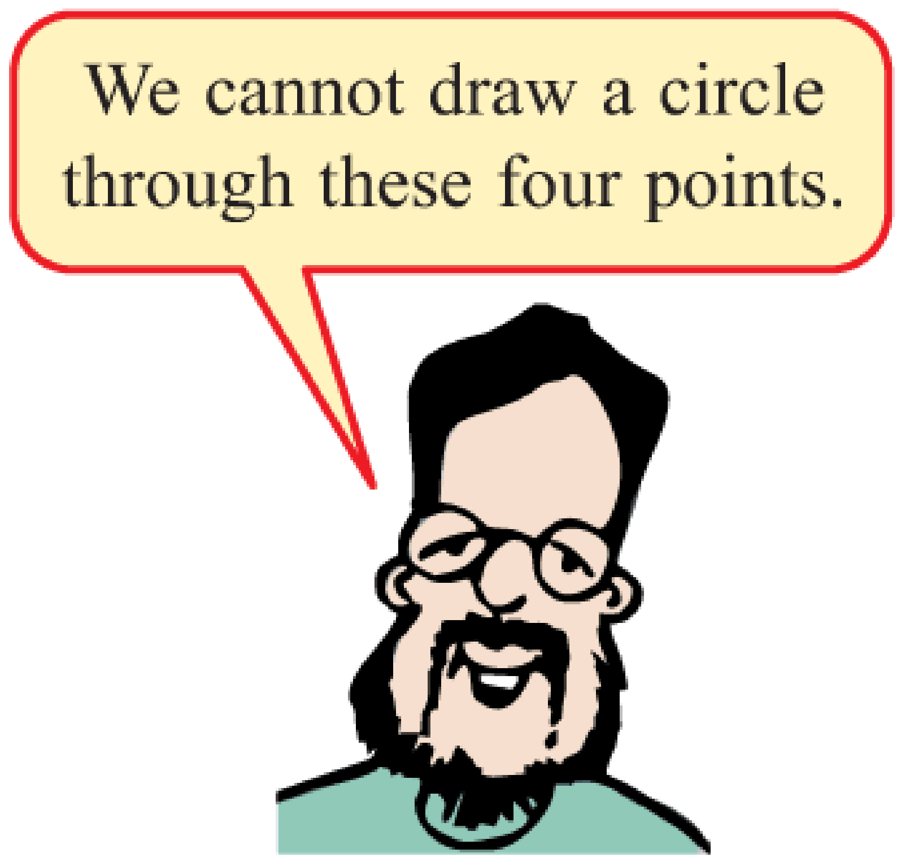
If a circle can be drawn through four points we say that the **points are concyclic**.

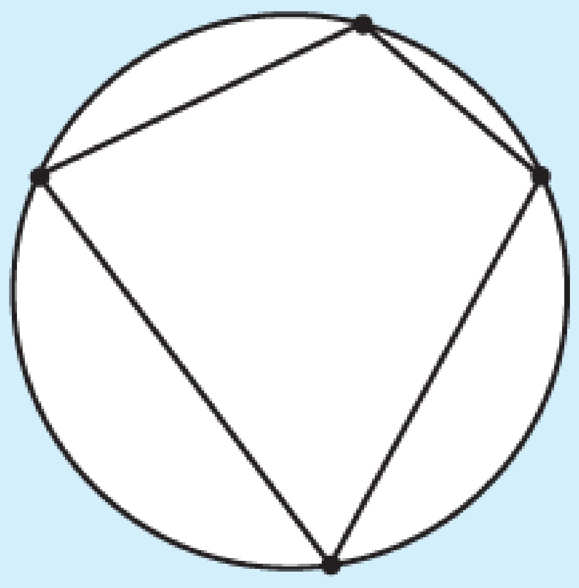


concyclic points



non-concyclic points

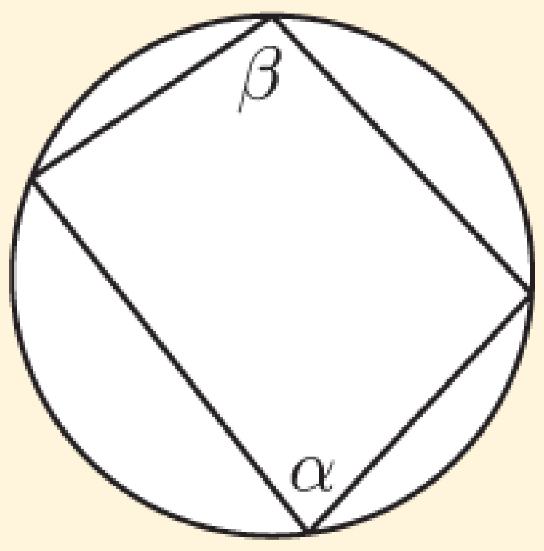

We cannot draw a circle through these four points.



If any four points on a circle are joined to form a convex quadrilateral, then the quadrilateral is called a **cyclic quadrilateral**.

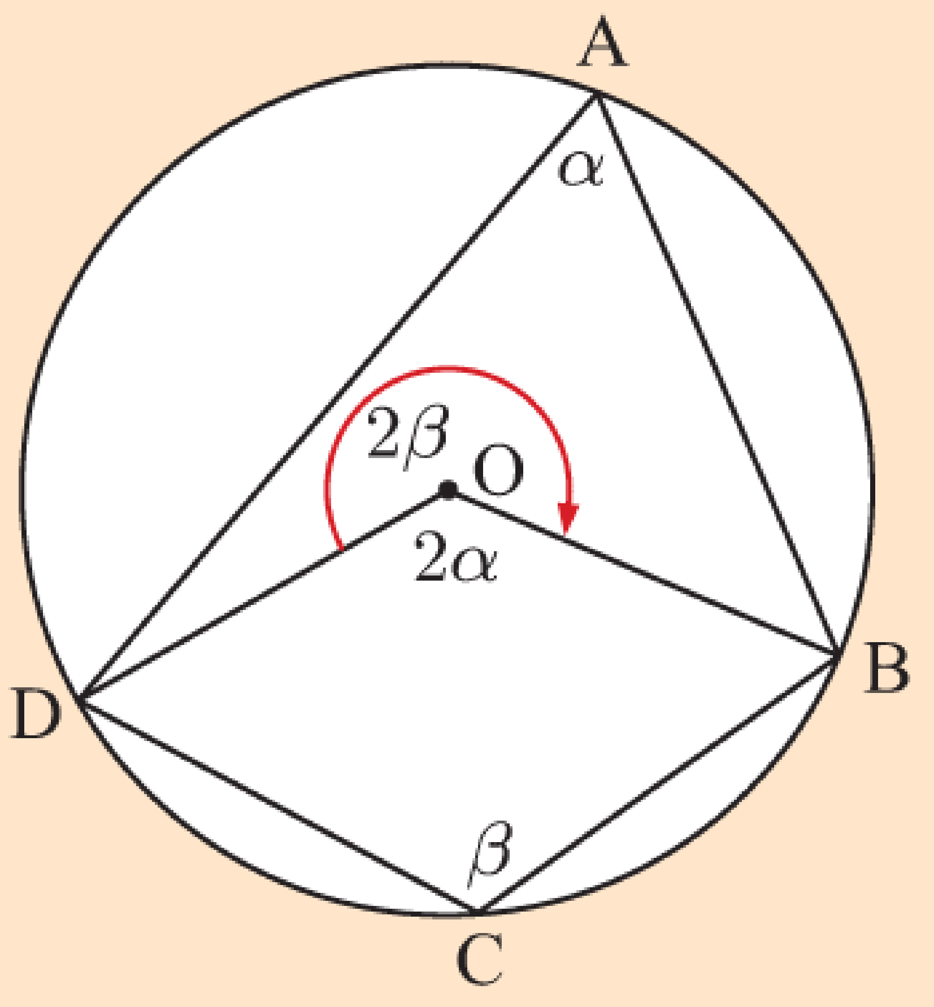
GEOMETRY PACKAGE

OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL THEOREM

Name of theorem	*Statement*	*Diagram*
Opposite angles of a cyclic quadrilateral	The opposite angles of a cyclic quadrilateral are supplementary.	 α β $\alpha + \beta = 180°$

GEOMETRY PACKAGE

Proof:



Consider a cyclic quadrilateral ABCD in a circle with centre O. Join [OD] and [OB].

If $D\widehat{A}B = \alpha$ and $D\widehat{C}B = \beta$ then $D\widehat{O}B = 2\alpha$

and reflex $D\widehat{O}B = 2\beta$ {angle at the centre}

$\therefore$ $2\alpha + 2\beta = 360°$ {angles at a point}

$\therefore$ $\alpha + \beta = 180°$

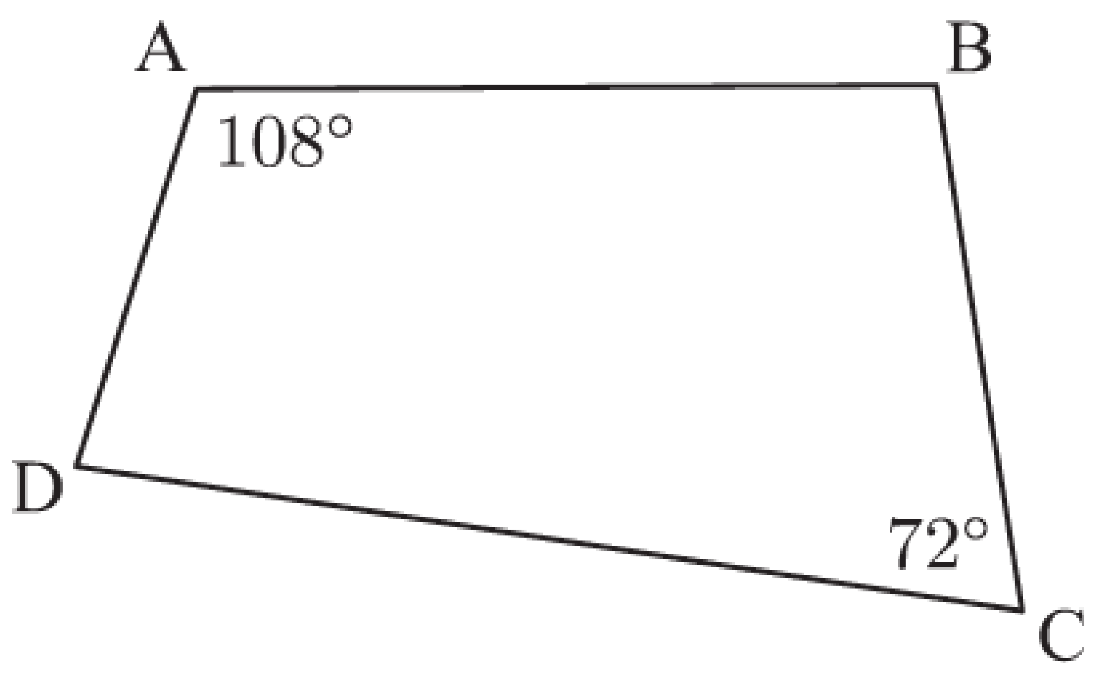
$\therefore$ angles DAB and DCB are supplementary.

Similarly, angles ADC and ABC are supplementary.

Converse:

If a pair of opposite angles of a quadrilateral are supplementary, then the quadrilateral is a cyclic quadrilateral.

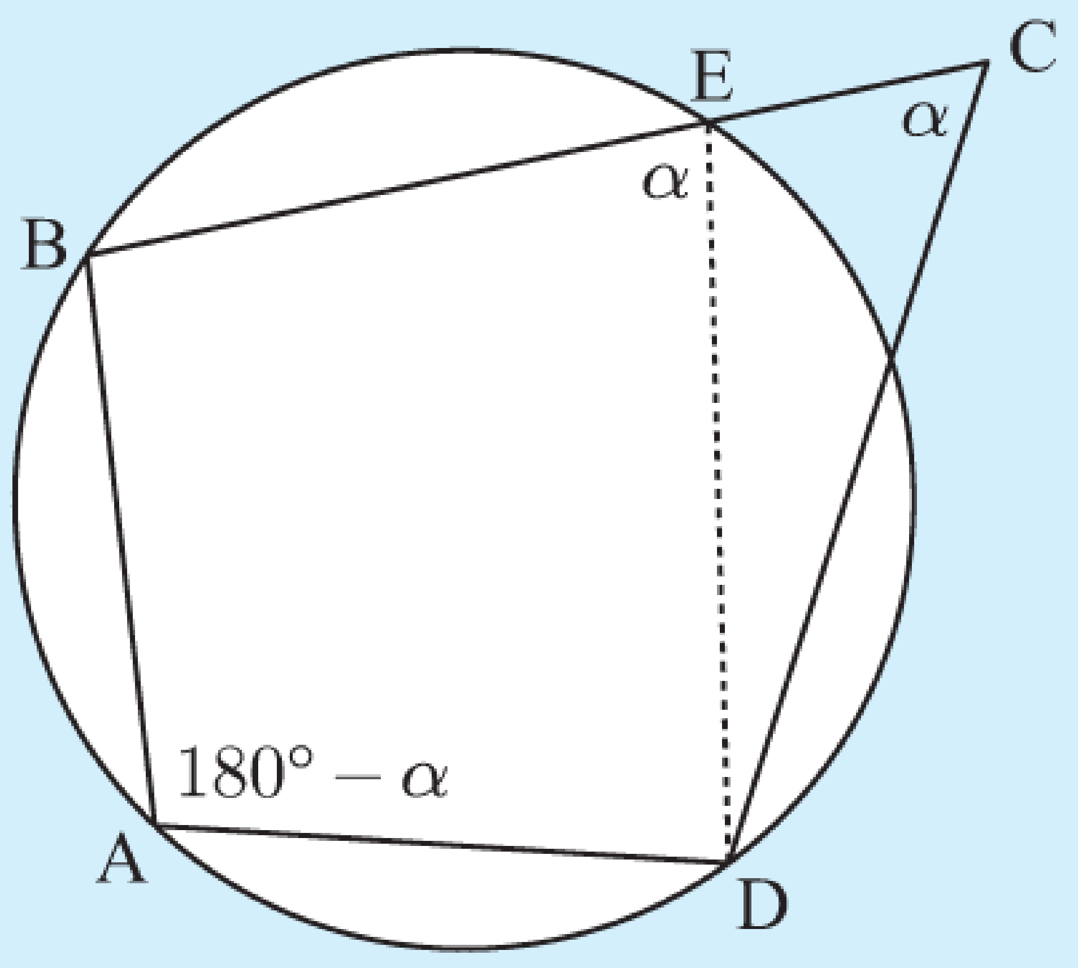
For example:



Since $D\widehat{A}B + B\widehat{C}D = 180°$,
ABCD is a cyclic quadrilateral.

Example 8

Prove the converse of the opposite angles of a cyclic quadrilateral theorem: If a pair of opposite angles of a quadrilateral are supplementary, then the quadrilateral is a cyclic quadrilateral.



Let ABCD be a quadrilateral with $B\widehat{C}D = \alpha$ and $B\widehat{A}D = 180° - \alpha$.

We draw a circle through A, B, and D.

The circle cuts [BC], or [BC] produced, at E.

Join [DE].

Clearly, ABED is a cyclic quadrilateral.
Consequently, $B\widehat{E}D = \alpha$ {opposite angles of a cyclic quadrilateral}

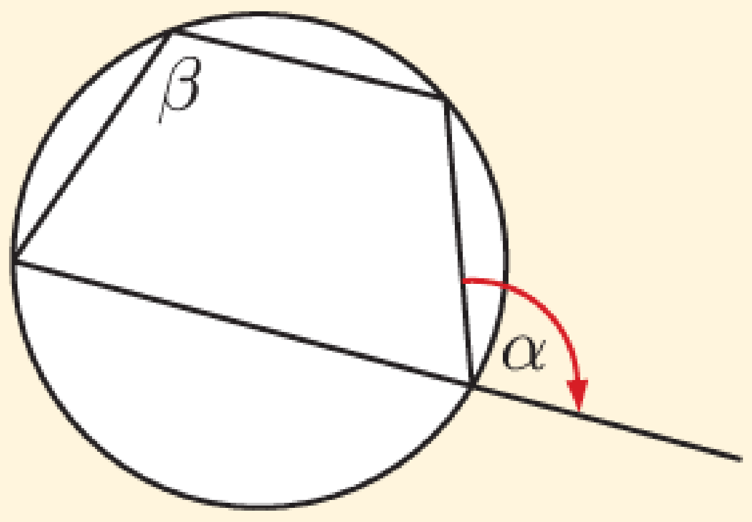
Now $B\widehat{E}D = B\widehat{C}D = \alpha$

$\therefore$ [ED] $\parallel$ [CD] {corresponding angles}

$\therefore$ E and C coincide

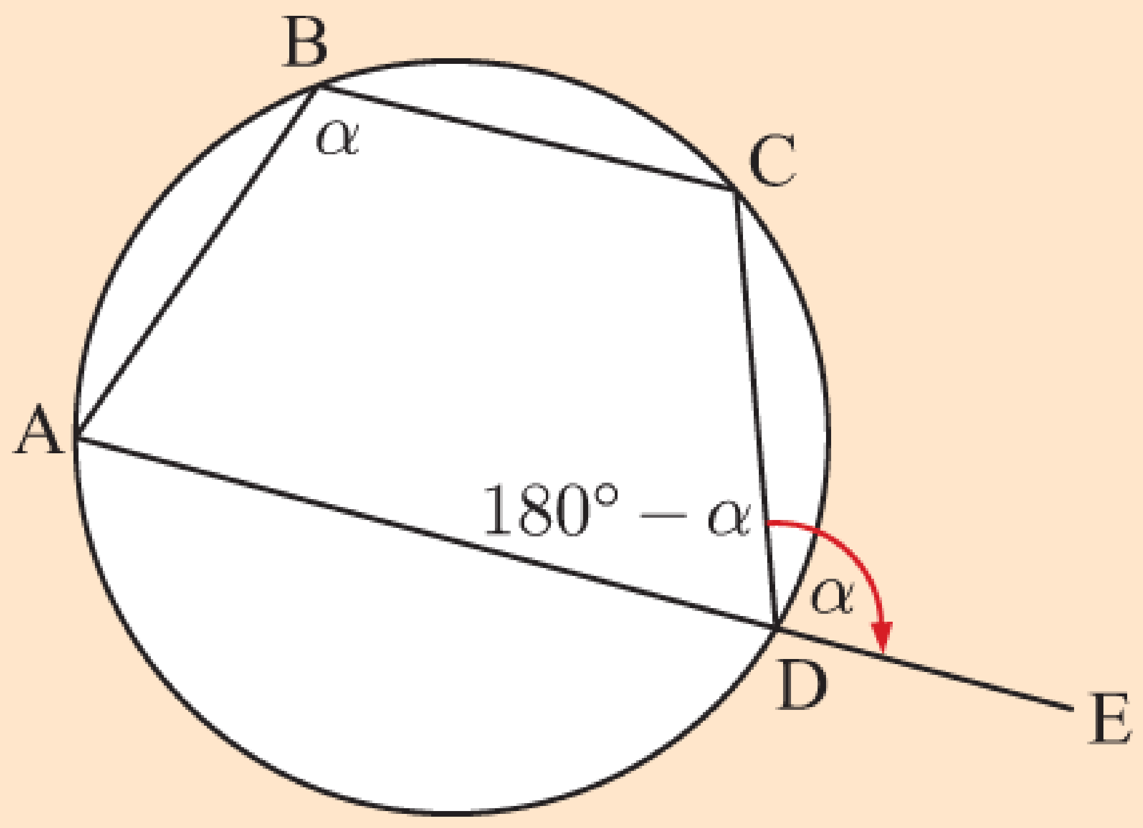
$\therefore$ ABCD is a cyclic quadrilateral.

EXTERIOR ANGLE OF A CYCLIC QUADRILATERAL THEOREM

Name of theorem	*Statement*	*Diagram*
Exterior angle of a cyclic quadrilateral	The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.	$\alpha = \beta$ 

GEOMETRY PACKAGE

Proof:



Consider the cyclic quadrilateral ABCD with $A\widehat{B}C = \alpha$.

Now $C\widehat{D}A = 180° - \alpha$
{opposite angles of a cyclic quadrilateral}

$\therefore$ $C\widehat{D}E = 180° - (180° - \alpha) = \alpha$
{angles on a line}

Thus $A\widehat{B}C = C\widehat{D}E$.

Converse:

If an exterior angle of a quadrilateral is equal to the interior opposite angle, the quadrilateral is a cyclic quadrilateral.

TESTS FOR CYCLIC QUADRILATERALS

A quadrilateral is a **cyclic quadrilateral** if any of the following is true:

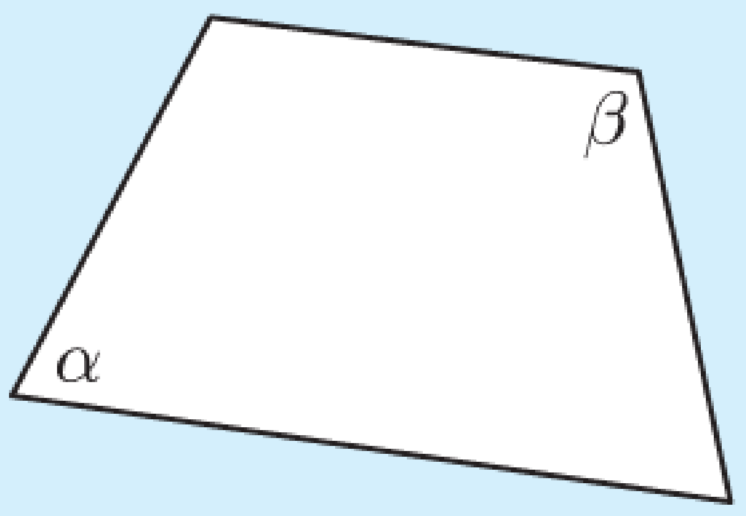
1 one pair of opposite angles are supplementary

2 one side subtends equal angles at the other two vertices

3 an exterior angle is equal to the interior opposite angle.

GEOMETRY PACKAGE

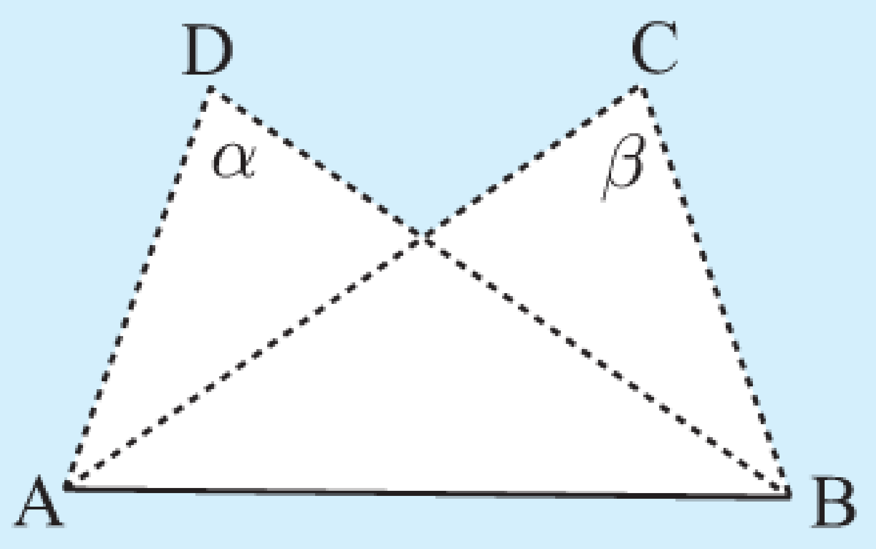
1



$\alpha + \beta = 180°$

$\Rightarrow$ the quadrilateral is a cyclic quadrilateral.

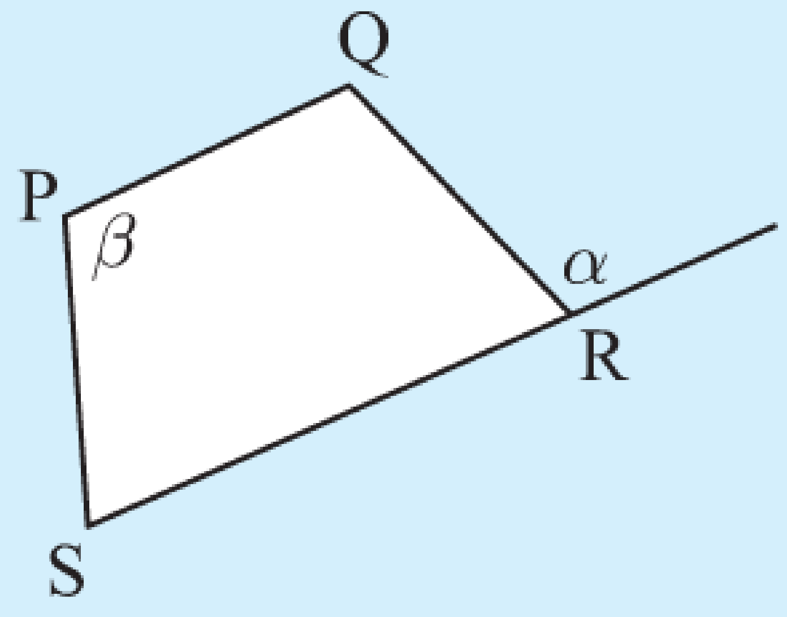
2



$\alpha = \beta$

$\Rightarrow$ ABCD is a cyclic quadrilateral.

3



$\alpha = \beta$

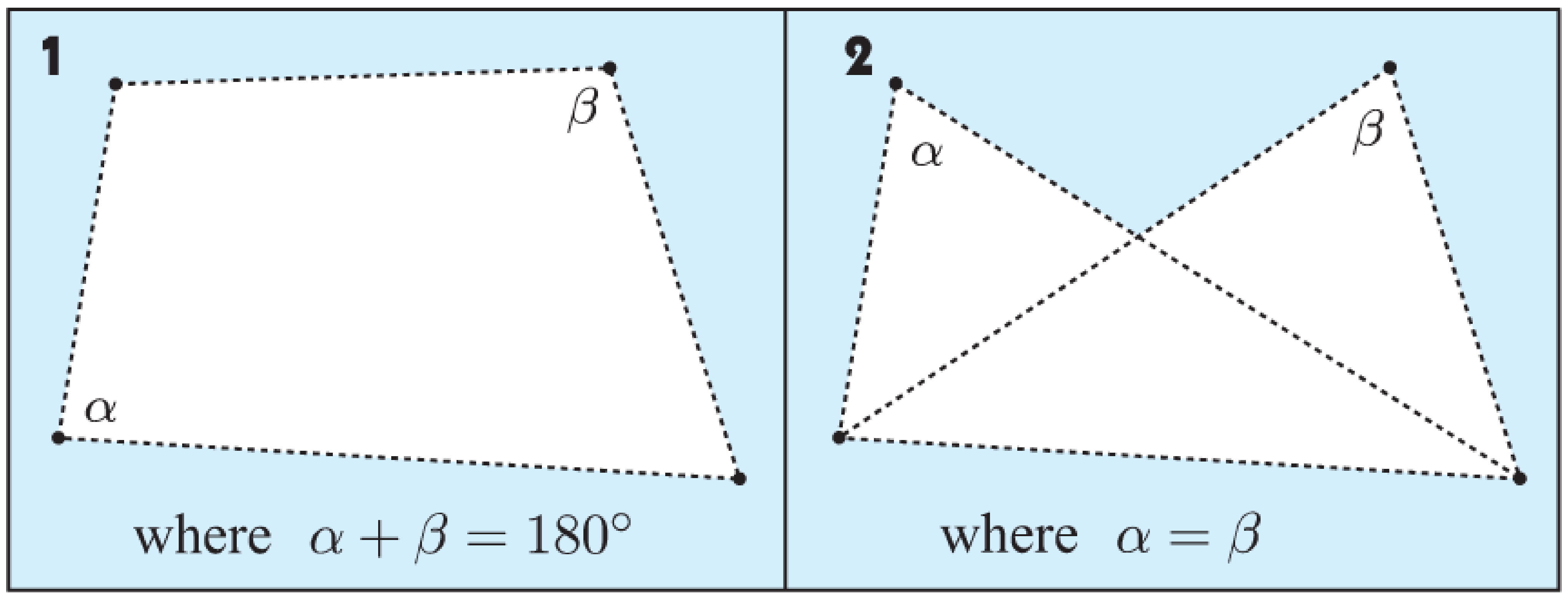
$\Rightarrow$ PQRS is a cyclic quadrilateral.

TEST FOR CONCYCLIC POINTS

Four points are concyclic if either of the following is true:

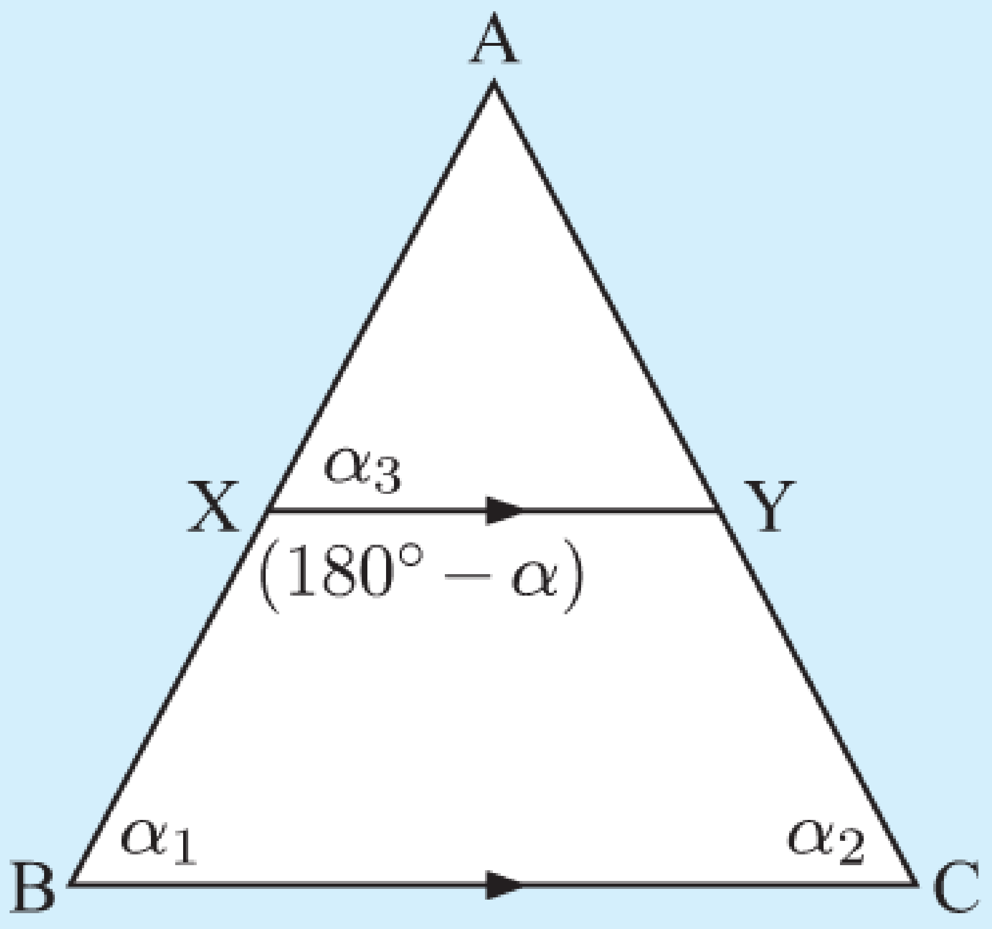
1 when the points are joined to form a convex quadrilateral, one pair of opposite angles are supplementary

2 when two points (defining a line) subtend equal angles at the other two points on the same side of the line.


1
β
α
where $\alpha+\beta=180°$
2
α
β
where $\alpha=\beta$

Example 9

Triangle ABC is isosceles with $AB = AC$. X and Y lie on [AB] and [AC] respectively such that [XY] is parallel to [BC]. Prove that XYCB is a cyclic quadrilateral.


A
X
Y
α_3
$(180°-\alpha)$
B
C
α_1
α_2

Since $\triangle ABC$ is isosceles with $AB = AC$,

$\alpha_1=\alpha_2$ {equal base angles}

Now $XY \parallel BC$, so $\alpha_1=\alpha_3$ {corresponding angles}

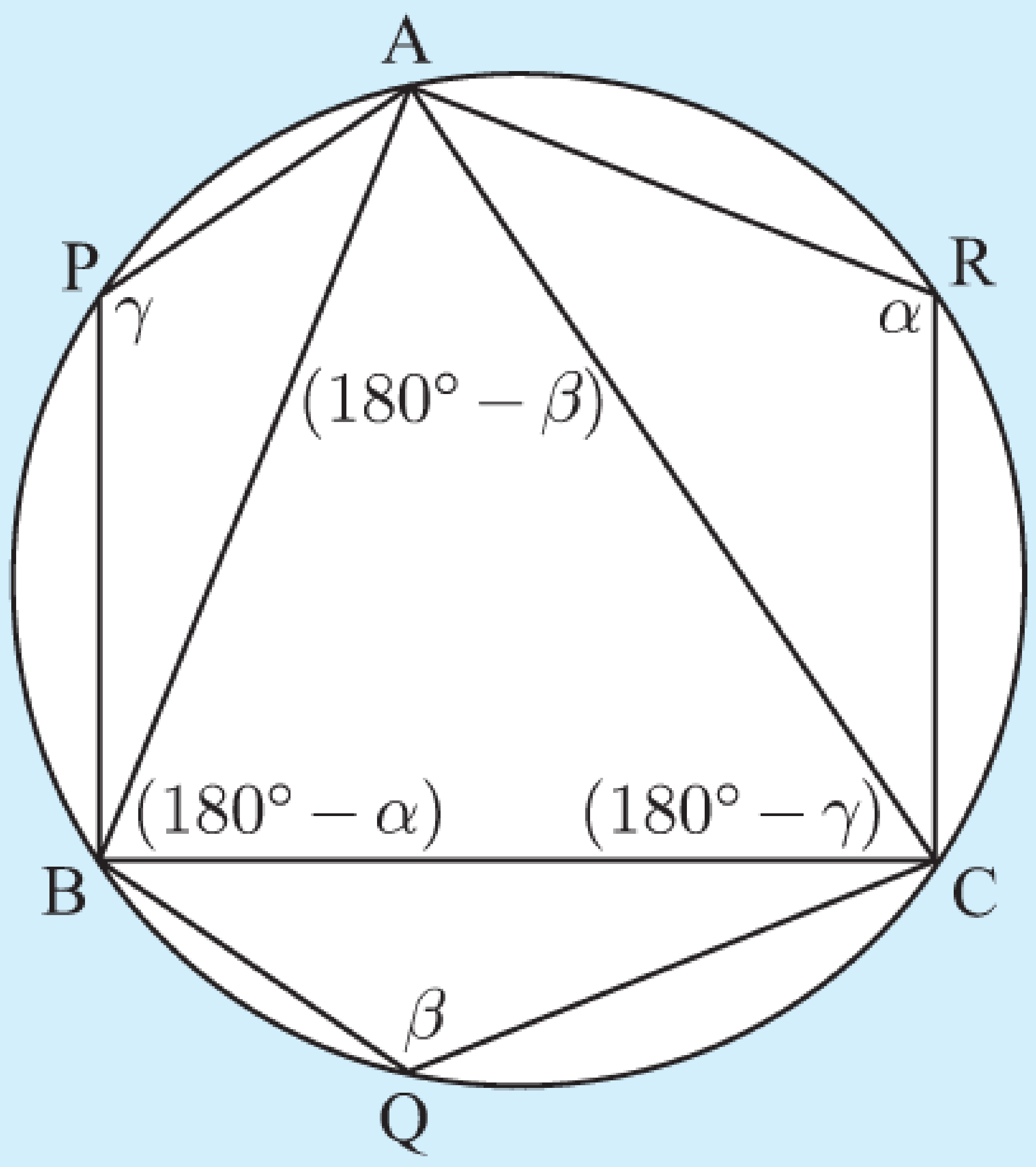
$\therefore$ $Y\widehat{X}B = 180°-\alpha$ {angles on a line}

$\therefore$ $Y\widehat{X}B + Y\widehat{C}B = (180°-\alpha)+\alpha = 180°$

$\therefore$ XYCB is a cyclic quadrilateral {opposite angles supplementary}

Example 10

Triangle ABC is inscribed in a circle. P, Q, and R are any points on arcs AB, BC, and AC respectively. Prove that $A\widehat{R}C$, $C\widehat{Q}B$, and $B\widehat{P}A$ have a sum of 360°.


A
P
R
γ
α
$(180°-\beta)$
$(180°-\alpha)$
$(180°-\gamma)$
B
C
β
Q

Let $A\widehat{R}C$, $C\widehat{Q}B$, and $B\widehat{P}A$ be α, β, and γ respectively.
Now BARC is a cyclic quadrilateral.

$\therefore$ $A\widehat{B}C = 180°-\alpha$

Likewise in cyclic quadrilaterals ABQC and CAPB,

$B\widehat{A}C = 180°-\beta$ and $A\widehat{C}B = 180°-\gamma$

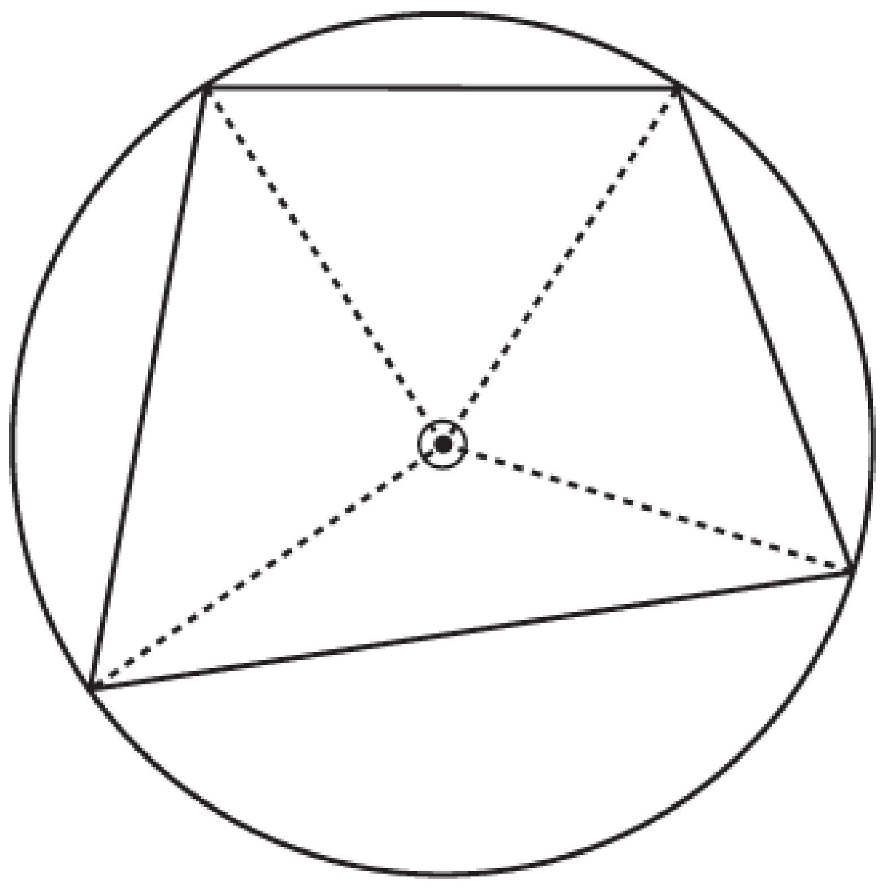
Thus $(180°-\alpha)+(180°-\beta)+(180°-\gamma)=180°$ {angles of a triangle}

$\therefore$ $540°-(\alpha+\beta+\gamma)=180°$

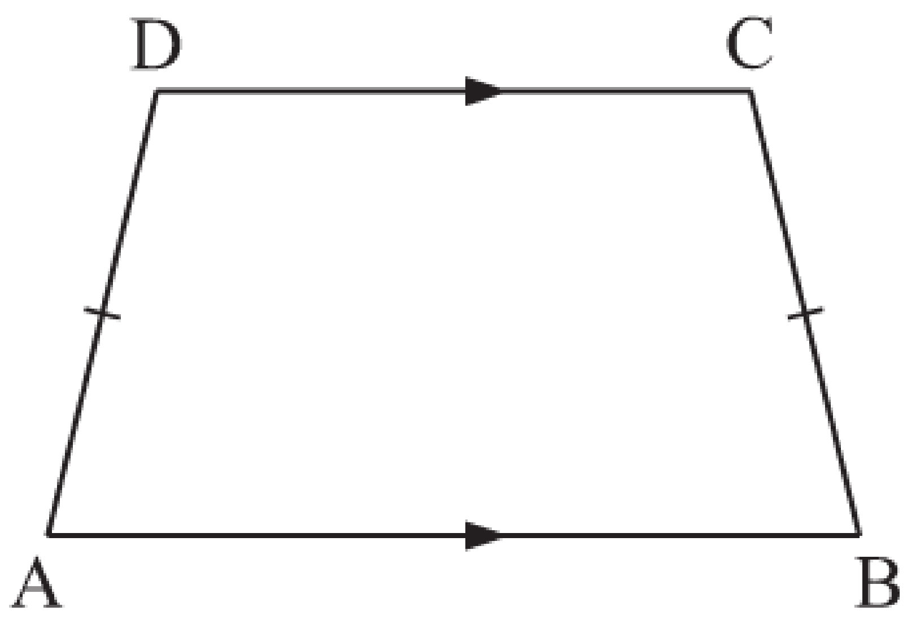
$\therefore$ $\alpha+\beta+\gamma=360°$, as required

EXERCISE 2E

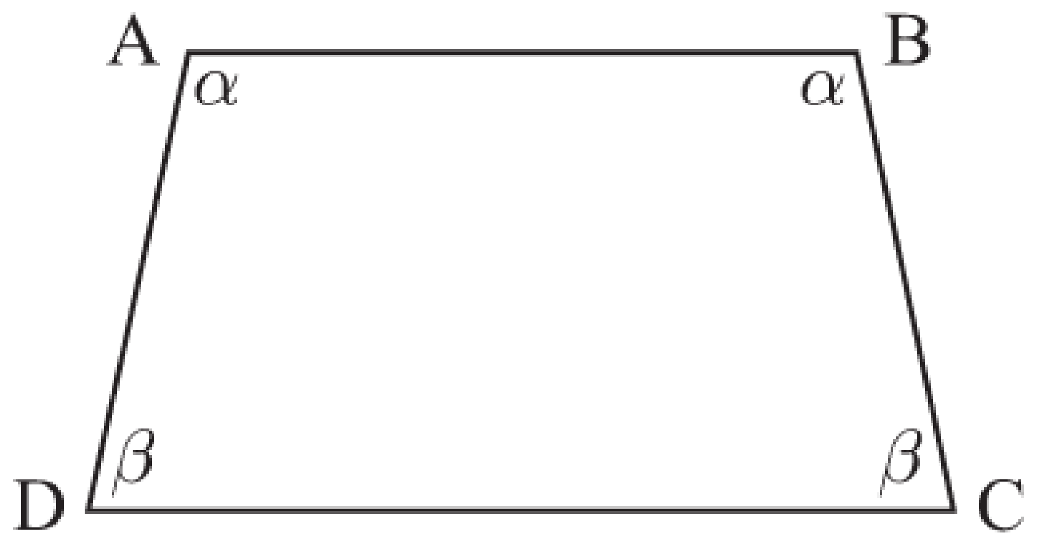
1 Use the given figure to prove that the opposite angles of a cyclic quadrilateral are supplementary.



2 Without assuming any properties of isosceles trapezia, prove that an isosceles trapezium is always a cyclic quadrilateral.
Hint: Draw [CX] parallel to [DA], and meeting [AB] at X.



3 What can be deduced about the quadrilateral ABCD? Explain your answer.



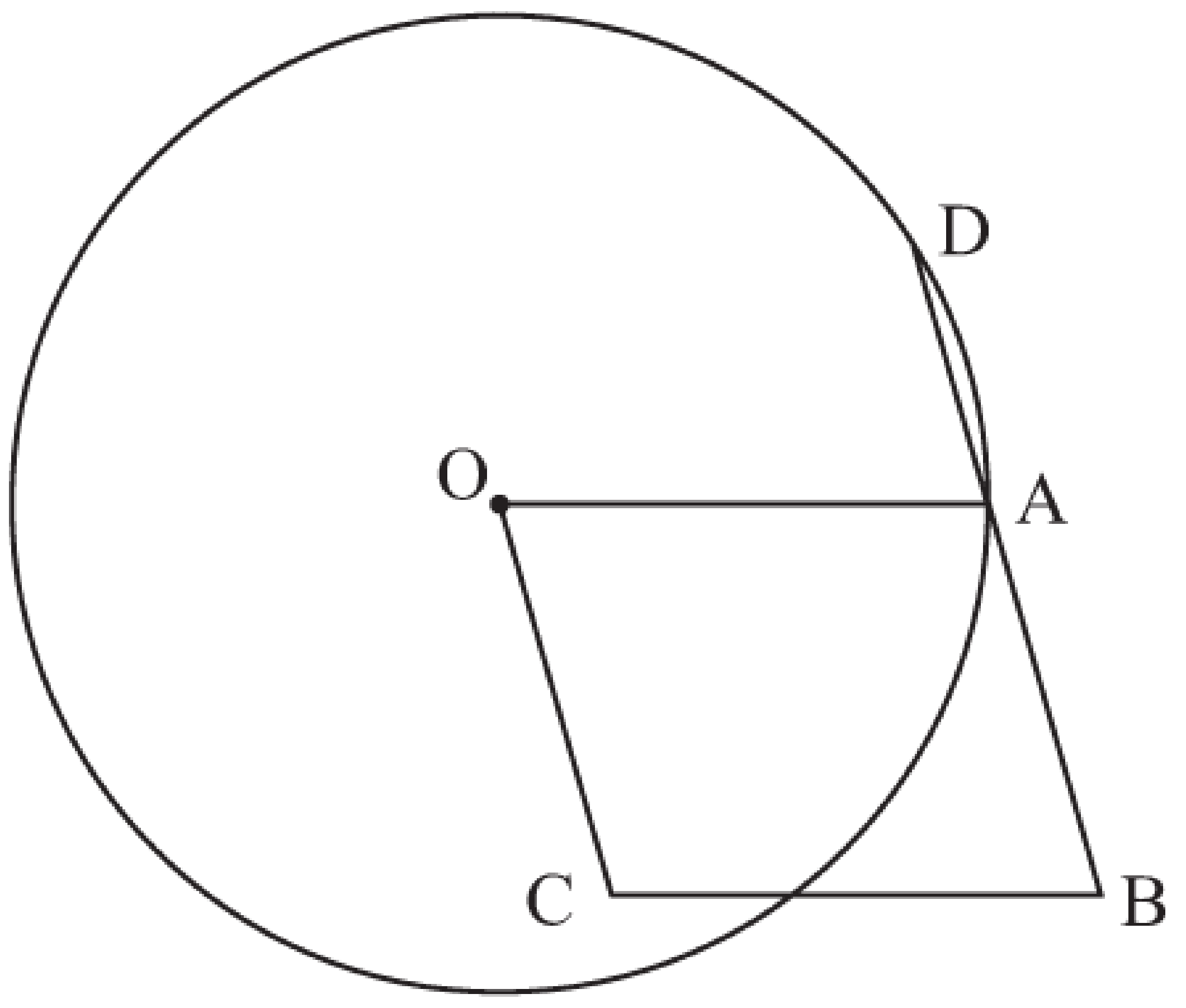
4 ABC is an isosceles triangle in which AB = AC. The angle bisectors at B and C meet the sides [AC] and [AB] at X and Y respectively. Show that BCXY is a cyclic quadrilateral.

5 Two circles meet at points X and Y. [AXB] and [CYD] are two line segments which meet one circle at A and C, and the other circle at B and D. Prove that [AC] is parallel to [BD].

6 Prove that a parallelogram inscribed in a circle is a rectangle.

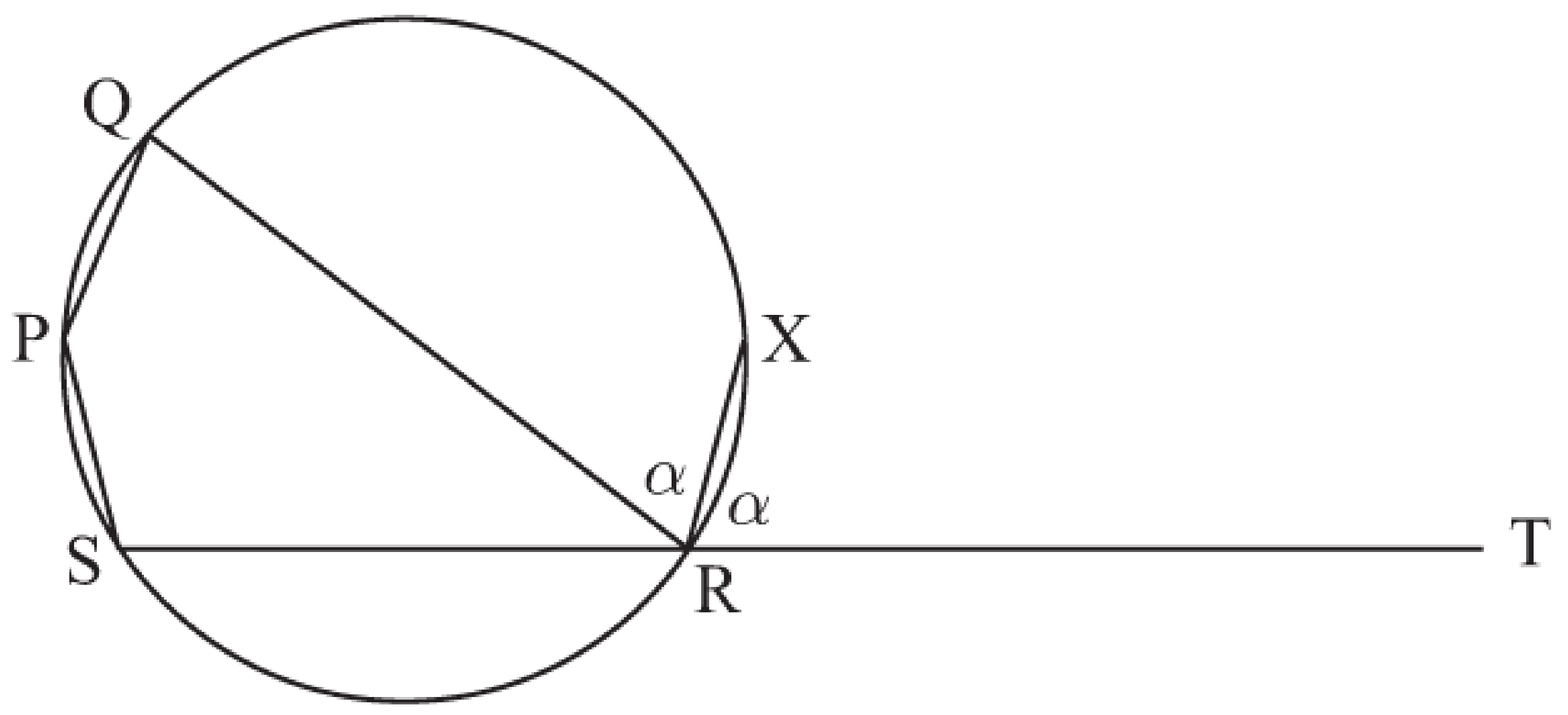
7 ABCD is a cyclic quadrilateral and X is any point on the diagonal [CA]. [XY] is drawn parallel to [CB] to meet [AB] at Y. [XZ] is drawn parallel to [CD] to meet [AD] at Z. Prove that XYAZ is a cyclic quadrilateral.

8 OABC is a parallelogram.
A circle with centre O and radius [OA] is drawn.
[BA] produced meets the circle at D.
Prove that DOCB is a cyclic quadrilateral.



9 Two circles intersect at X and Y. A line segment [AXB] is drawn cutting the circles at A and B respectively. The tangents at A and B meet at C.
Prove that AYBC is a cyclic quadrilateral.

10 [RX] is the bisector of angle QRT.
Prove that [PX] bisects angle QPS.

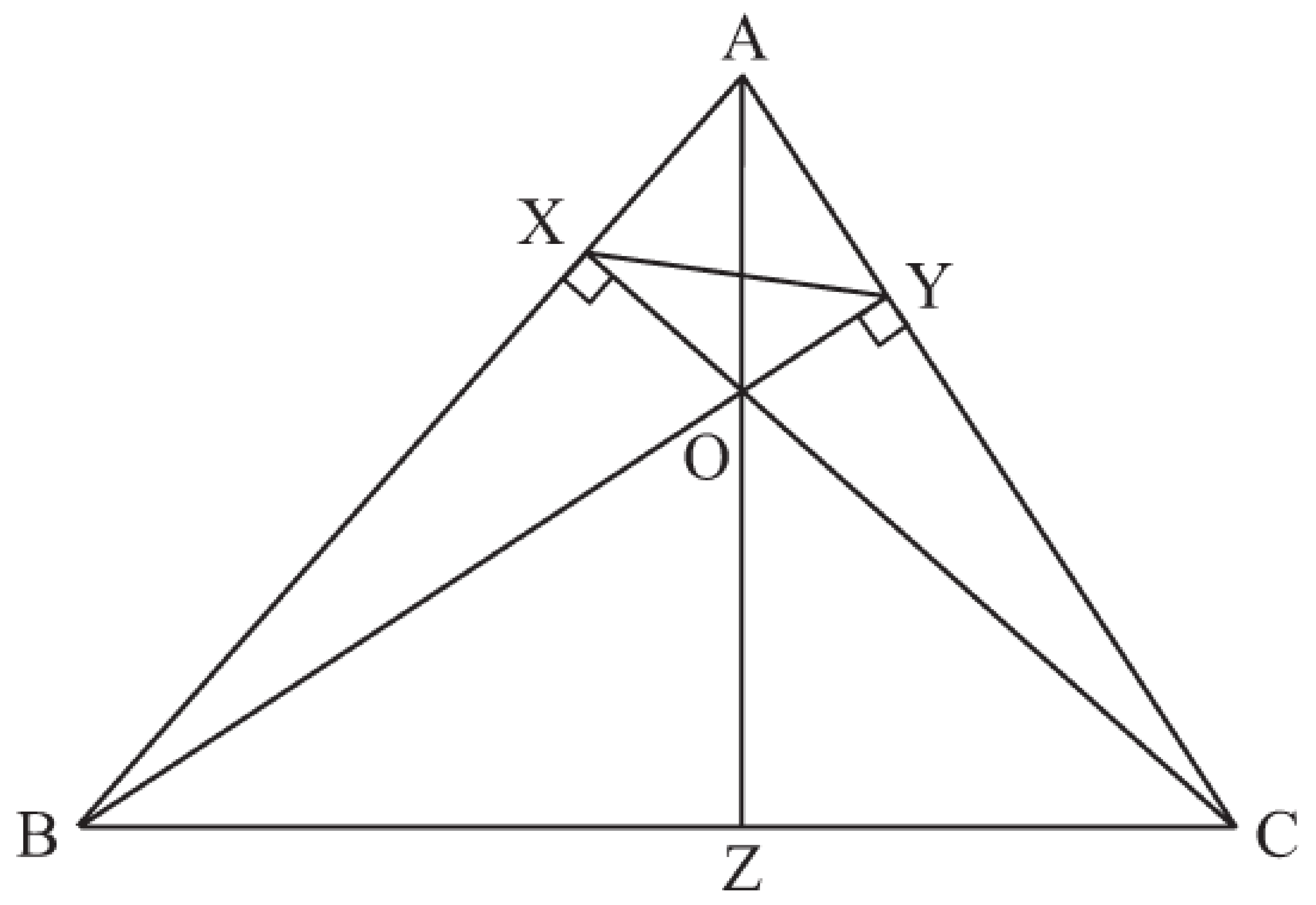


11 [AB] and [CD] are two parallel chords of a circle with centre O. [AD] and [BC] meet at E. Prove that A, E, O, and C are concyclic points.

12 [AB] and [AC] are chords of a circle with centre O. X and Y are the midpoints of [AB] and [AC] respectively. Prove that O, X, A, and Y are concyclic points.

13 Triangle ABC has perpendiculars [CX] and [BY] as shown.

- **a** What can be said about quadrilaterals AXOY and BXYC? Explain your answers.
- **b** Prove that $X\widehat{A}O = X\widehat{Y}O = X\widehat{C}B$.
- **c** Prove that [AZ] is perpendicular to [BC].



14 Two circles intersect at P and Q. [APB] and [CQD] are two parallel lines which meet the circles at A, B, C, and D. Prove that AB = CD.

15 In triangle PQR, PQ = PR. If S and T are the midpoints of [PQ] and [PR] respectively, show that S, Q, R, and T are concyclic points.

16 Triangle ABC is acute angled. Squares ABDE and BCFG are drawn externally to the triangle. If [GA] and [CD] meet at P, show that:

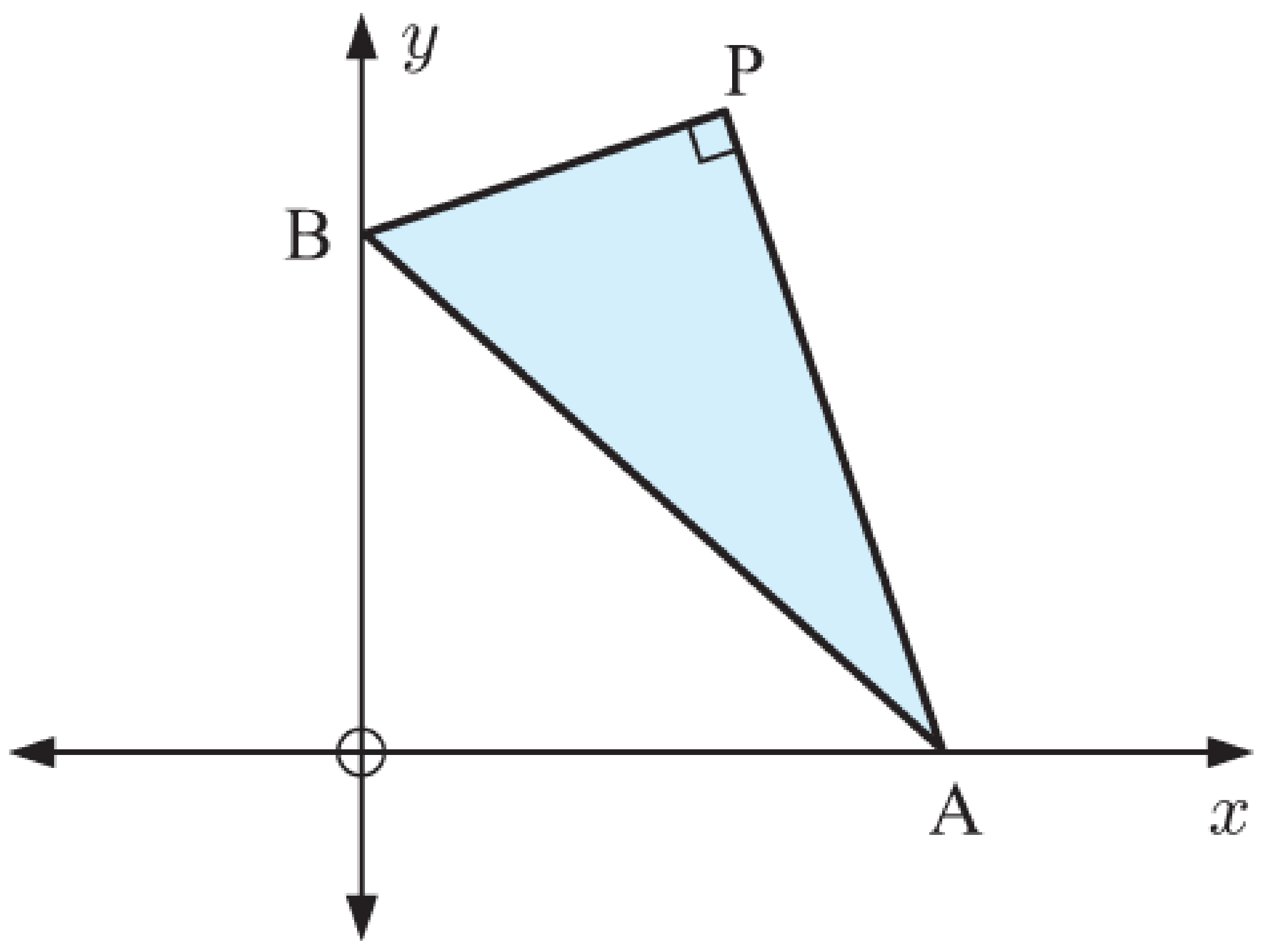
- **a** B, G, C, and P are concyclic
- **b** [DC] and [AG] are perpendicular
- **c** [BP] bisects angle DPG.

17 [AOB] is a diameter of a circle with centre O. C is any other point on the circle, and the tangents at B and C meet at D. Prove that [OD] and [AC] are parallel.

18 Triangle PQR is inscribed in a circle. [ST] is parallel to the tangent at P, intersecting [PQ] at S and [PR] at T. Prove that SQRT is a cyclic quadrilateral.

19 Prove the converse of the exterior angle of a quadrilateral theorem.

20



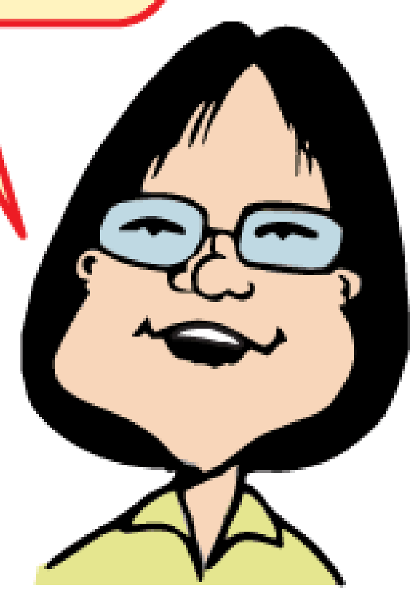
DEMO

PAB is a wooden set square in which $A\widehat{P}B$ is a right angle. The set square is free to move so that A is always on the x-axis and B is always on the y-axis.

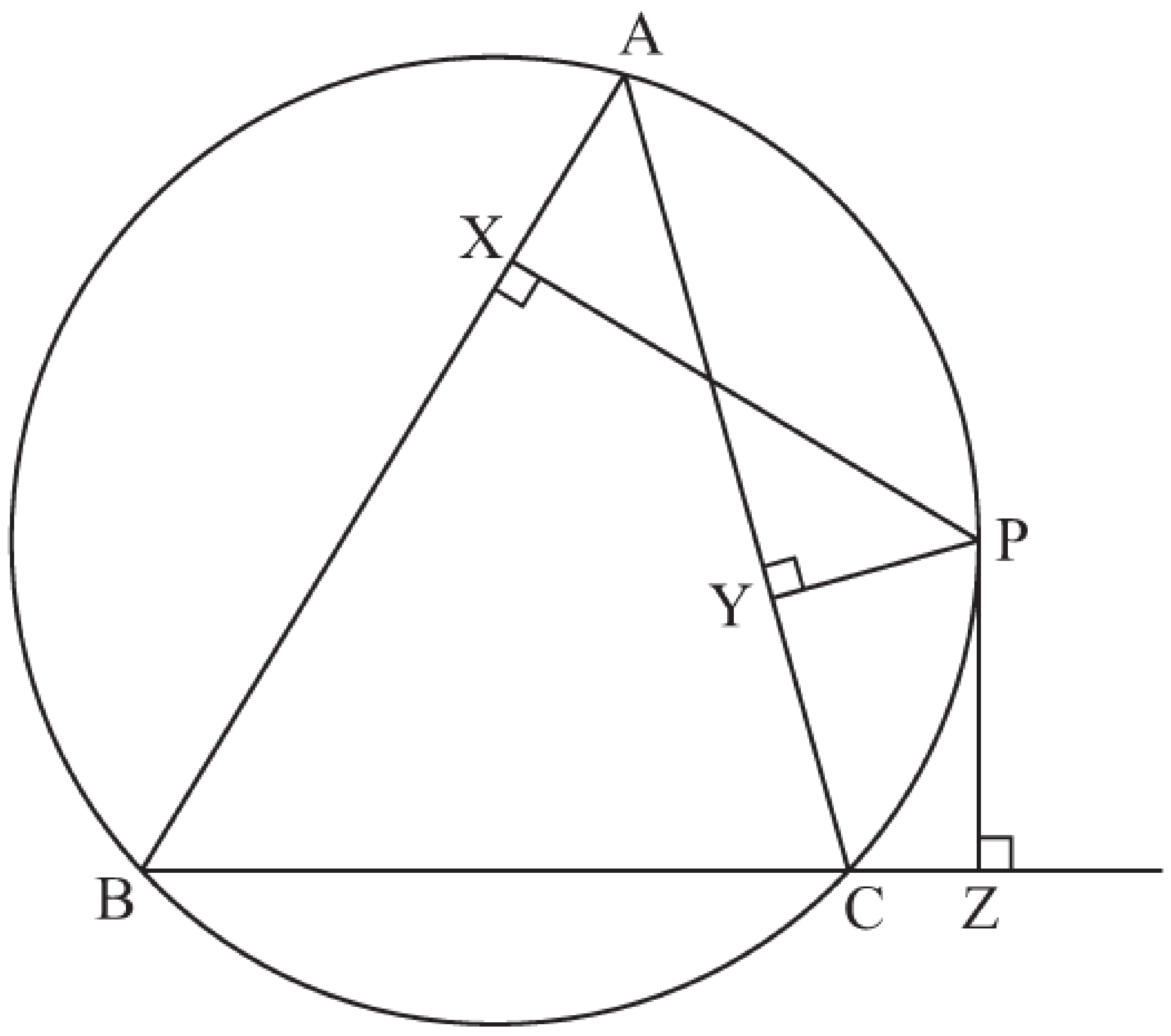
Without using coordinate geometry methods, show that the point P always lies on a straight line segment which passes through O.

21 Prove that if a line segment [AB] subtends equal angles at C and D, then A, B, C, and D are concyclic.

22 P is any point on the circumcircle of $\triangle ABC$ other than at A, B, or C. Altitudes [PX], [PY], and [PZ] are drawn to the sides of $\triangle ABC$ (or the sides produced).
Prove that X, Y, and Z are collinear.



[XYZ] is known as **Simson's line**.



23 Triangle ABC has altitudes [AX] and [BY]. P and Q are the midpoints of [AC] and [BC] respectively. Prove that points P, Q, X, and Y are concyclic.

INVESTIGATION 1 HART'S INVERSOR

The mechanism in the picture is called **Hart's inversor**, invented by Harry Hart in about 1874.

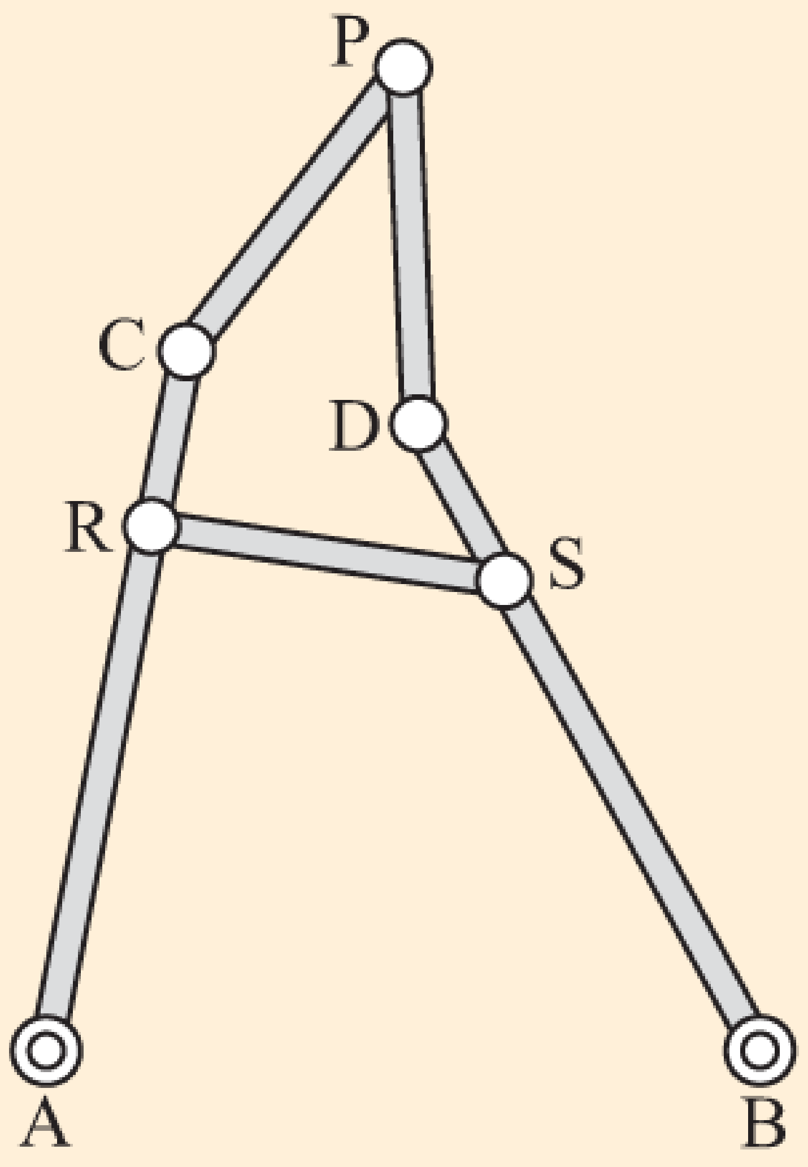
A and B are fixed ends of bars [AC] and [BD] with AC = BD = a.

Bars [PC], [PD], and [RS] are equal in length with PC = PD = RS = b.

R and S are located such that CR = DS = c.

The bars are made so that $b^2 = ac$.

A pencil is placed at P, and as P moves, a path is traced out.



What to do:

1. Discuss with your class how you can draw a straight line. How do you know that a ruler is straight?
2. Make Hart's inversor from wood (or metal) with $a = 16$ cm, $b = 8$ cm, $c = 4$ cm. What is the locus of points described by the movement of P?
3. Use deductive geometry to prove your proposition when $a = 8$ cm, $b = 4$ cm, $c = 2$ cm.
4. Prove the general case of your proposition with $b^2 = ac$.

F INTERSECTING CHORDS AND SECANTS THEOREMS

A **secant** of a circle is a line that intersects the circle twice.
A **chord** is the line segment which connects the points of intersection of the secant.

INVESTIGATION 2 INTERSECTING CHORDS AND SECANTS

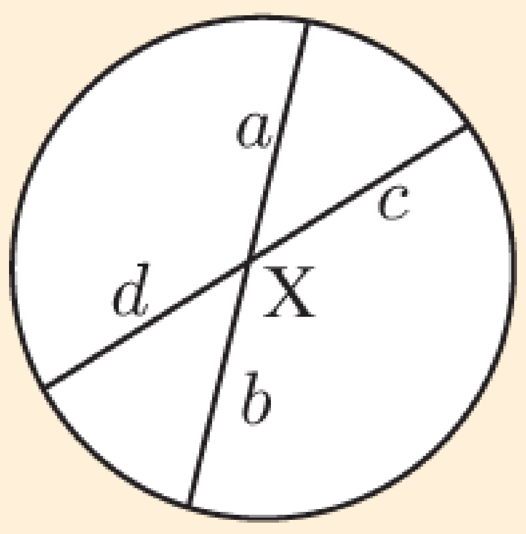
INTERSECTING CHORDS AND SECANTS THEOREMS

Click on the icon to access software for investigating the intersection of chords and secants.

What to do:

For each of the following cases, use the software to find the connection between the variables.

1



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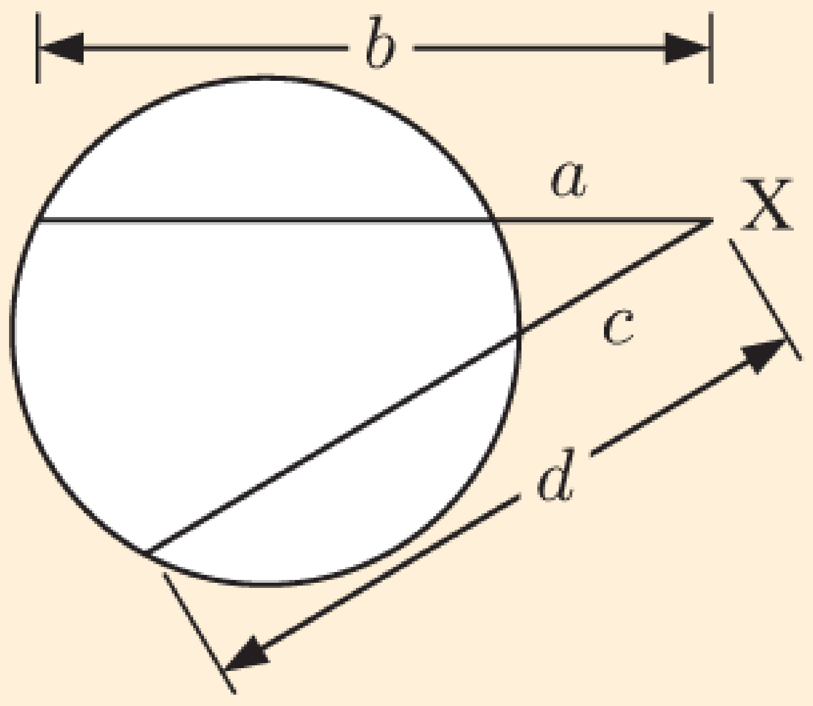

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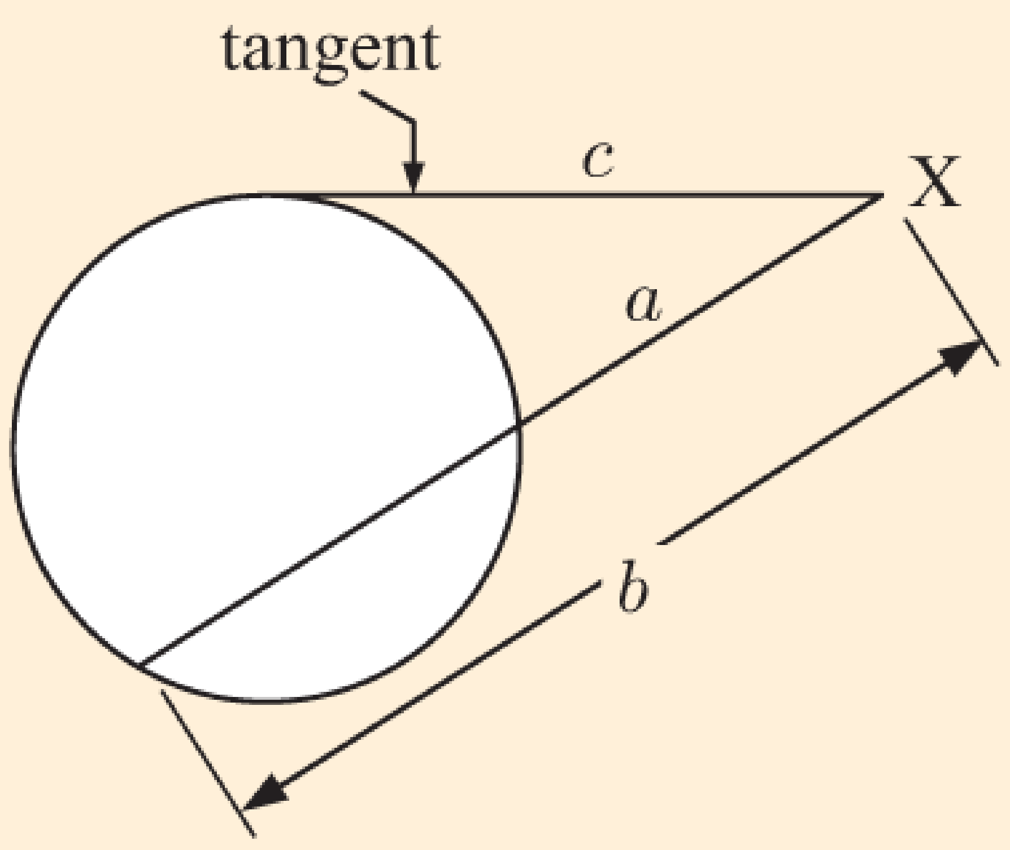
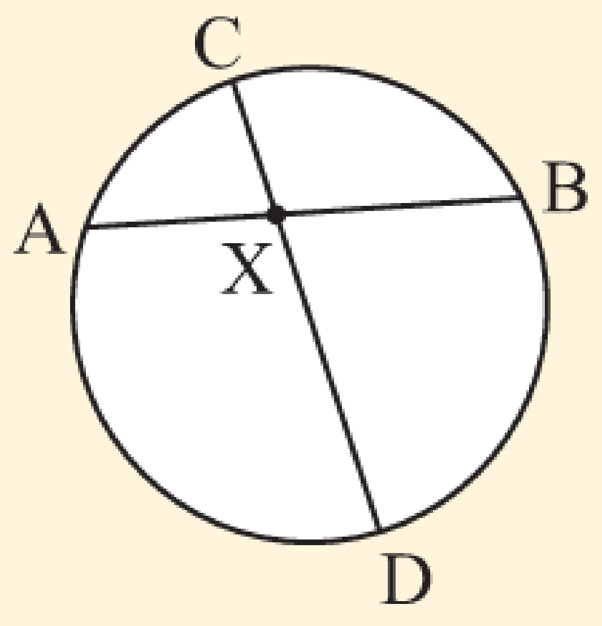
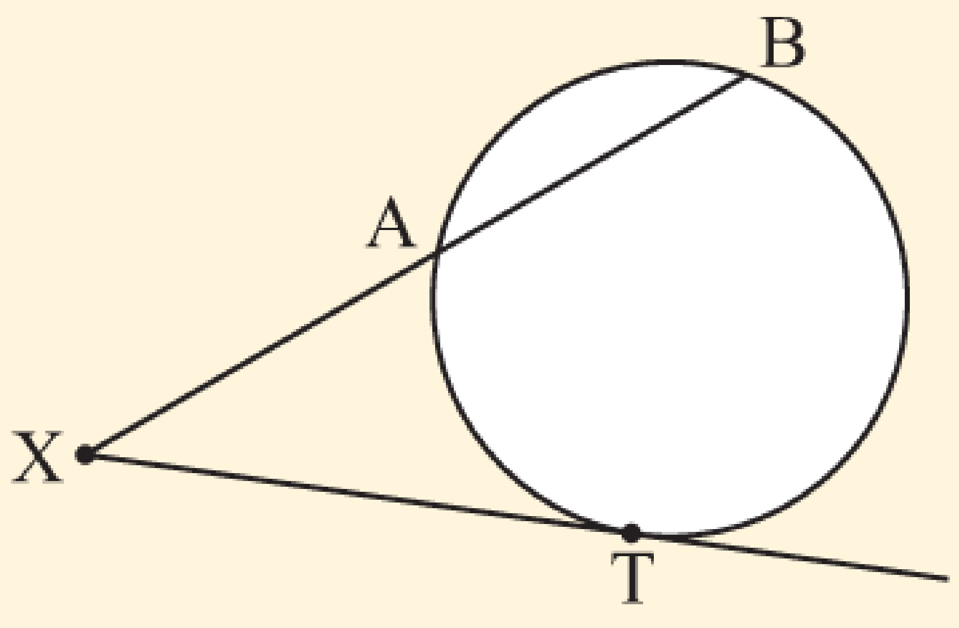
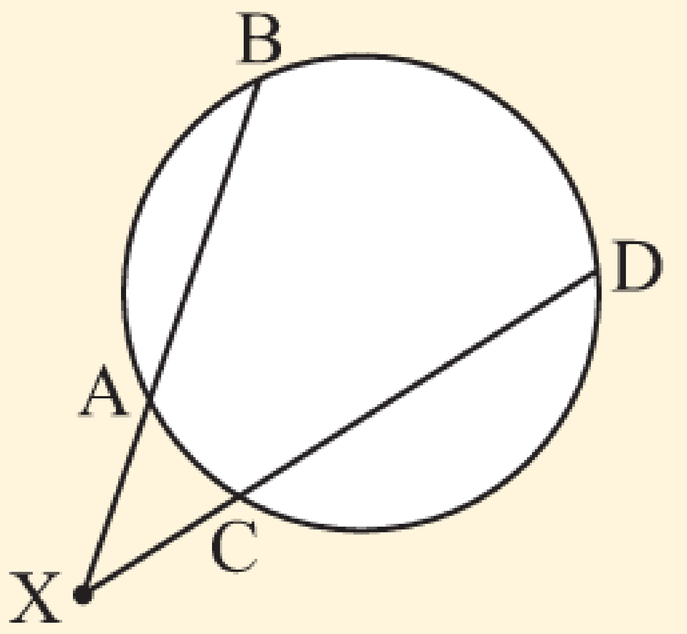


From the **Investigation** you should have discovered:

Name of theorem	*Statement*	*Diagram*
Intersecting chords or chord-chord	If chords [AB] and [CD] intersect at X, then $AX.BX = CX.DX$.	 $AX.BX = CX.DX$
Secant-tangent	If the tangent at T to a circle meets the chord [BA] produced at X, then $(XT)^2 = XA.XB$.	 $(XT)^2 = XA.XB$

Name of theorem	*Statement*	*Diagram*
Secant-secant	If X is outside a circle, and XAB and XCD are two secants, then XA.XB = XC.XD.	 XA.XB = XC.XD

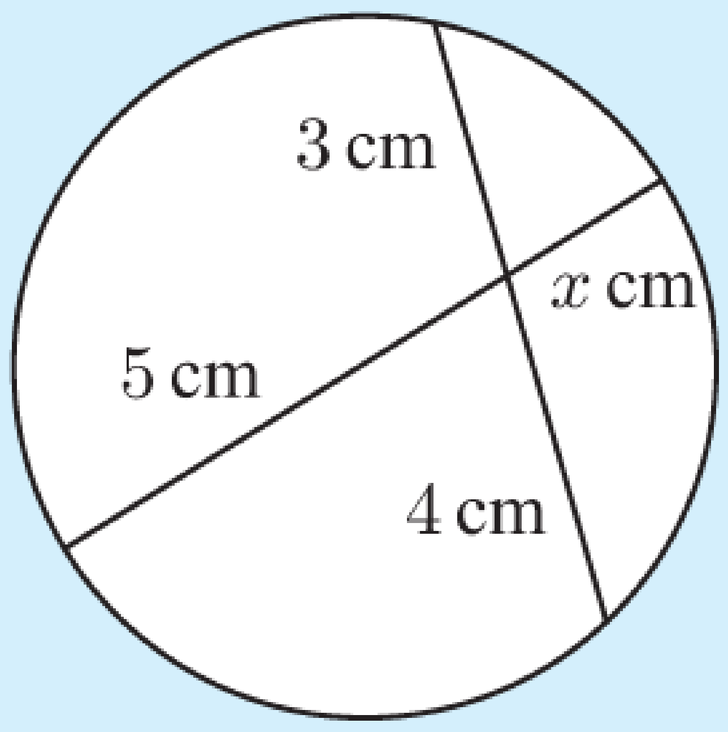
The converse of each theorem also holds:

- Suppose [AB] and [CD] (or their line extensions) meet at X. If AX.BX = CX.DX, then ABCD is a cyclic quadrilateral.
- Suppose A and B are distinct points on a circle, and X is a point outside the circle that is collinear with A and B. If T is a point on the circle such that $(XT)^2 = XA.XB$, then [XT] is a tangent.

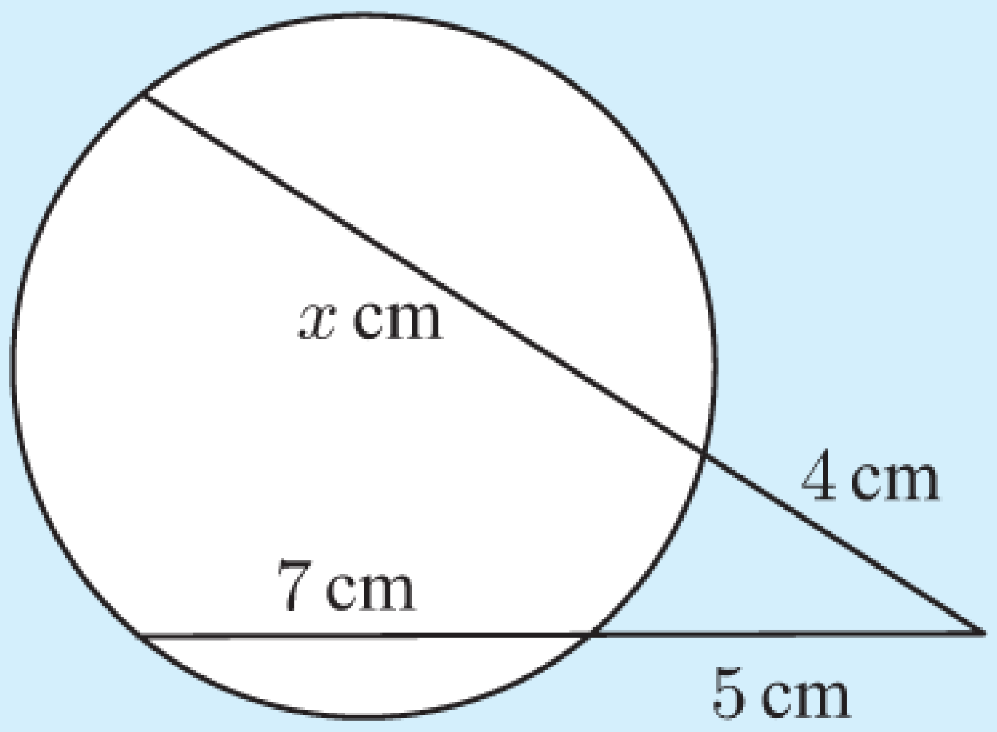
Example 11

Find x in:

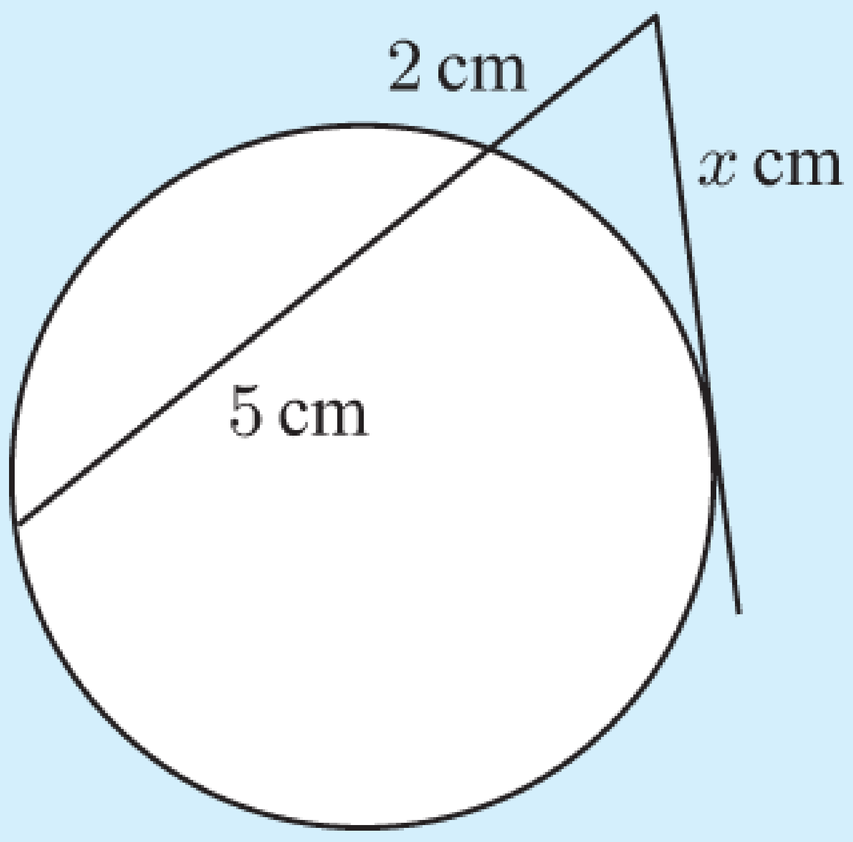
a



b



c



a By the intersecting chords theorem,

$$x \times 5 = 3 \times 4$$
$$\therefore \quad 5x = 12$$
$$\therefore \quad x = 2.4$$

b By the secant-secant theorem,

$$4(4+x) = 5 \times (5+7)$$
$$\therefore \quad 4(4+x) = 5 \times 12$$
$$\therefore \quad 4+x = 15$$
$$\therefore \quad x = 11$$

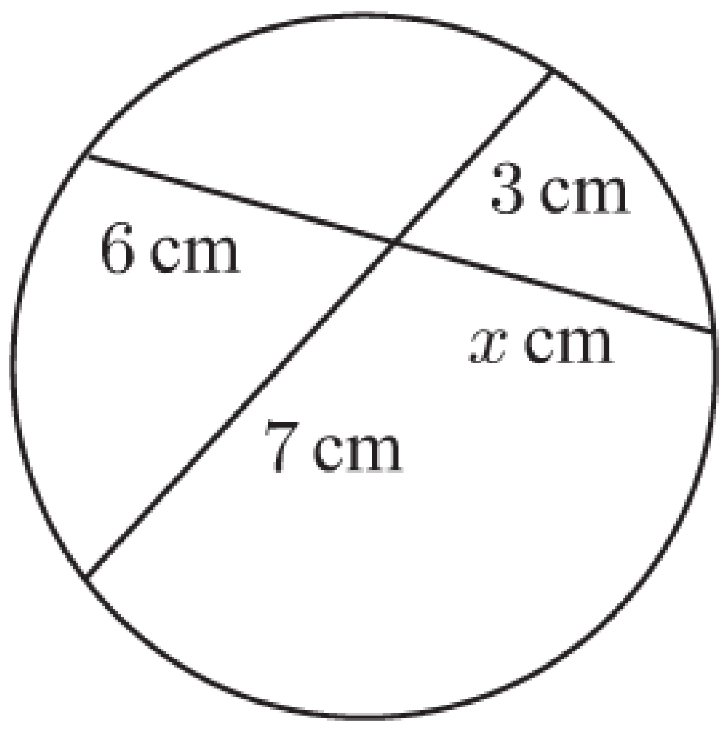
c By the secant-tangent theorem,

$$x^2 = 2 \times 7$$
$$\therefore \quad x^2 = 14$$
$$\therefore \quad x = \sqrt{14} \quad \{\text{as } x > 0\}$$

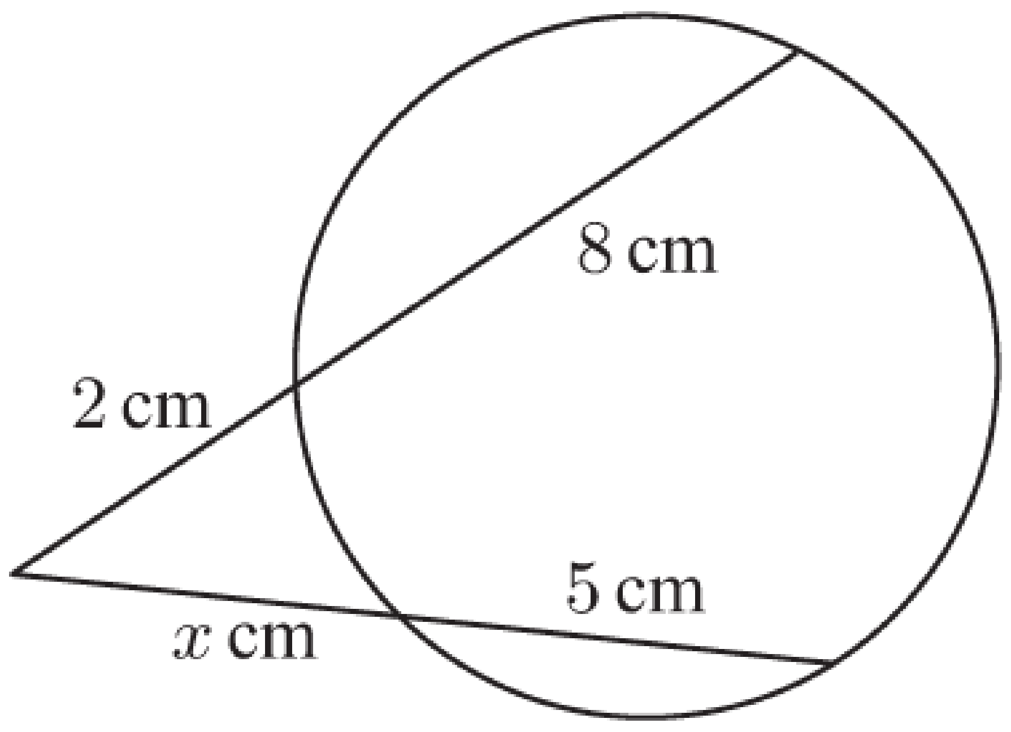
EXERCISE 2F

1 Find x:

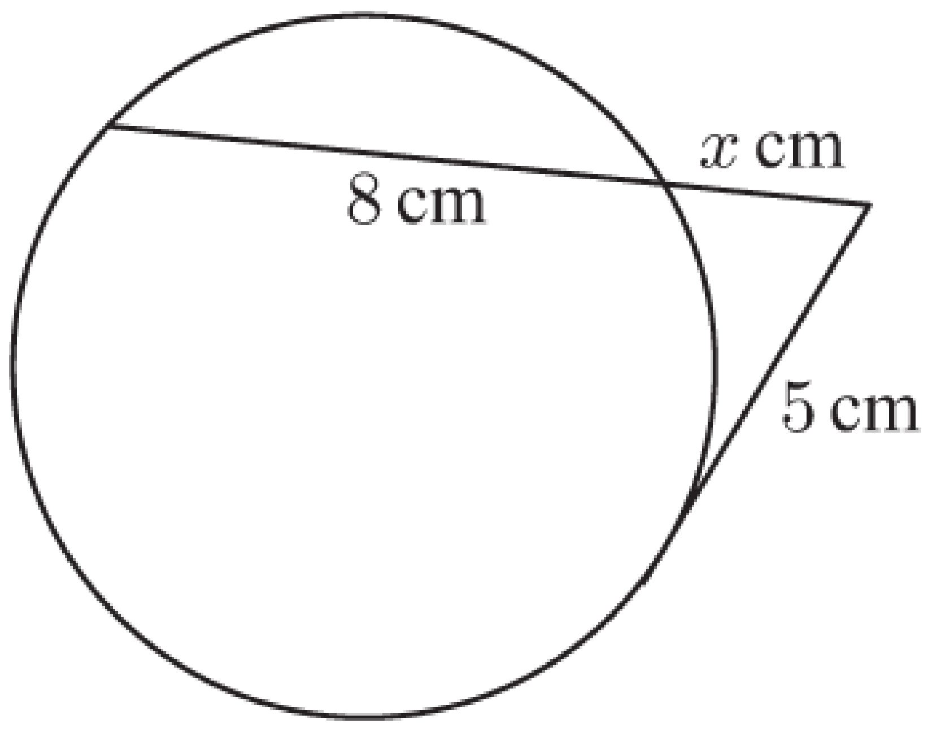
a



b



c



2 Chords [AB] and [CD] meet at X inside the circle.

- **a** If $AX = 4$ cm, $BX = 6$ cm, and $CX = 5$ cm, find the length of [DX].
- **b** If $AX = 2$ cm, $AB = 8$ cm, and $CX = 3$ cm, find the length of [CD].
- **c** If $AX = 3$ cm, $BX = 5$ cm, and $CD = 9$ cm, find the length of [CX].
- **d** If $BX = 2 \times AX$, $DX = 3$ cm, and $CD = 7$ cm, find the length of [AB].

3 X lies on a chord [AB] of a circle. $AX = 3$ cm and $BX = 5$ cm. If O is the circle's centre and $OX = 4$ cm, find the radius of the circle.

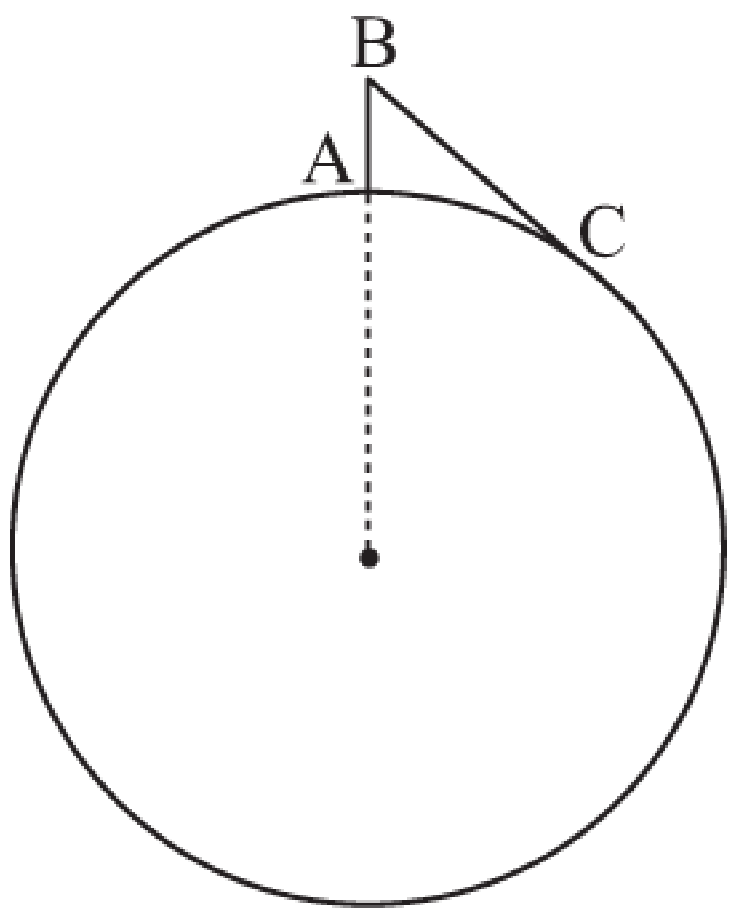
4 Chords [AB] and [CD] of a circle are produced to X, where X is outside the circle.

- **a** If $BX = 4$ cm, $BA = 2$ cm, and $DX = 3$ cm, find the length of [CD].
- **b** If $AX = 3 \times BX$, $DX = 3$ cm, and $CX = 11$ cm, find the length of [AB].

5 Consider a point X outside a circle with centre O. Secant XAB is drawn cutting the circle at points A and B. [XT] is a tangent, with T the point of contact.

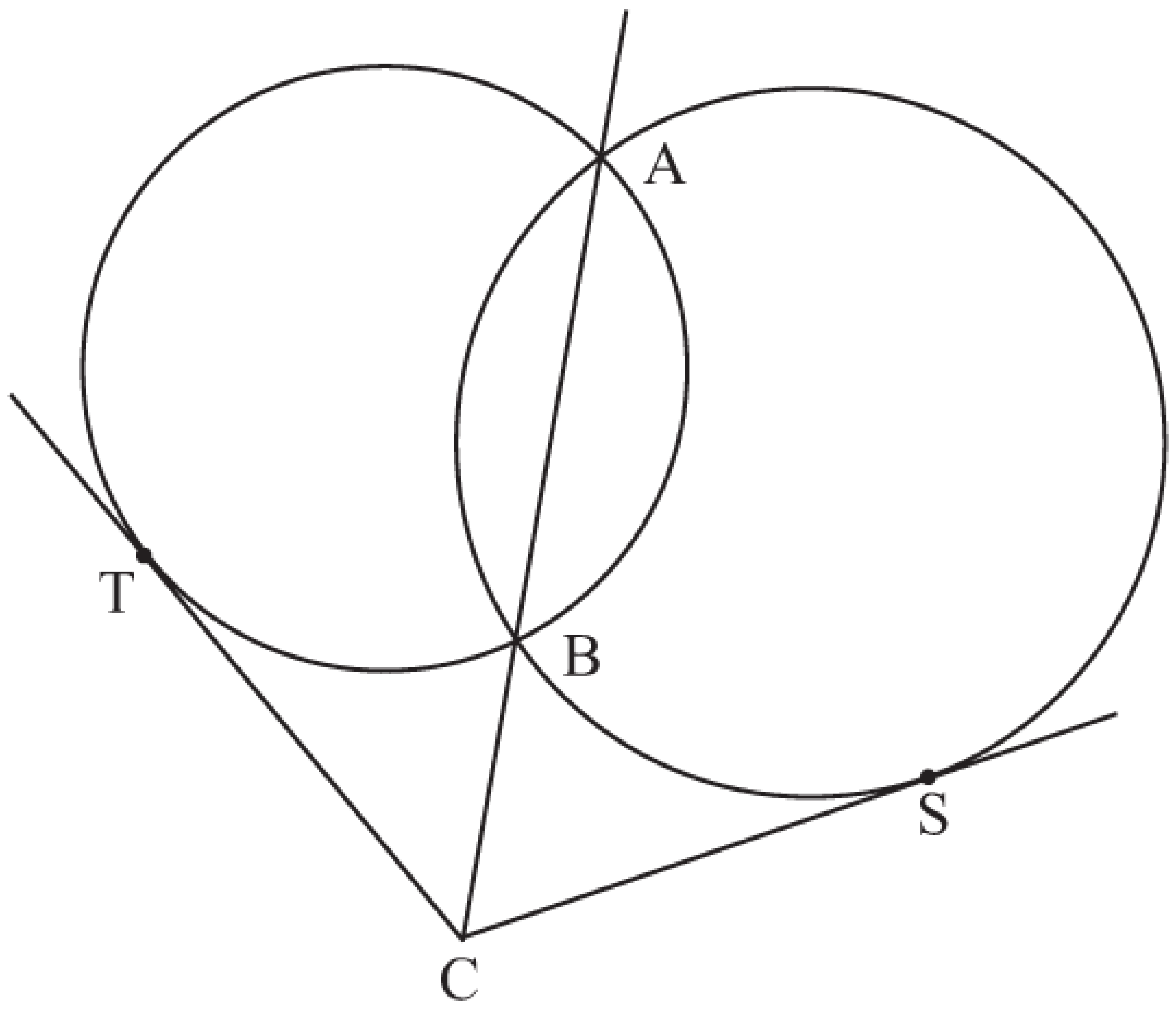
- **a** If $XT = 6$ cm and $XA = 4$ cm, find the length of [BX].
- **b** If $XA = 2$ cm and $AB = 3$ cm, find the length of [XT].
- **c** If $XA = 8$ cm, $AB = 2$ cm, and $OA = 5$ cm, find the length of [OX].

6 The radius of the Earth is about 6370 km. Point B is directly above point A on the Earth's surface. The distance from point B to the visible horizon is the length of the tangent [BC].



- **a** Find the distance to the visible horizon from the observers in a space shuttle 400 km above the Earth's surface.
- **b** Show that if B is height h km above the Earth's surface, then the distance to the visible horizon is given by $D \approx \sqrt{h^2 + 12\,740h}$ km.

7 Two circles intersect at A and B.
C is any point on the common chord [AB] produced.
Prove that the tangents [CS] and [CT] are equal in length.



8 [AXB] and [CXD] are two intersecting line segments.
Prove that points A, B, C, and D are concyclic when:

- **a** $AX = 8$ cm, $BX = 7$ cm, $CX = 14$ cm, and $DX = 4$ cm
- **b** $AX = 5$ cm, $BX = 3.2$ cm, $CX = 8$ cm, and $DX = 2$ cm.

9 [XAB] and [XC] are two intersecting straight line segments.
Given that $BX = 6.4$ m, $AB = 5.5$ m, and $XC = 2.4$ m, prove that [CX] is a tangent to the circle through A, B, and C.

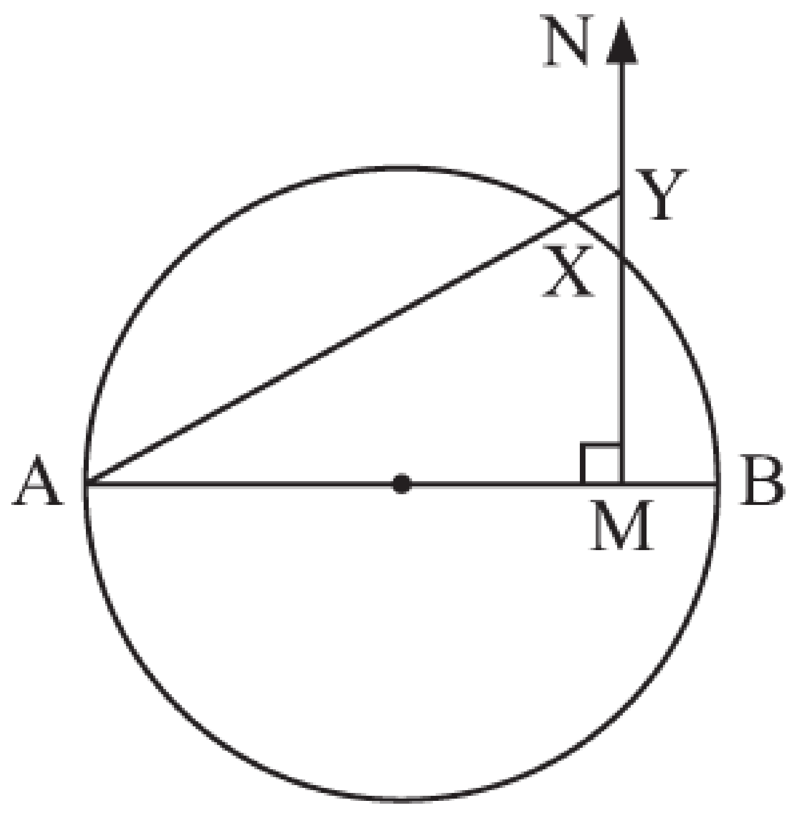
10 Point P is 7 cm from the centre of a circle with radius 5 cm. A secant is drawn from P which cuts the circle at A and B, A being closer to P. If $AB = 5$ cm, find the length of [AP].

11 Two circles have a common chord [CD]. [AB] is a common tangent to the circles. [DC] produced meets [AB] at X. Prove that X bisects [AB].

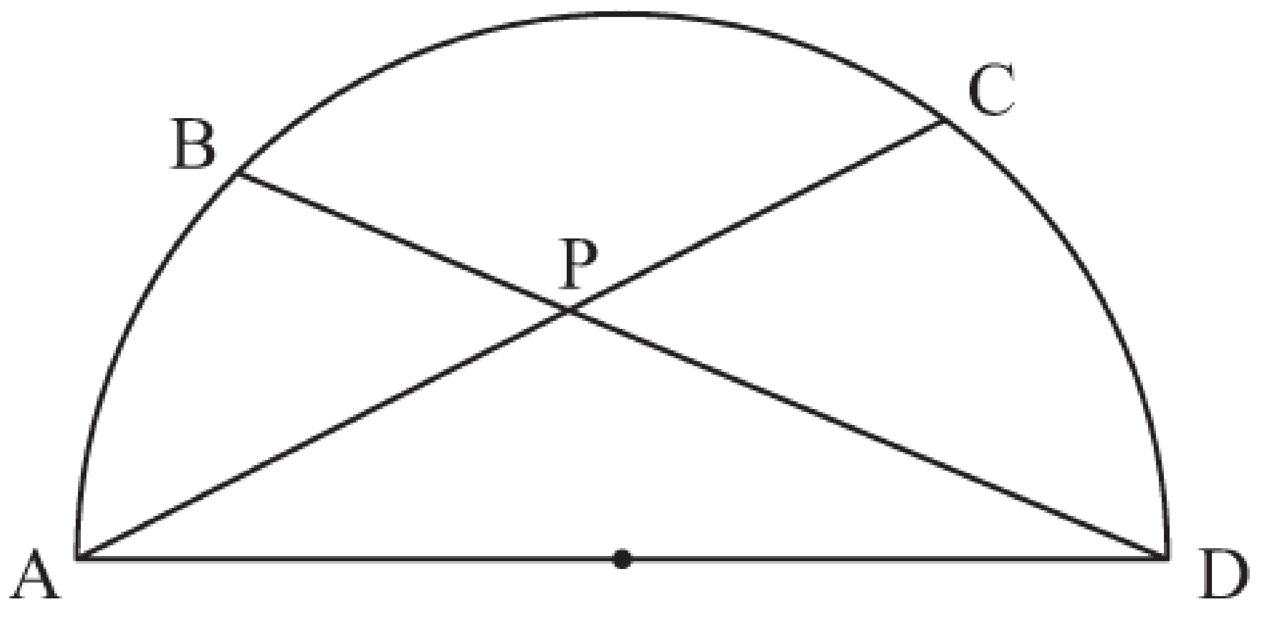
12 Two circles meet at P and Q. X lies on [PQ] produced. Line segment [XAB] is drawn to cut the first circle at A and B. Likewise, line segment [XCD] is drawn to cut the second circle at C and D. Prove that ACDB is a cyclic quadrilateral.

13 Two non-intersecting circles are cut by a third circle. The first circle is cut at A and B. The second circle is cut at C and D. When the common chords are extended, they meet at X. Prove that the tangents from X to all three circles are equal in length.

14 [AB] is a fixed diameter of a circle. The ray [MN) is a fixed perpendicular to [AB]. A line from point A cuts the circle at X and meets [MN) at Y. X is a moving point, and consequently Y moves on [MN).
Prove that $AX.AY$ is constant.



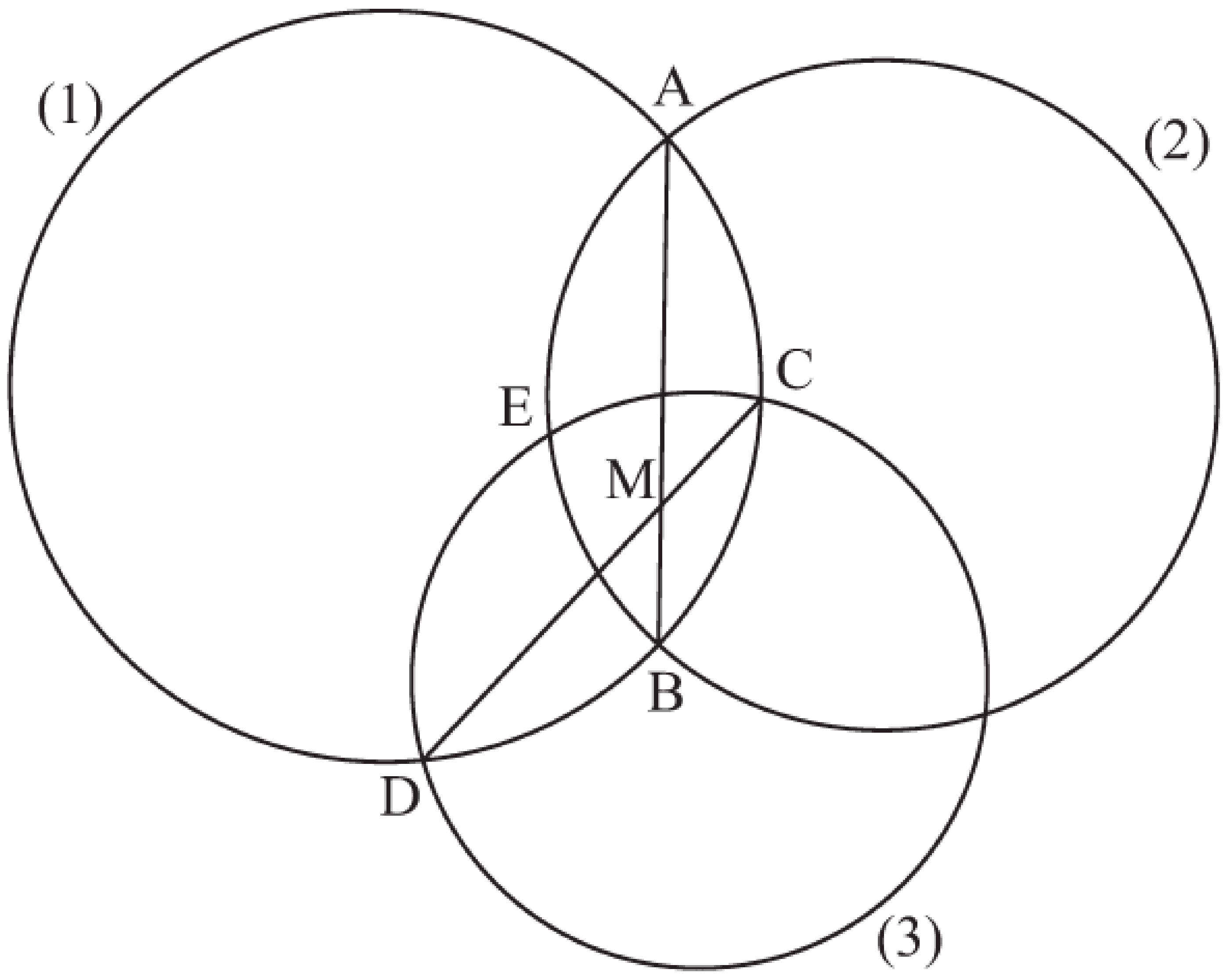
15



ABCD is a semi-circle with diameter [AD].
P is the point of intersection of [AC] and [BD].
Prove that $AP.AC + DP.DB = AD^2$.

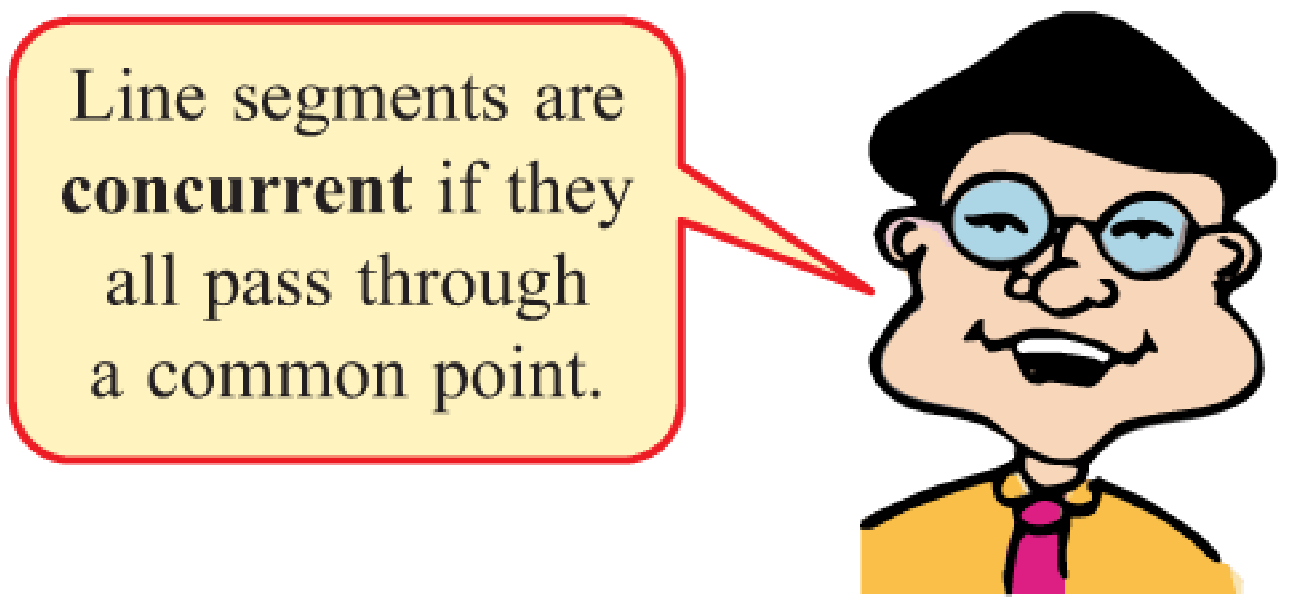
16 Suppose A, B, C, and D are points such that [AB] and [CD] intersect at X. If $XA.XB = XC.XD$, show that A, B, C, and D are concyclic.

17



Three circles intersect each other as shown.
Prove that the three common chords are concurrent.

Hint: Draw two common chords [AB] and [CD], and let them meet at M. From E, draw a chord [EMF] of circle (2), and a chord [EMG] of circle (3). Then show that F and G coincide.


Line segments are **concurrent** if they all pass through a common point.

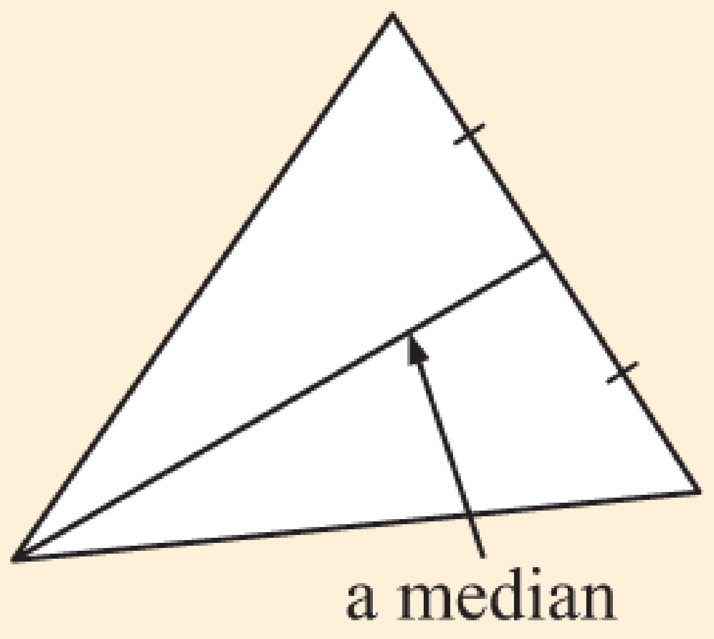
G CENTRES OF A TRIANGLE

INVESTIGATION 3 THE DIFFERENT CENTRES OF A TRIANGLE

In this Investigation we will use software demonstrations to discover properties of any general triangle.

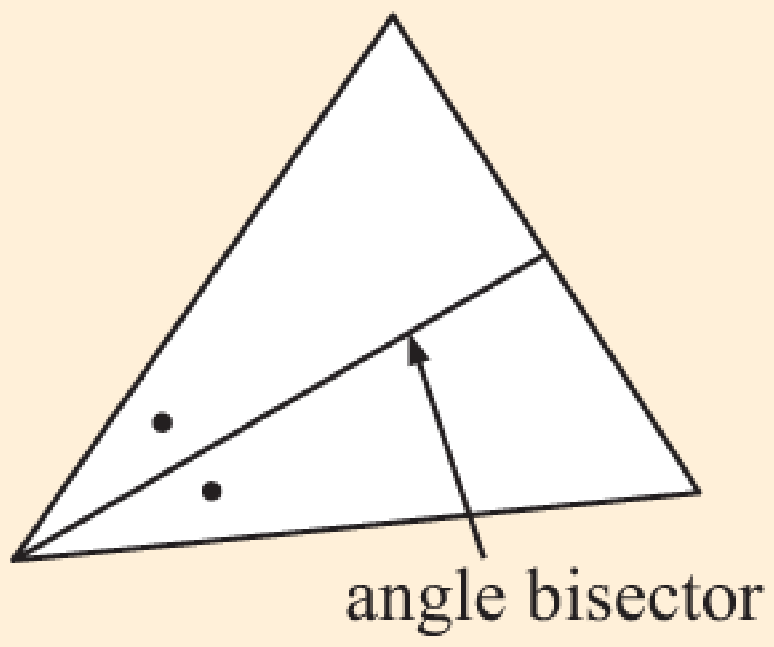
What to do:

1. A **median** of a triangle is any line segment from a vertex to the midpoint of the opposite side.
   Click on the icon and follow the instructions. Do not forget to change the triangle by clicking and dragging the vertices.
   Write down your observations and conclusions.



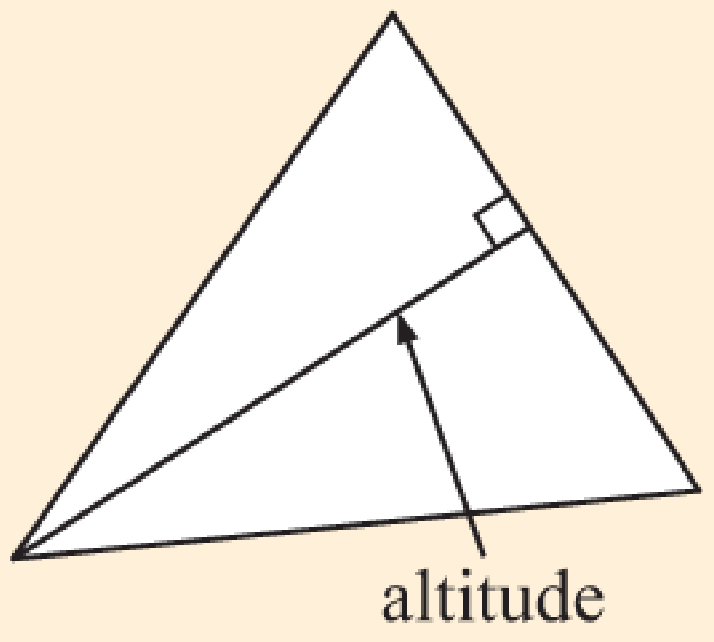
DEMO

2. An **angle bisector** of a triangle is any line segment from a vertex to the opposite side, which bisects the angle at the vertex.
   Click on the icon and follow the instructions.
   Write down your observations and conclusions.



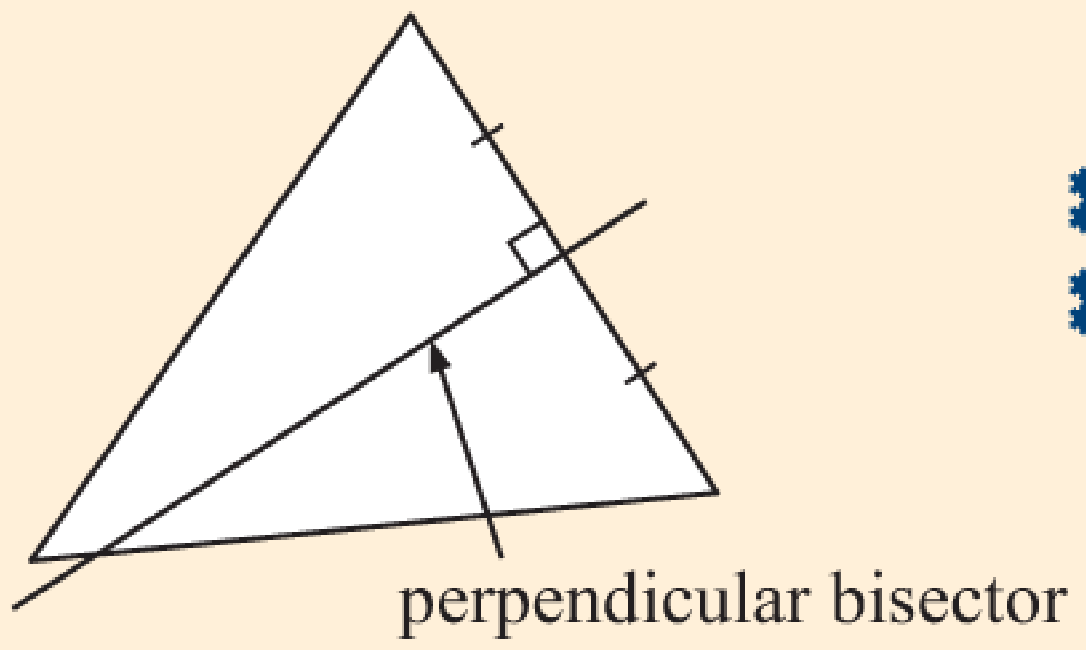
DEMO

3. An **altitude** of a triangle is any line segment from a vertex which meets the base (or the base extended) in a right angle.
   Click on the icon and follow the instructions. Make sure you consider obtuse angled triangles.
   Write down your observations and conclusions.



DEMO

4. A **perpendicular bisector** of a triangle is any line which is a perpendicular bisector of one of its sides.
   Click on the icon and follow the instructions. Make sure you consider obtuse angled triangles.
   Write down your observations and conclusions.



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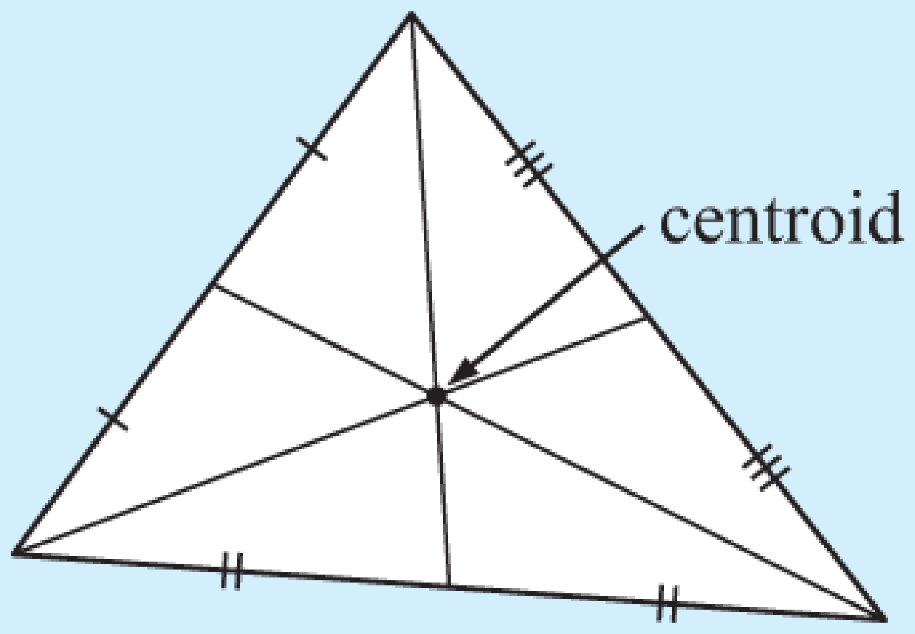
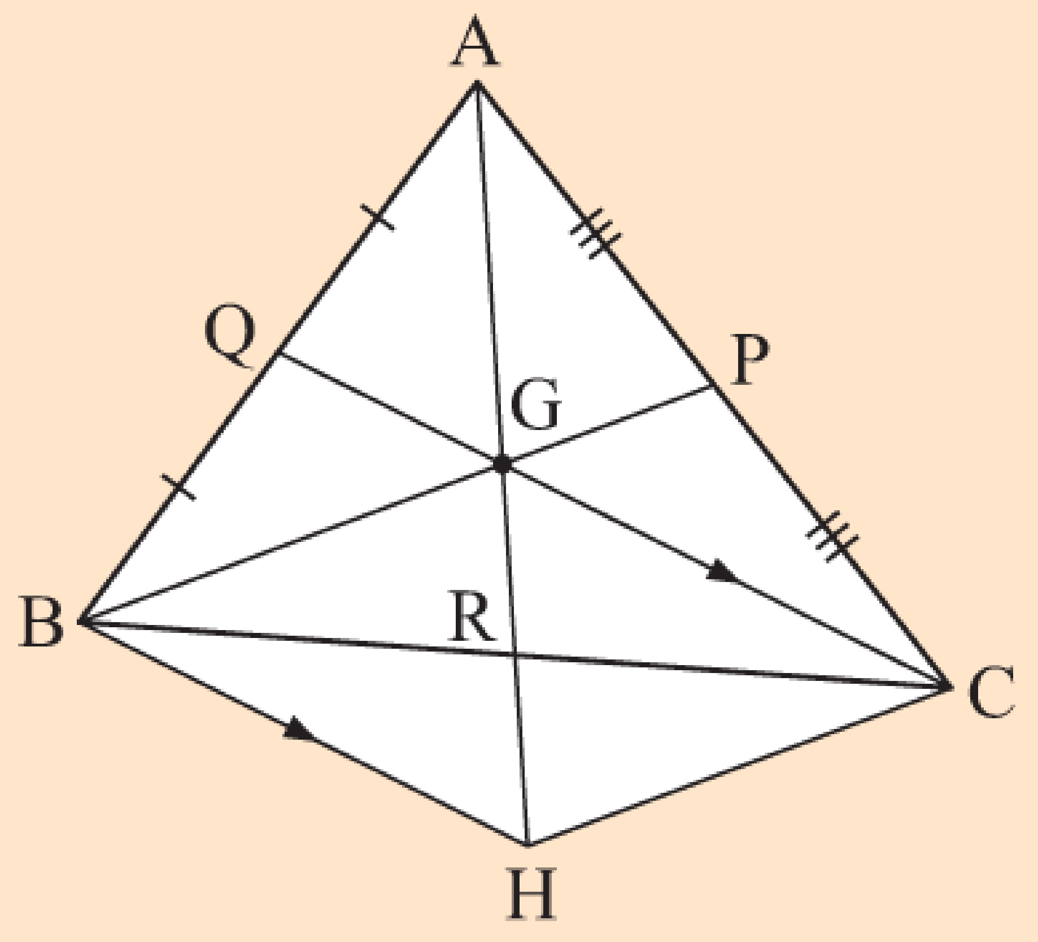
You should find that your observations from the **Investigation** are summarised in the theorems which follow.

Theorem:

The **medians** of a triangle are concurrent at a point called the **centroid**, and this point divides each median in the ratio 2 : 1.



Proof:



We start with $\triangle$ABC.

Let P be the midpoint of [AC], and Q be the midpoint of [AB].

Let [BP] and [CQ] intersect at G.

We now have to prove that:

BR $=$ RC, GR $= \frac{1}{3}$AR, GQ $= \frac{1}{3}$CQ, and GP $= \frac{1}{3}$BP.

We draw [BH] parallel to [QC], to meet [AR] produced at H. We then join [CH].

In $\triangle$ABH, [QG] $\parallel$ [BH].

$\therefore$ since Q is the midpoint of [AB], G is the midpoint of [AH]. {converse of midpoint theorem}

$\therefore$ [GP] is the line joining the midpoints of two sides of $\triangle$AHC

$\therefore$ [GP] $\parallel$ [HC] {midpoint theorem}

$\therefore$ [BG] $\parallel$ [HC]

$\therefore$ BGCH is a parallelogram

$\therefore$ BR $=$ RC {diagonals of a parallelogram}

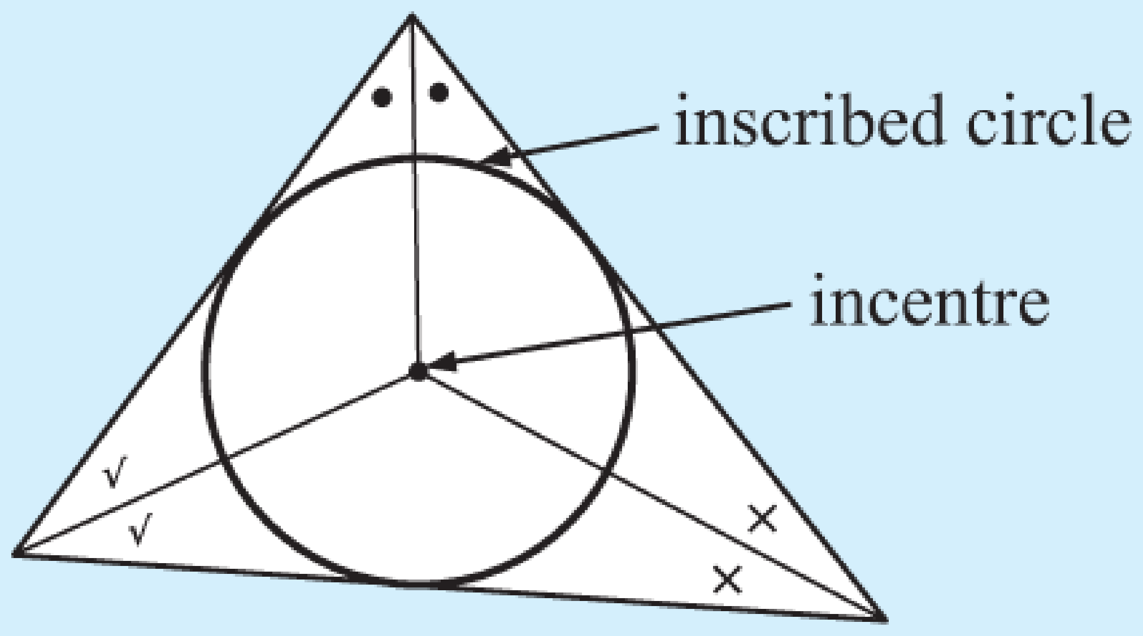
If RG $= a$ units, then RH $= a$ units and so AG $=$ GH $= 2a$ units

$\therefore$ AR $= 3a$ units and so RG $= \frac{1}{3}$AR

Using the same method, we can show that GQ $= \frac{1}{3}$CQ and GP $= \frac{1}{3}$BP.

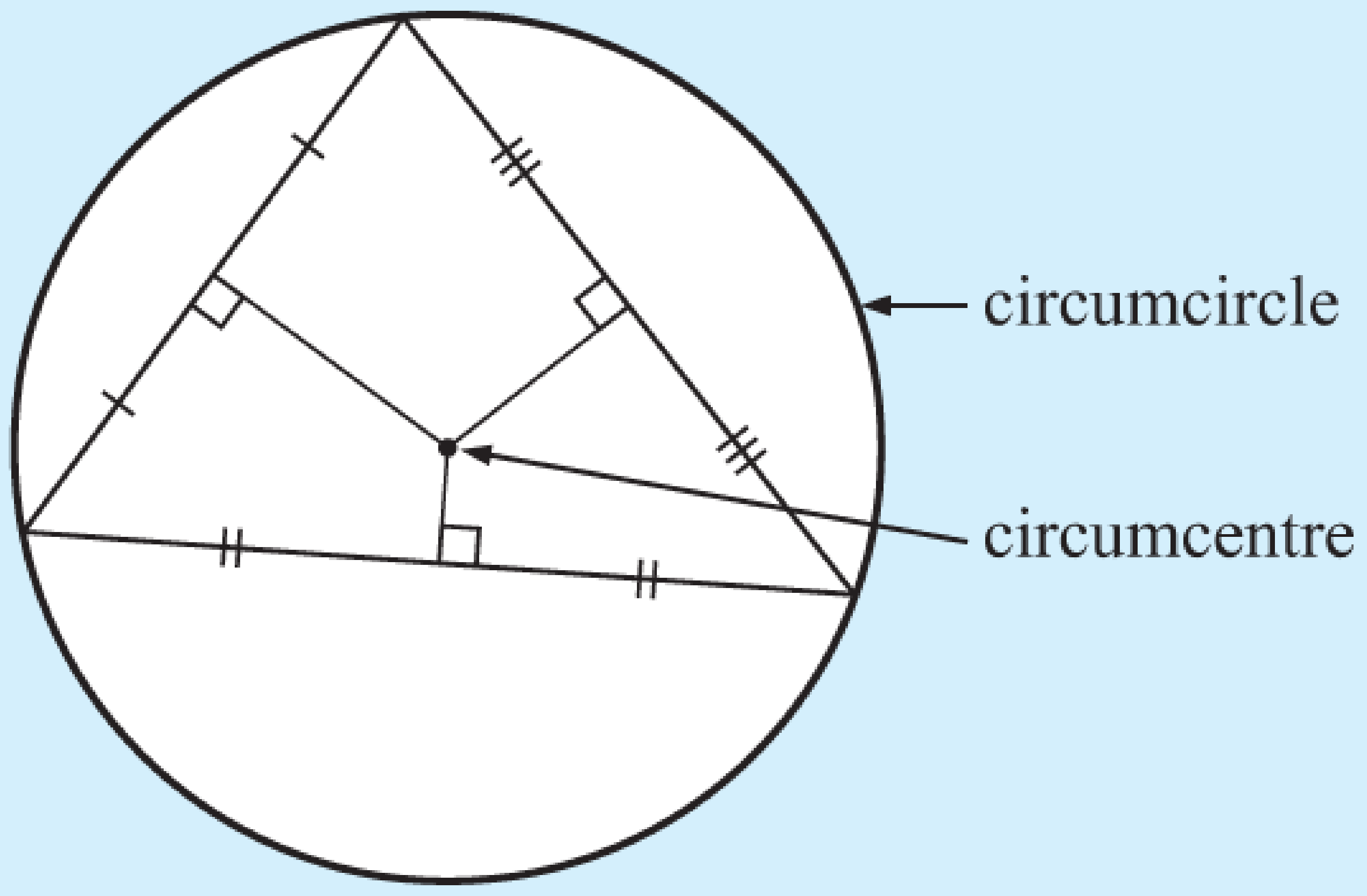
Theorem:

The **angle bisectors** of a triangle are concurrent at a point called the **incentre**, and a circle with this centre can be inscribed in the triangle.



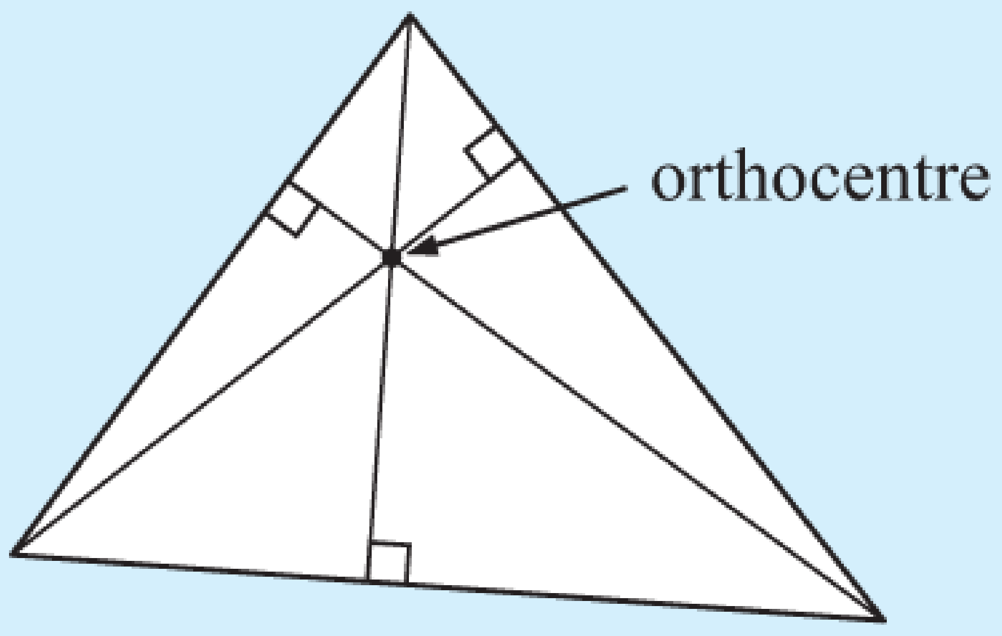
Theorem:

The **perpendicular bisectors** of the sides of a triangle are concurrent at a point called the **circumcentre**, and a circle with this centre can be drawn through the triangle's vertices.

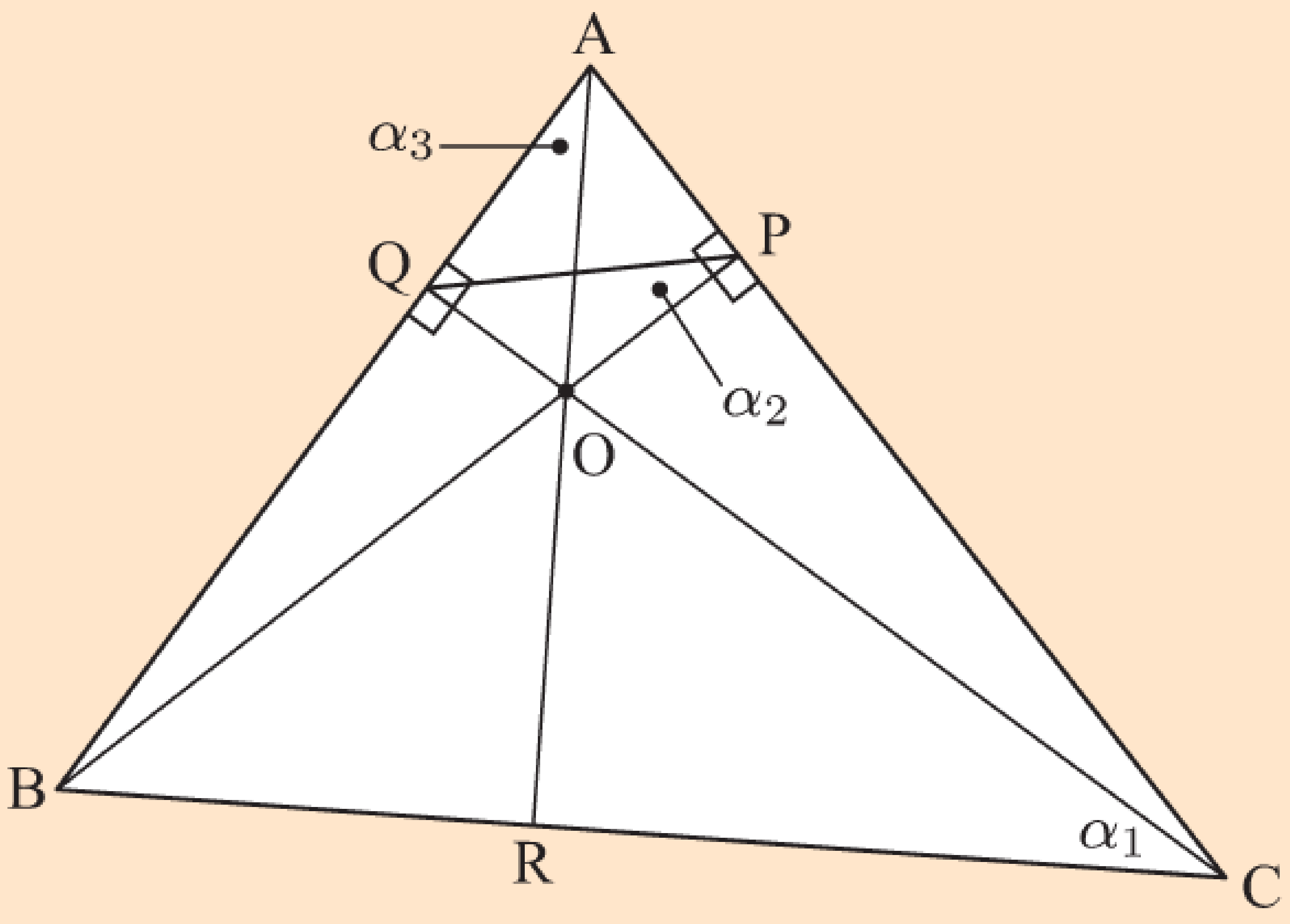


Theorem:

The three **altitudes** from vertices to opposite sides of a triangle are concurrent at a point called the **orthocentre**.



Proof:



We draw two of the altitudes, [BP] and [CQ].

Let O be the point where they meet.

We draw [AO], and produce it to meet [BC] at R.

We now need to prove that [AR] $\perp$ [BC].

We join [PQ].

Since [BC] subtends equal angles at P and Q, BCPQ is a cyclic quadrilateral. {cyclic quadrilateral theorem}

$\therefore$ $\alpha_1 = \alpha_2$ {angles subtended by the same arc}

But APOQ is also a cyclic quadrilateral as its opposite angles at P and Q are supplementary (both right angles).

$\therefore$ $\alpha_2 = \alpha_3$ {angles subtended by the same arc}

$\therefore$ $\alpha_1 = \alpha_3$

$\therefore$ [QR] subtends equal angles at C and A.

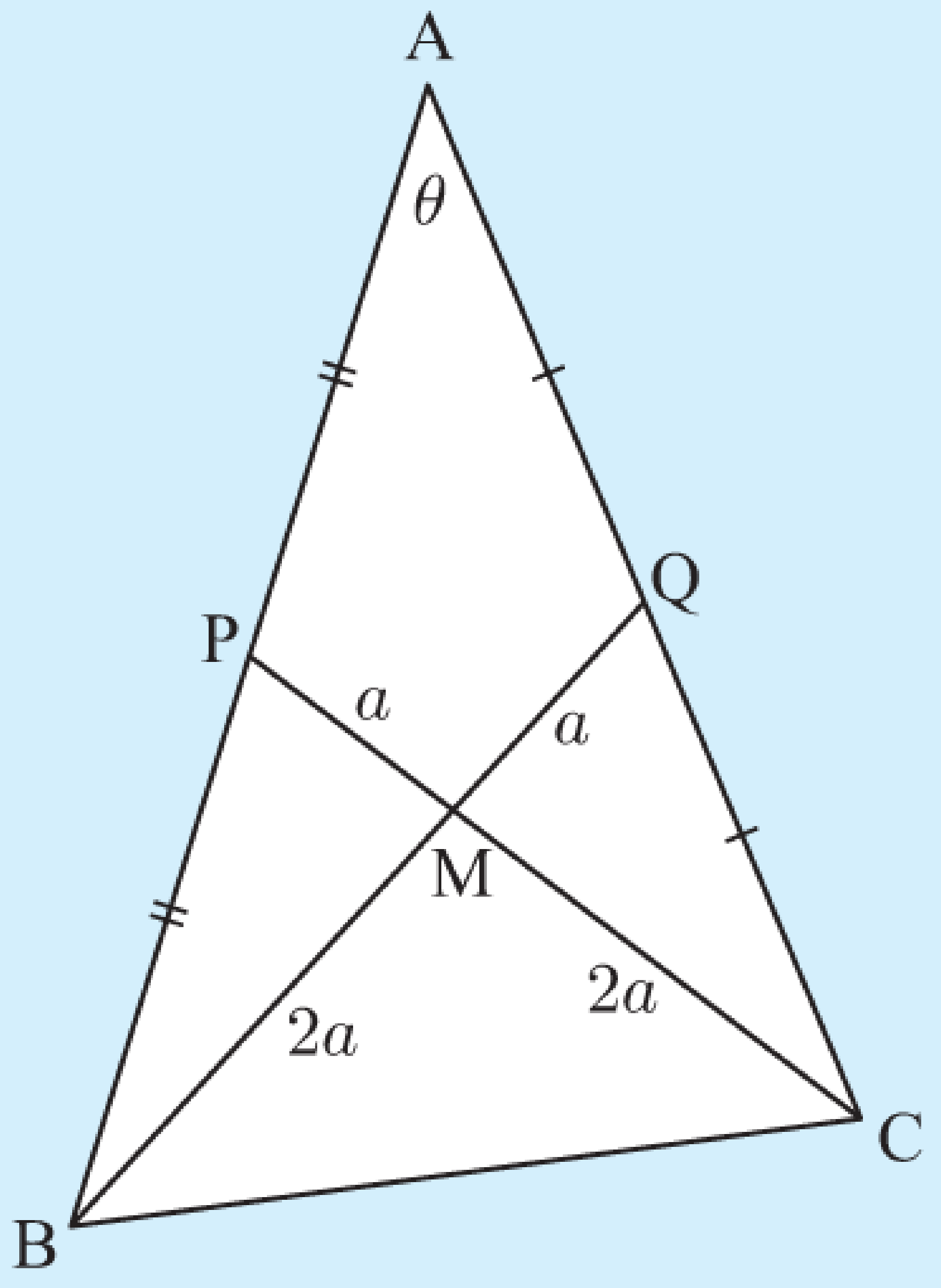
$\therefore$ QRCA is a cyclic quadrilateral.

Thus, [AC] subtends equal angles at Q and R, and since the angle at Q is a right angle, $A\widehat{R}C$ is a right angle also.

Thus [AR] $\perp$ [BC].

Example 12

Prove that if two medians of a triangle are equal in length then the triangle is isosceles.



Consider triangle ABC with medians [BQ] and [CP] of equal length.

Let these medians meet at the centroid M, and let PM $= a$.

Since M is a point of trisection of the medians, MC $= 2a$, MQ $= a$, and BM $= 2a$.

In triangles BMP and CMQ:

- MP = MQ
- BM = CM
- $B\widehat{M}P = C\widehat{M}Q$ {vertically opposite angles}

$\therefore$ the triangles are congruent.

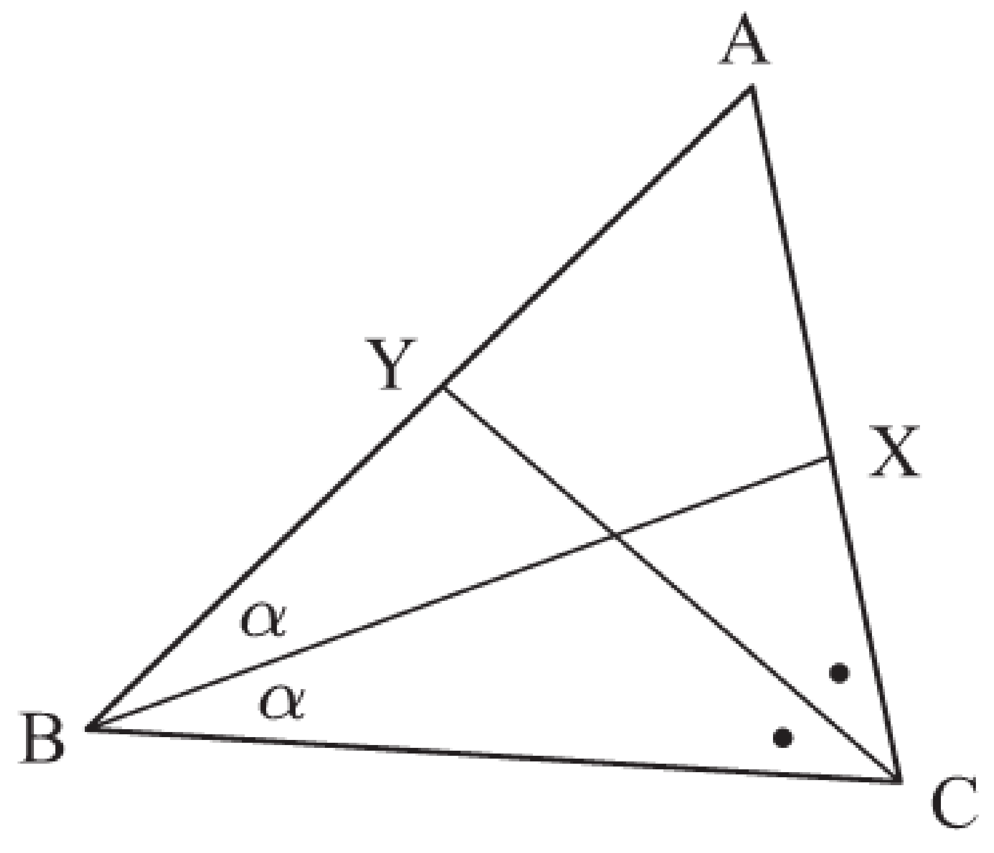
$\therefore$ BP = CQ

$\therefore$ AB = AC

$\therefore$ $\triangle$ABC is isosceles.

EXERCISE 2G

1



a How many circles can be drawn with a centre on [BX], which *touch* [BA] and [BC]?

b How many circles can be drawn with a centre on [CY], which *touch* [CB] and [CA]?

c What can you conclude from **a** and **b**?

d Explain how **a**, **b**, and **c** can be used to prove the angle bisectors of a triangle theorem.

2 X is the midpoint of side [CD] of parallelogram ABCD, and [BX] meets [AC] at Y. Prove that [DY] produced bisects [BC].

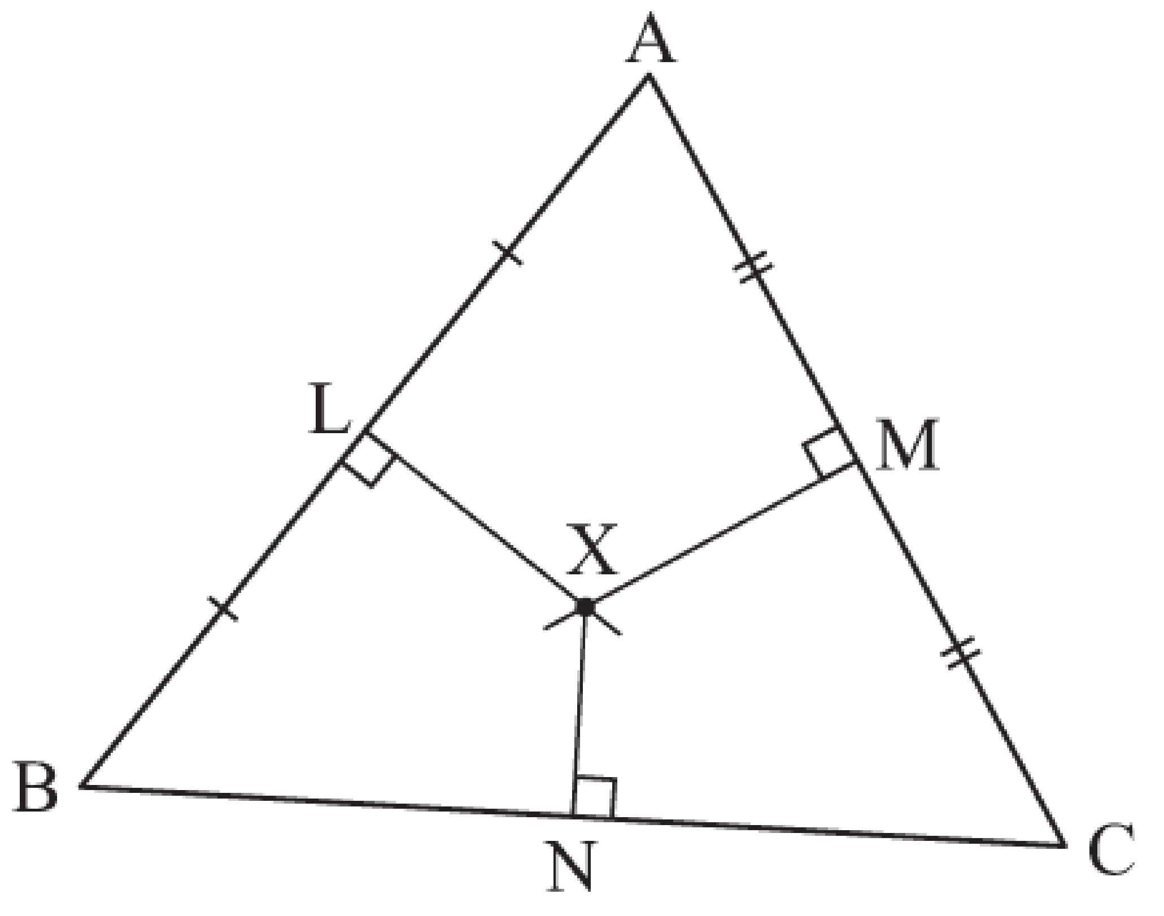
3 Consider the given figure.

a What can you conclude from $\triangle$ABX?

b What can you conclude from $\triangle$ACX?

c What do **a** and **b** tell us about $\triangle$BCX? Give reasons for your answer.

d What can be deduced from **c**?



4 Triangle ABC has centroid G. [AX] is a median of the triangle. Prove that $\triangle$GBX has $\frac{1}{6}$ of the area of $\triangle$ABC.

5 Through the centroid of a triangle, lines are drawn parallel to two sides of the triangle. Prove that these lines trisect the third side.

6 A circle with centre O has diameter [AB]. P is a point outside the circle, not on the line through A and B, such that AP = AB. If [PB] cuts the circle at R, and [OP] and [AR] meet at X, prove that [XP] is twice as long as [OX].

7 Two circles of equal radius touch externally at B. [AB] is the diameter of one circle, and [CD] is any diameter of the other circle. Prove that [CB] produced bisects [AD].

8 Through the vertices of $\triangle$PQR, lines are drawn which are parallel to the opposite sides of the triangle. The new triangle formed is $\triangle$ABC. Prove that $\triangle$s ABC and PQR have the same centroid.

9 Triangle ABC has centroid G. [BC] is fixed, and A moves such that $\widehat{CGB}$ is always a right angle. Find the locus of A.

10 PQRS is a rhombus. [PM] is perpendicular to [QR], and meets [QS] at Y and [QR] at M. Prove that [RY] is perpendicular to [PQ].

11 PQRS is a parallelogram. A and B are the orthocentres of triangles PQR and PSR respectively. Prove that PARB is also a parallelogram.

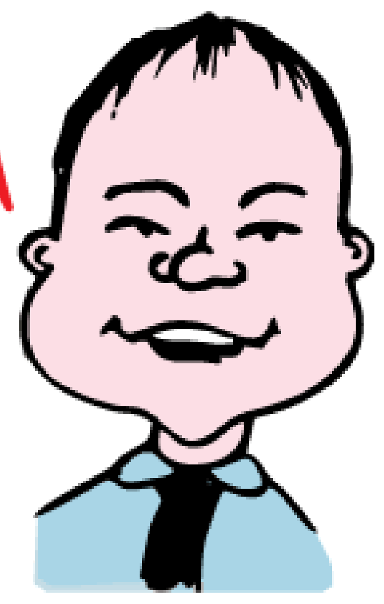
12 Triangle PQR has altitudes [PA], [QB], and [RC] which meet at H. Prove that:

a PH.PA = PB.PR **b** PH.HA = QH.HB = RH.HC

The **locus** of A is the set of all points where A could be.

13 [AP] and [BQ] are altitudes of $\triangle$ABC, and O is the orthocentre of the triangle. X and Y are the midpoints of [AB] and [OC] respectively. Prove that [XY] bisects [PQ] at right angles.

14 Triangle PQR has orthocentre O, and [RS] is a diameter of the circumcircle of the triangle. Prove that SQOP is a parallelogram.



The **circumcircle** is the circle through the vertices of the triangle.

15 Prove that in any triangle, the centroid, orthocentre, and circumcentre are collinear, and that the centroid divides the line joining the circumcentre and orthocentre in the ratio 2 : 1.

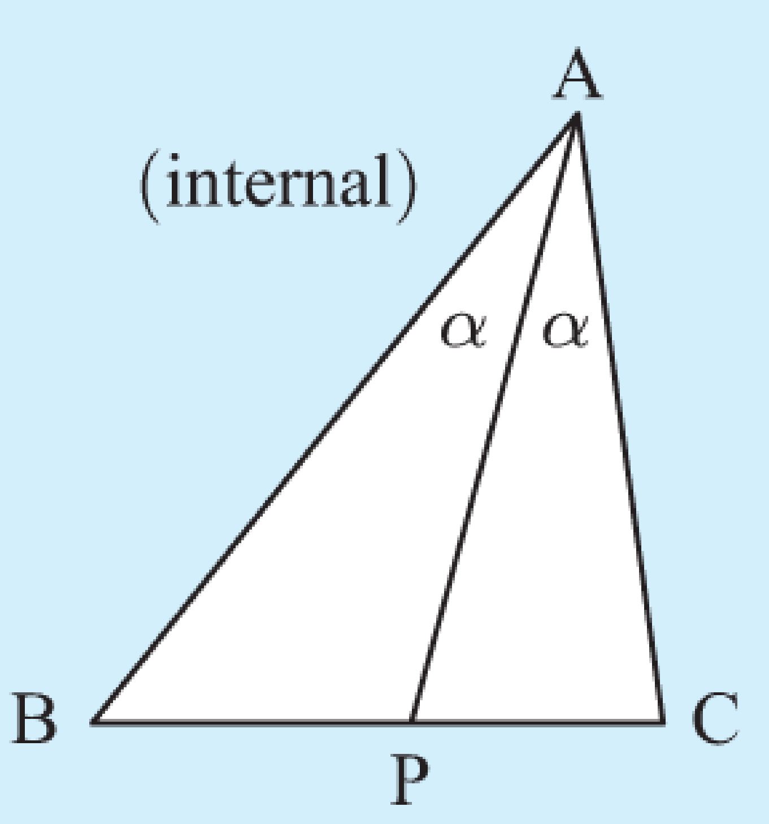
Hint: In $\triangle$ABC, locate the centroid G and the circumcentre O. Let [OG] be produced to X such that OG : GX = 1 : 2. Now prove that X is the orthocentre.

The line which joins these three centres is called the **Euler line**.

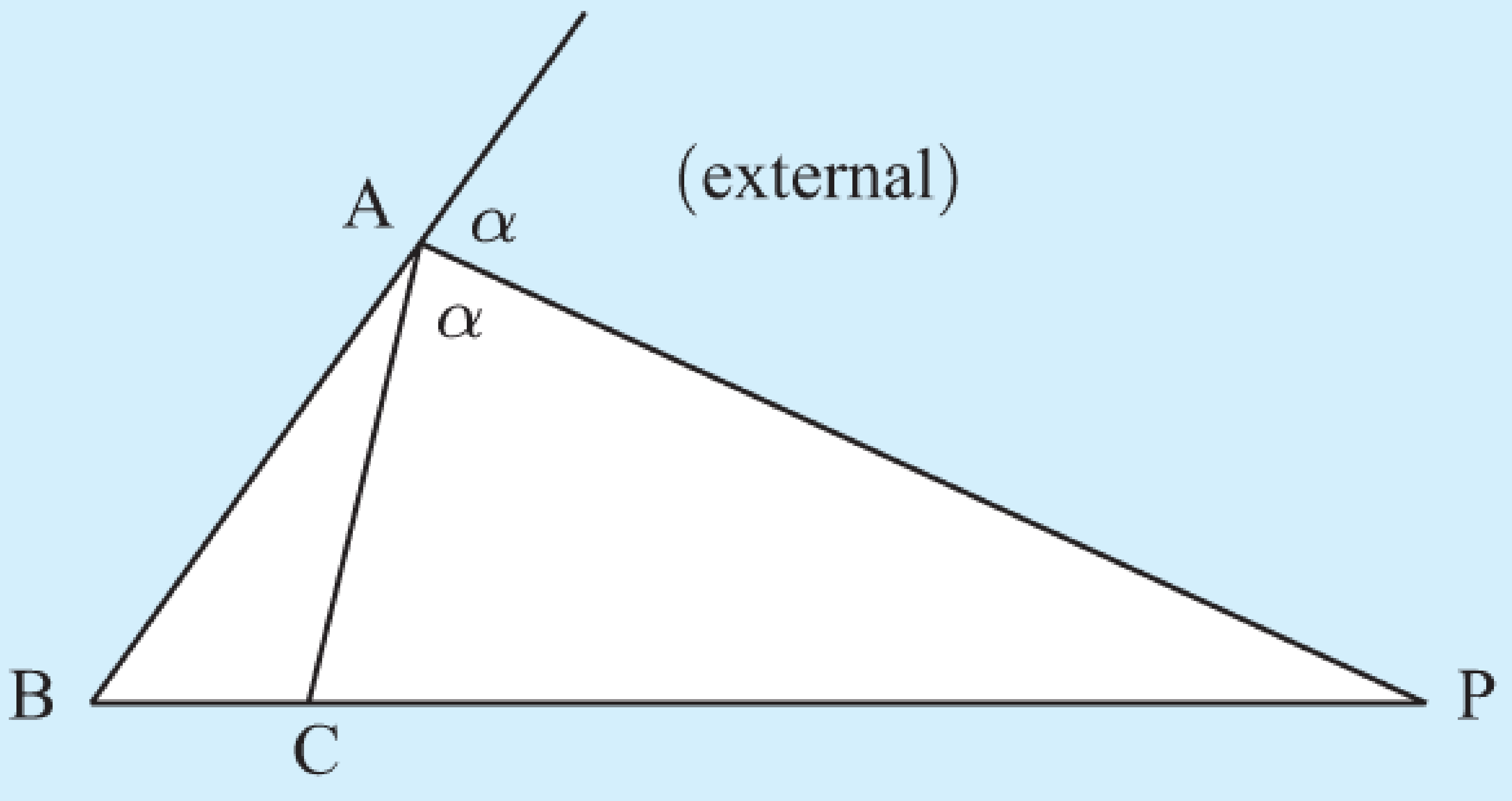
H EUCLID'S ANGLE BISECTOR THEOREM

The bisectors of the angles of a triangle divide the opposite side in the same ratio as the ratio of the lengths of the sides containing that angle.

$\dfrac{AB}{AC} = \dfrac{BP}{CP}$ for either case

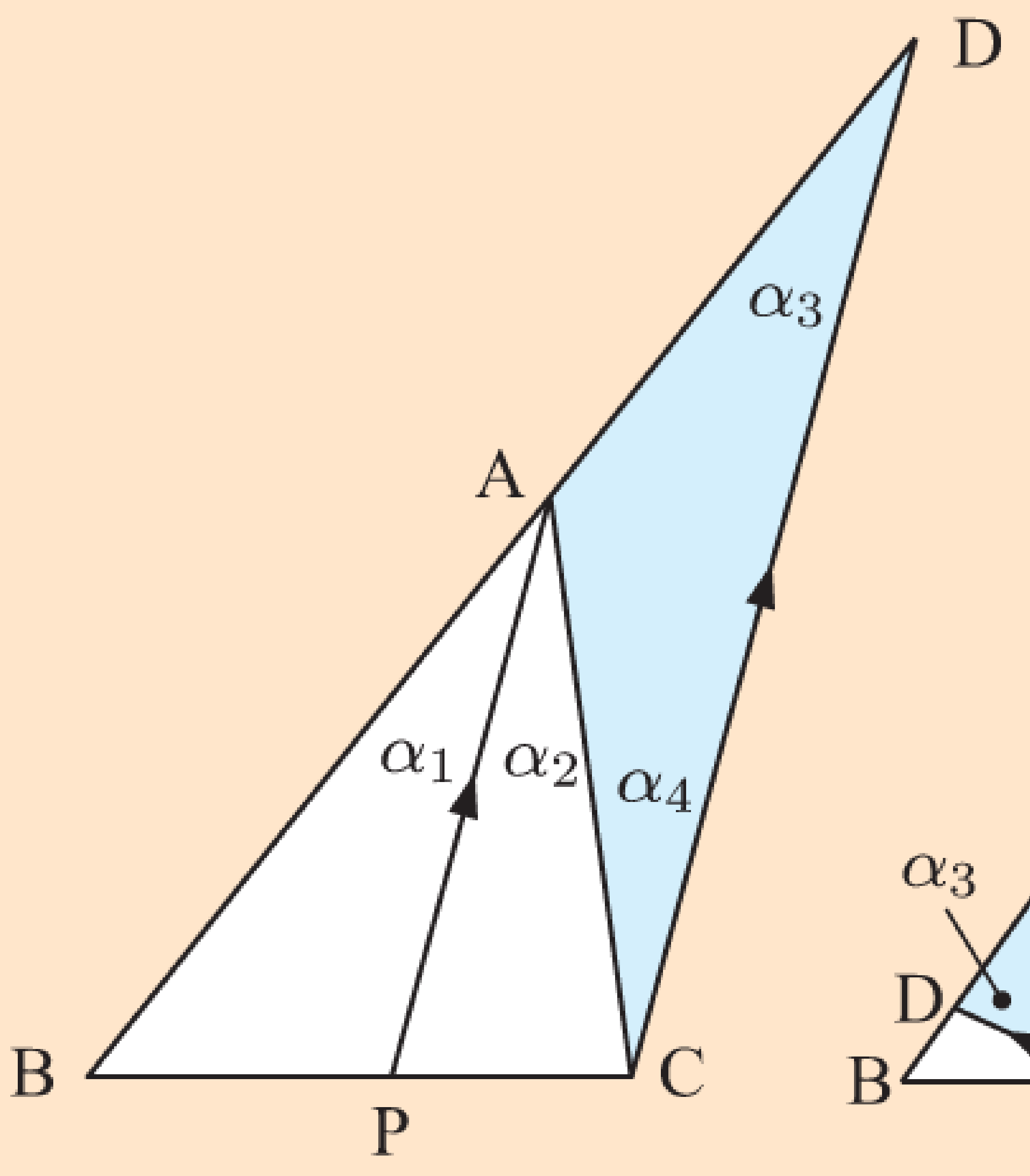

(internal)
A
α
α
B
P
C

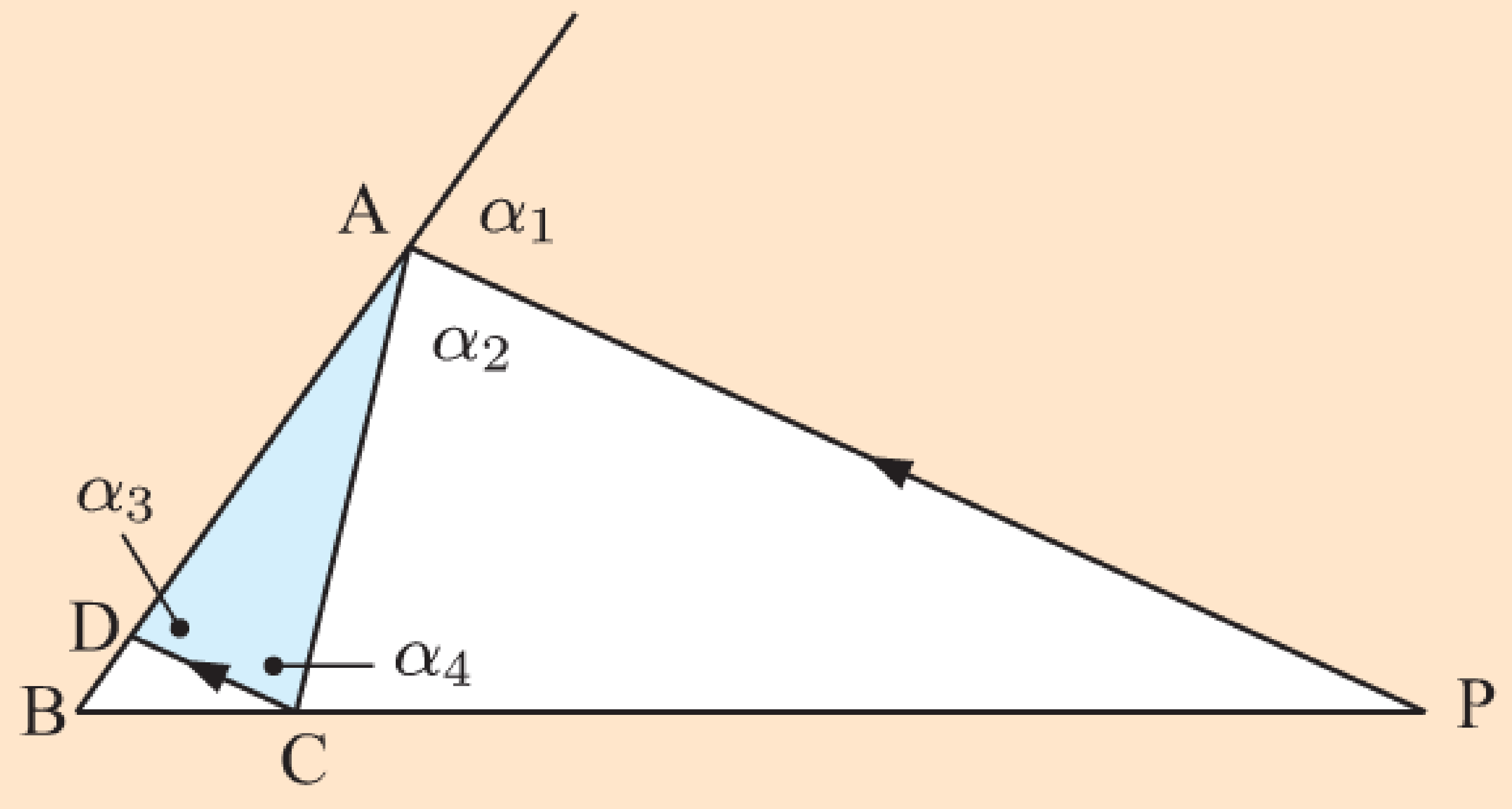
or


(external)
A
α
α
B
C
P

Proof 1: (Classical)

We draw [CD] parallel to [PA] to meet [BA] or [BA] produced at D.


D
α_3
A
α_1
α_2
α_4
B
P
C


A
α_1
α_2
α_3
D
α_4
B
C
P

$\alpha_1 = \alpha_2$ {given}

$\alpha_2 = \alpha_4$ {alternate angles}

$\alpha_1 = \alpha_3$ {corresponding angles}

$\therefore$ $\alpha_3 = \alpha_4$

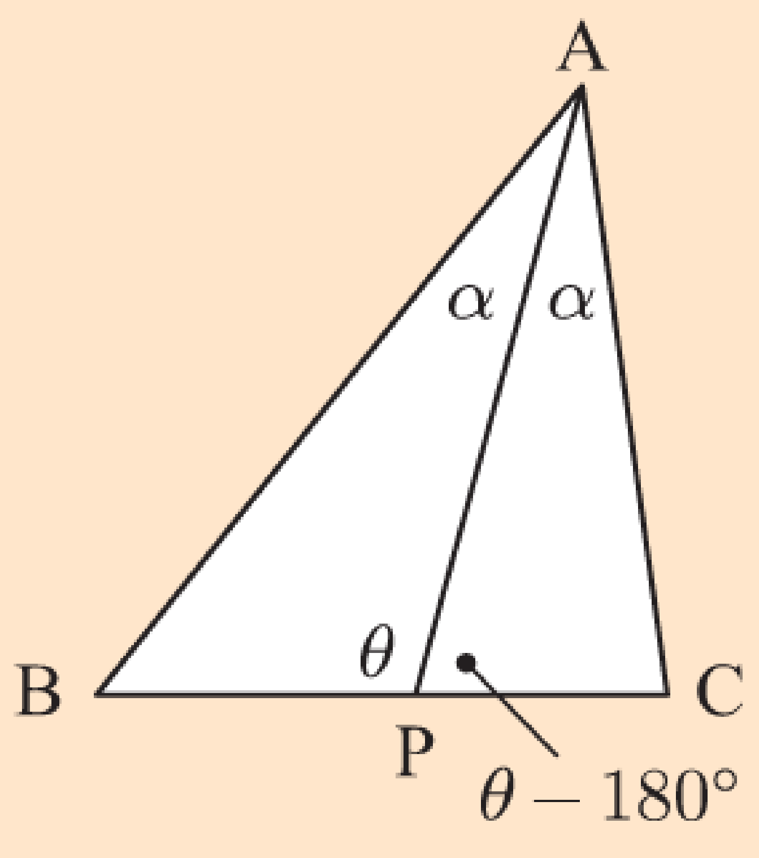
$\therefore$ $\triangle$ACD is isosceles

$\therefore$ AD = AC {isosceles triangle} .... (1)

Since [PA] ∥ [CD], then $\dfrac{BA}{AD} = \dfrac{BP}{PC}$ {parallel lines within a triangle theorem}

Using (1), $\dfrac{AB}{AC} = \dfrac{BP}{PC}$

Proof 2: (Modern - Internal case only)


A
α
α
θ
B
P
C
$\theta - 180°$

Using the Sine Rule in $\triangle$s ABP and ACP,

$$\frac{\sin\theta}{AB} = \frac{\sin\alpha}{BP} \quad \text{and} \quad \frac{\sin(180° - \theta)}{AC} = \frac{\sin\alpha}{PC}$$

But $\sin\theta = \sin(180° - \theta)$

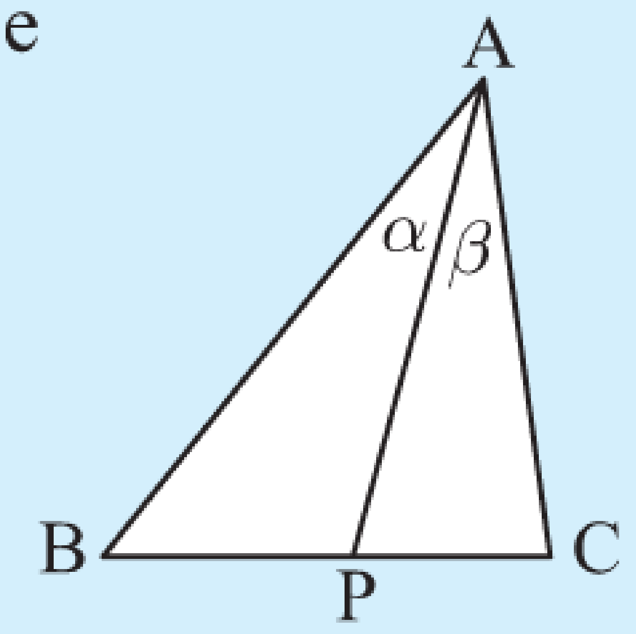
$$\therefore \quad \frac{AB \times \cancel{\sin\alpha}}{BP} = \frac{AC \times \cancel{\sin\alpha}}{PC}$$

$$\therefore \quad \frac{AB}{AC} = \frac{BP}{PC}$$

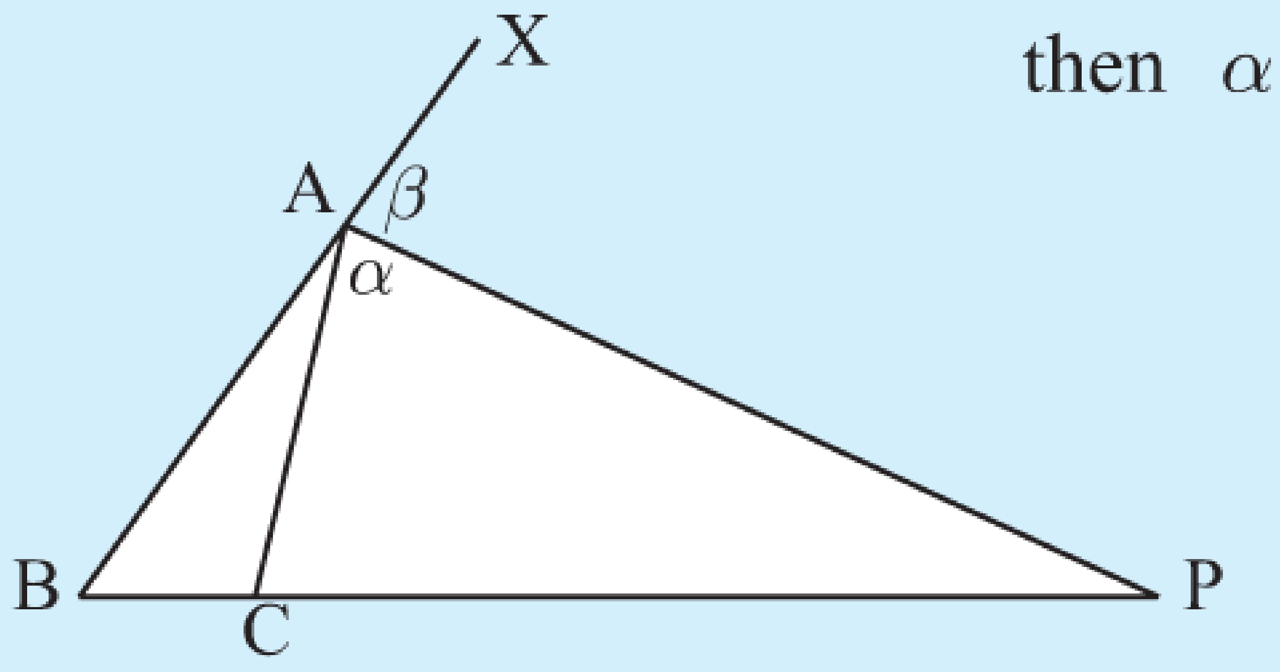
THE CONVERSE TO THE ANGLE BISECTOR THEOREM

In $\triangle$ABC, suppose P lies on [BC] or [BC] produced. If $\dfrac{AB}{AC} = \dfrac{BP}{PC}$, $C\widehat{A}B$ is bisected by [AP].

If $\dfrac{AB}{AC} = \dfrac{BP}{PC}$ for either case

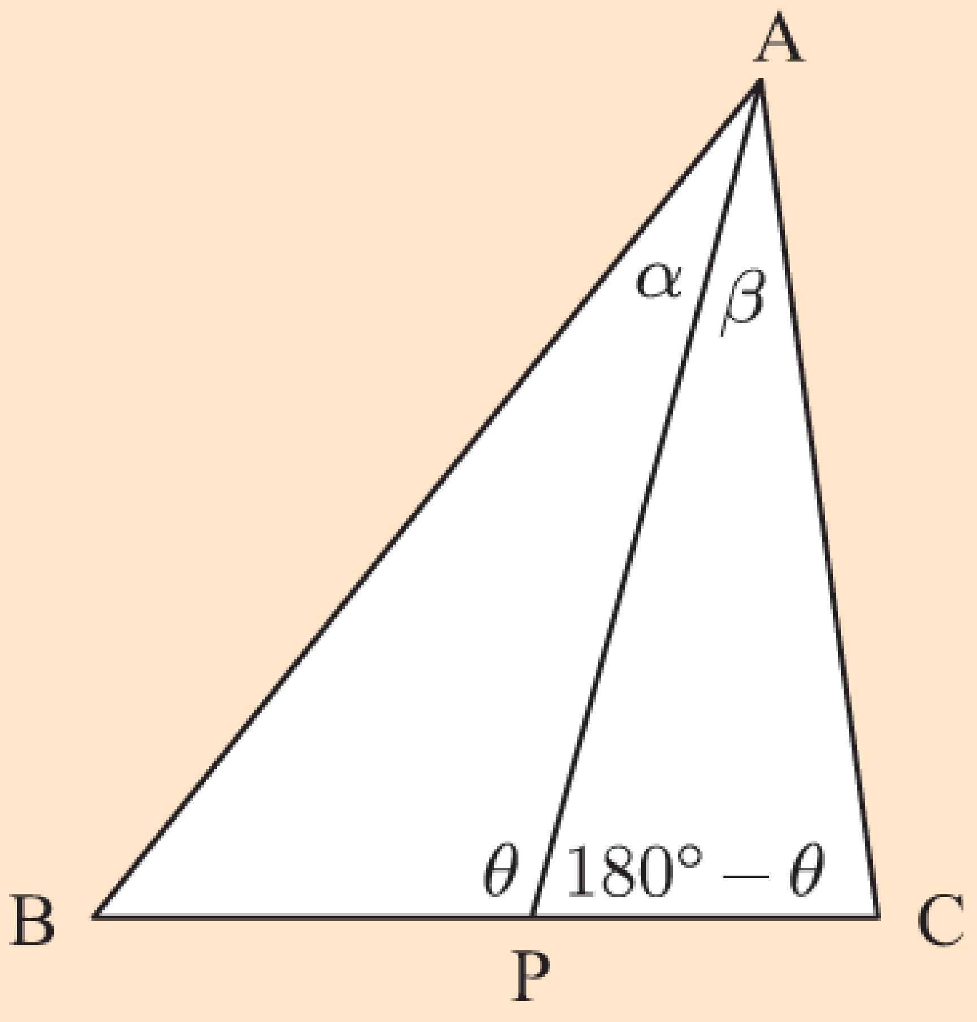


or



then $\alpha = \beta$.

Proof: (Internal case only)



Using the Sine Rule,

$$\frac{\sin\alpha}{BP} = \frac{\sin\theta}{AB} \quad\text{and}\quad \frac{\sin\beta}{PC} = \frac{\sin(180^\circ-\theta)}{AC}$$

But $\sin(180^\circ-\theta) = \sin\theta$

$$\therefore\ \frac{AC}{PC}\sin\beta = \frac{AB}{BP}\sin\alpha$$

$$\therefore\ \frac{BP}{PC}\sin\beta = \frac{AB}{AC}\sin\alpha$$

$$\therefore\ \sin\alpha = \sin\beta \quad \left\{\text{since } \frac{AB}{AC} = \frac{BP}{PC}\right\}$$

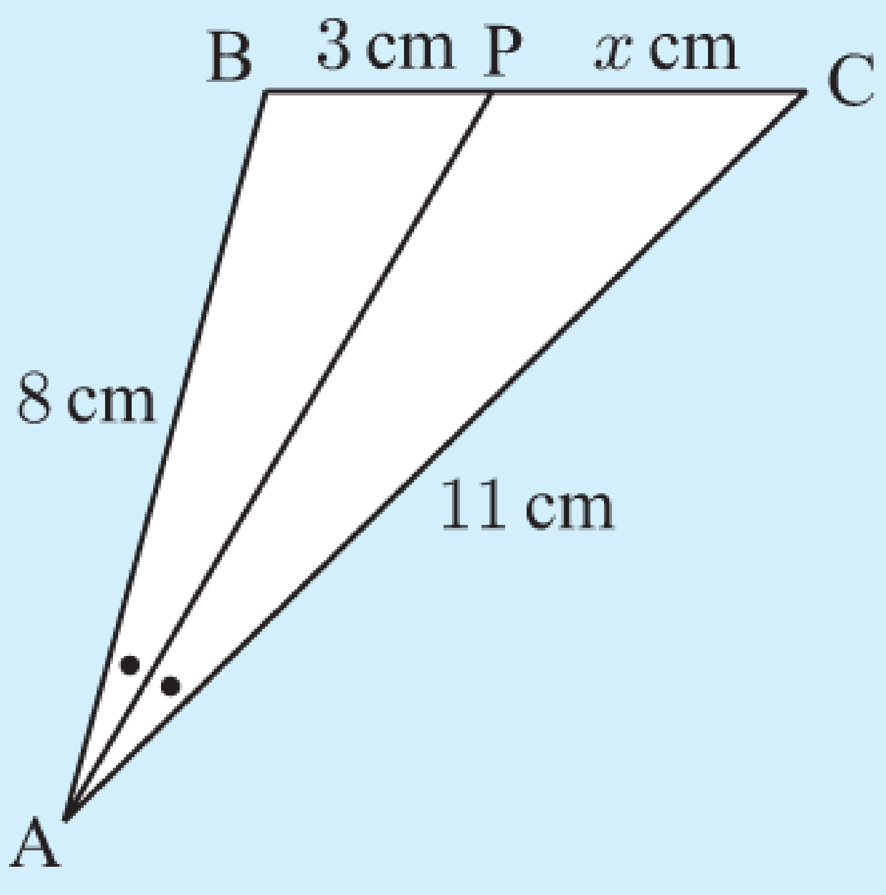
$$\therefore\ \alpha = \beta \quad\text{or}\quad \alpha = 180^\circ - \beta$$

$$\therefore\ \alpha = \beta \qquad \{\text{as } \alpha + \beta < 180^\circ\}$$

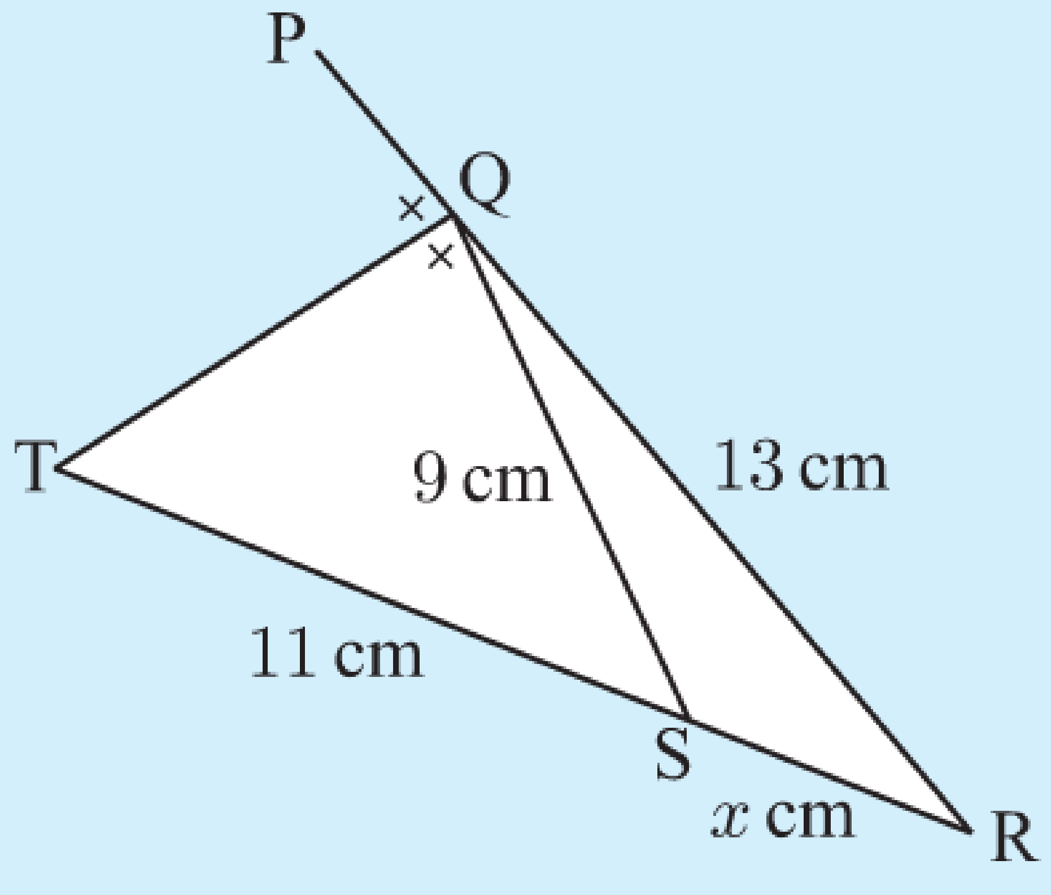
Example 13

Find x:

a



b



a $\dfrac{AB}{AC} = \dfrac{BP}{PC}$ {angle bisector theorem}

$\therefore\ \dfrac{8}{11} = \dfrac{3}{x}$

$\therefore\ x = \frac{33}{8}$

b $\dfrac{QR}{QS} = \dfrac{RT}{TS}$ {angle bisector theorem}

$\therefore\ \dfrac{13}{9} = \dfrac{x+11}{11}$

$\therefore\ 143 = 9x + 99$

$\therefore\ x = \frac{44}{9}$

EXERCISE 2H

1 The side lengths of a triangle are 12 cm, 15 cm, and 18 cm. Find the lengths of the segments cut from the sides of the triangle by the internal angle bisectors.

2 **a** Triangle ABC has sides of length BC = 6 cm, AC = 8 cm, and AB = 10 cm. The bisector of $A\widehat{B}C$ meets [AC] at D. Find:

i the lengths of [AD] and [DC]

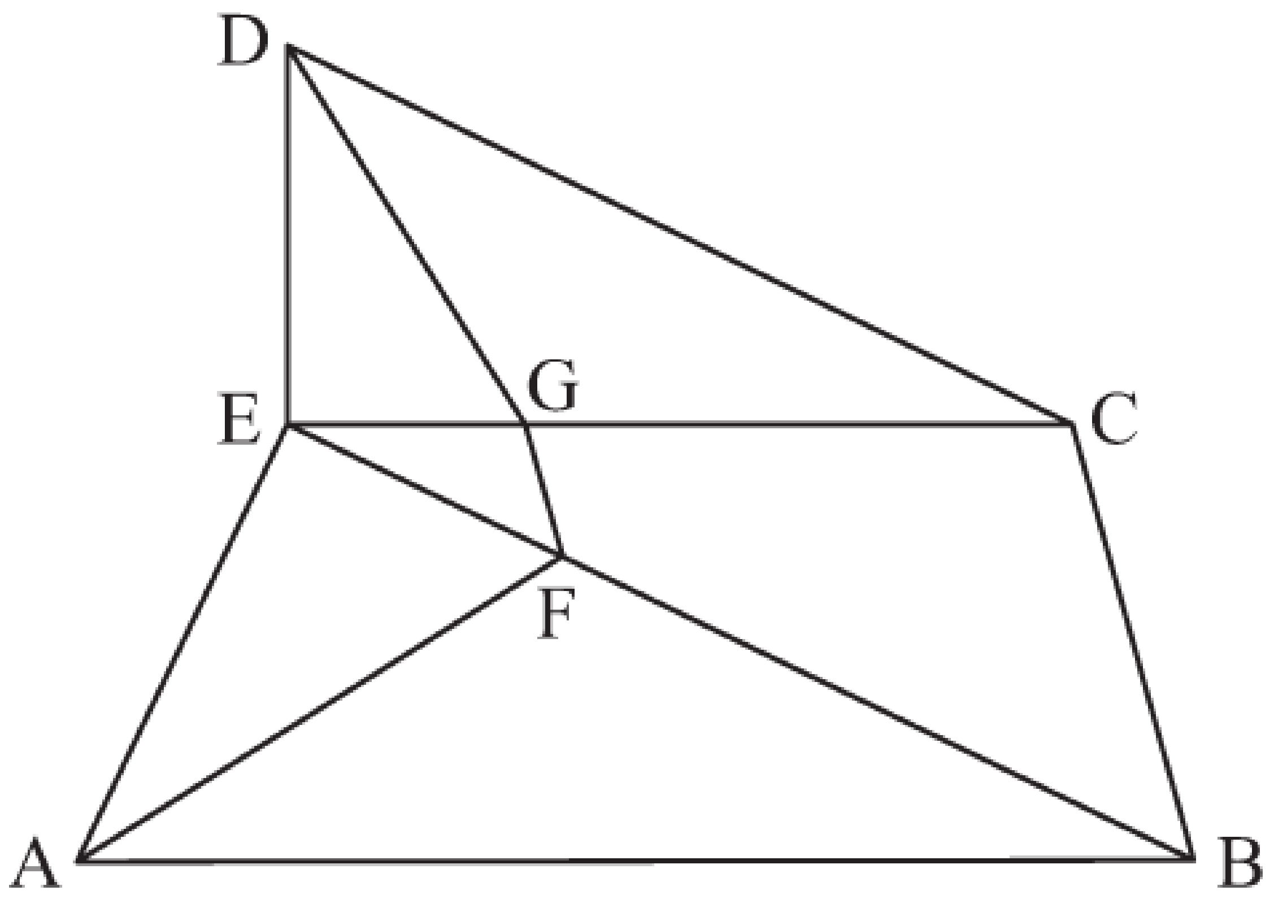
ii $\tan\left(\frac{A\widehat{B}C}{2}\right)$

b Find $\tan(\frac{\pi}{8})$ using the technique in **a**.

3 In triangle PQR, [PX] is a median. The internal bisectors of the angles at X meet [PQ] and [PR] at Y and Z respectively. Prove that [YZ] is parallel to [QR].

4 Triangle ABC is isosceles with AB = AC. Suppose P is any point within the triangle. The bisector of $P\widehat{A}B$ meets [BP] at H. The bisector of $C\widehat{A}P$ meets [CP] at K. Prove that [HK] is parallel to [BC].

5



EAB and EDC are similar triangles with corresponding vertices in that order.

[AF] bisects $B\widehat{A}E$ and meets [BE] at F.

[DG] bisects $C\widehat{D}E$ and meets [EC] at G.

Prove that [GF] is parallel to [CB].

6 P is the midpoint of [BC] of triangle ABC. [PQ] is the bisector of $A\widehat{P}B$, and cuts [AB] at Q. [QR] is drawn parallel to [BC], meeting [AC] at R. Prove that $Q\widehat{P}R$ is a right angle.

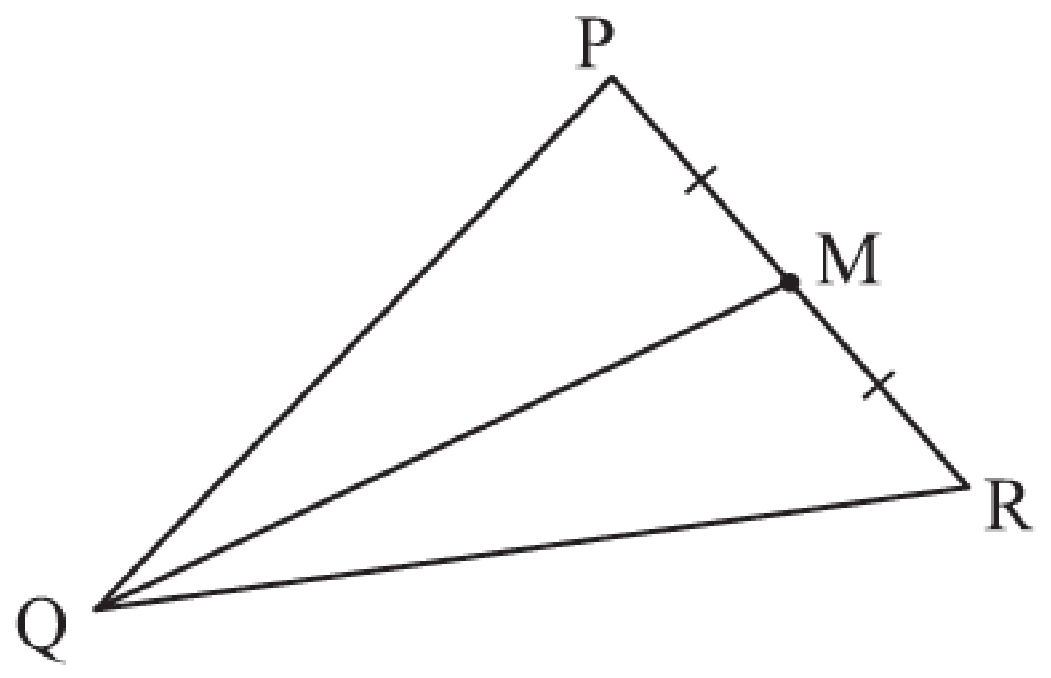
7 A semi-circle has diameter [AB]. P lies on the semi-circle and [PQ] bisects $A\widehat{P}B$, cutting [AB] at Q. [PC] is drawn perpendicular to [AB], cutting [AB] at C.

Prove that $\frac{AQ}{QB} = \frac{AC}{PC}$.

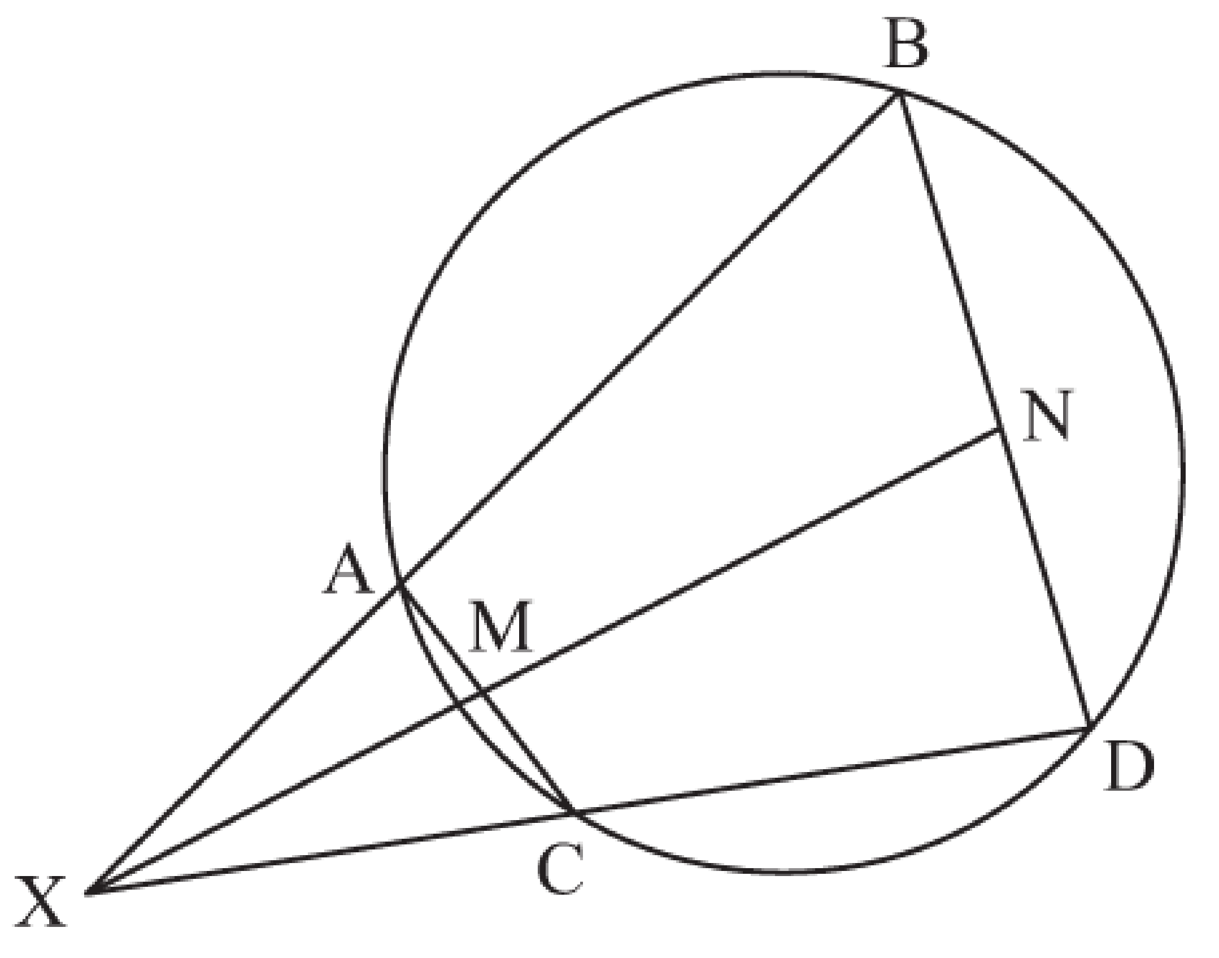
8 M is the midpoint of [PR] of triangle PQR, and [QM] bisects $P\widehat{Q}R$.

a Prove that $\triangle$PQR is isosceles using the angle bisector theorem.

b Why can we not use congruence in this figure?



9



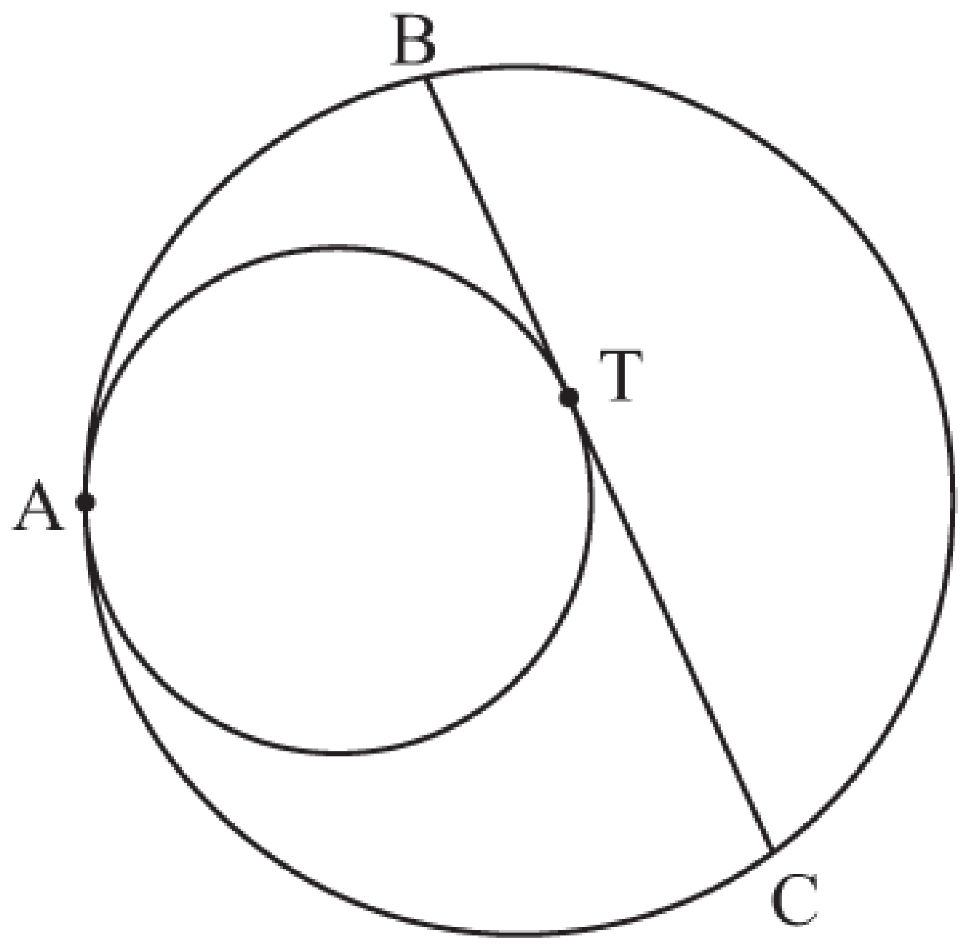
If XMN is the angle bisector of $B\widehat{X}D$, prove that MA.BN = ND.CM.

10 Triangle ABC has interior angle bisectors which meet [BC], [CA], and [AB] at points P, Q, and R respectively. Prove that AR.BP.CQ = AQ.BR.CP.

11 Triangle PQR has incentre O. [PO] produced meets [RQ] at S.
Prove that PO : OS = (PQ + PR) : QR.

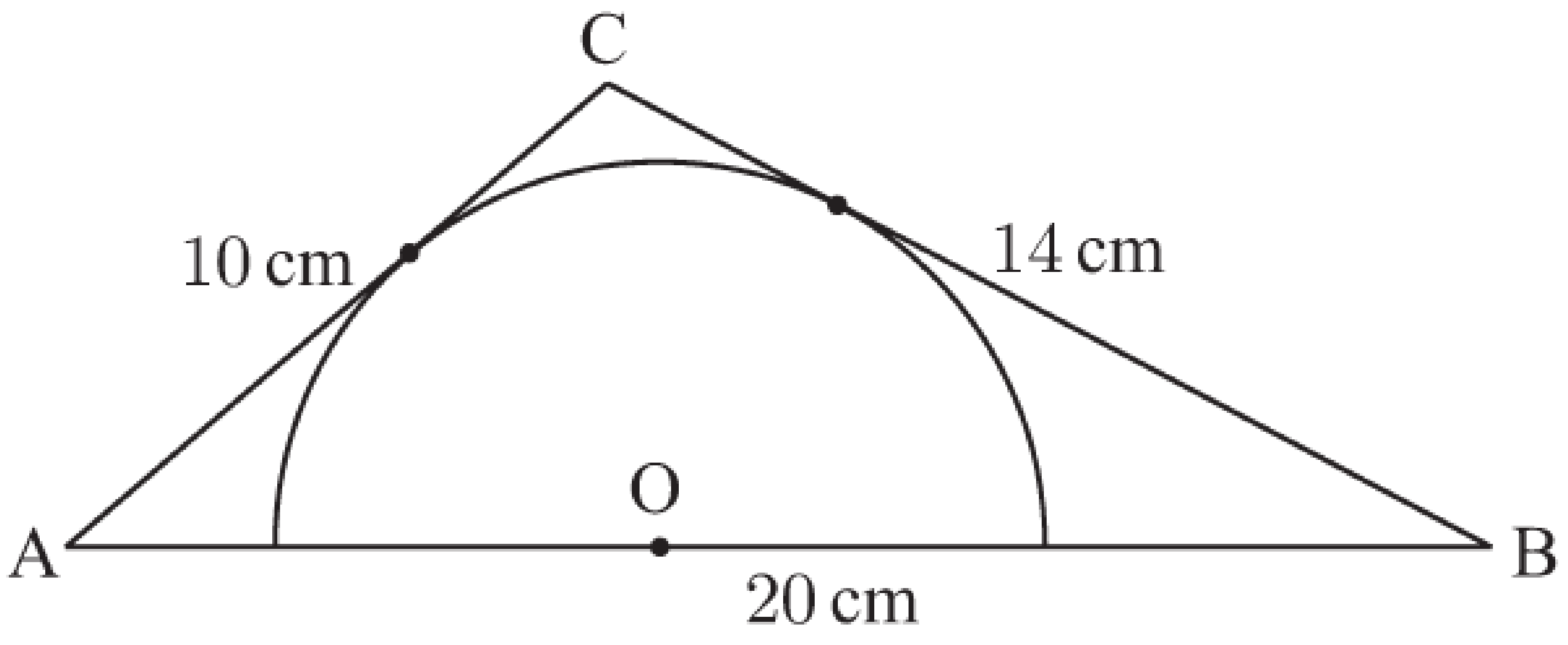
12 A circle has diameter [PQ]. [RS] is any chord perpendicular to [PQ]. T lies on [RS]. [PT] produced and [QT] produced meet the circle at A and B respectively. Prove that BR.AS = RA.BS.

13



Two circles touch internally at A. [BC] is a chord of the larger circle and is a tangent to the smaller circle at T.
Prove that AB : AC = TB : TC.

14



A semi-circle with centre O is inscribed within a triangle with sides AB = 20 cm, BC = 14 cm, and AC = 10 cm.
Find the radius of the semi-circle.

I APOLLONIUS' CIRCLE THEOREM

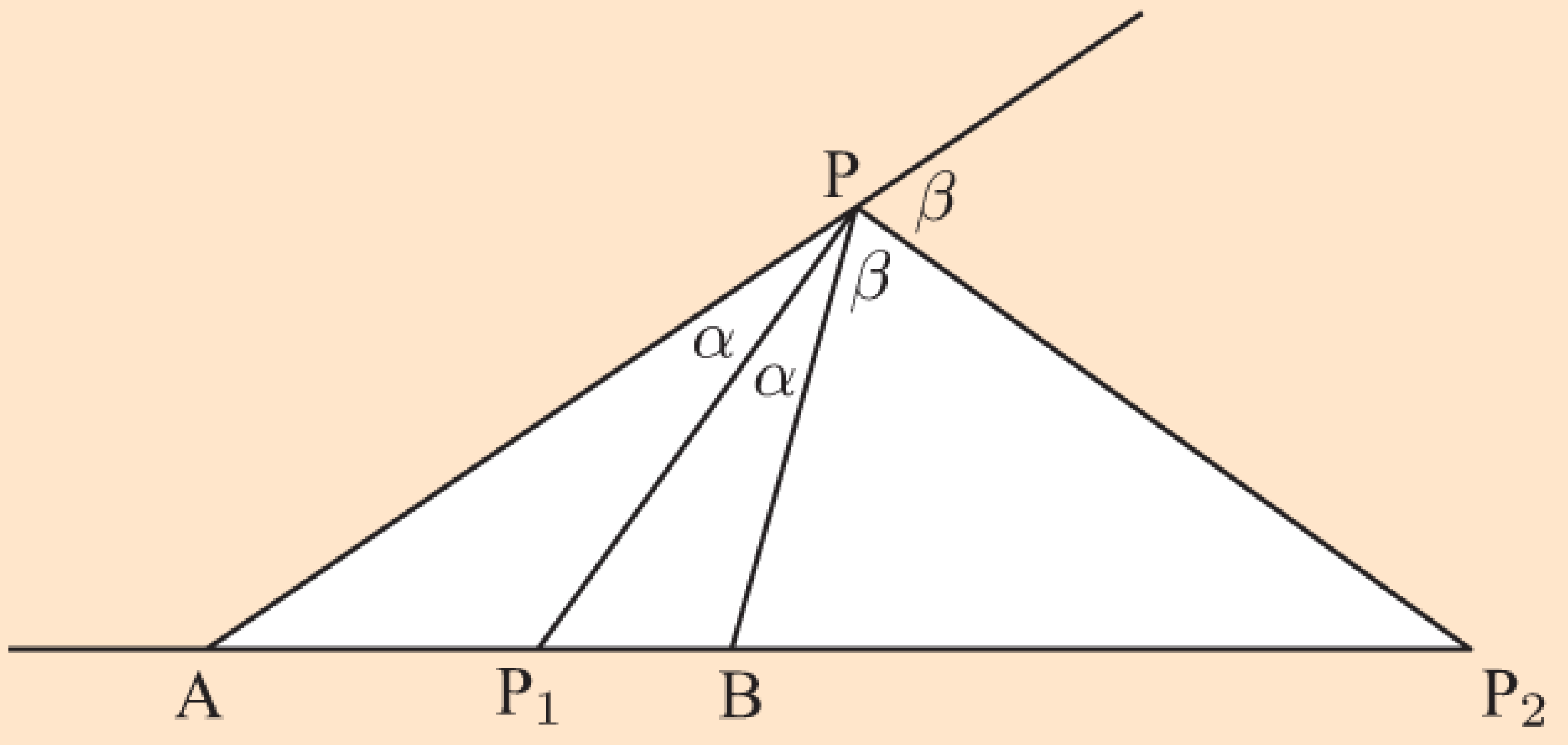
If A and B are fixed points such that $\frac{\mathrm{PA}}{\mathrm{PB}} = k$, where k is a positive constant, $k \neq 1$, then the locus of points P is a circle.

DEMO

Proof:

Let P be a point not collinear with A and B such that $\frac{\mathrm{PA}}{\mathrm{PB}} = k$ where k is a fixed positive constant, $k \neq 1$.

We draw the internal and external bisectors of $\mathrm{A}\widehat{\mathrm{P}}\mathrm{B}$. We let them meet [AB], and [AB] produced, at P_1 and P_2 respectively.



Now $2\alpha + 2\beta = 180°$ {angles on a line}

$\therefore\ \alpha + \beta = 90°$

$\therefore\ \mathrm{P}_1\widehat{\mathrm{P}}\mathrm{P}_2$ is a right angle

Now $\frac{\mathrm{AP}_1}{\mathrm{BP}_1} = \frac{\mathrm{AP}}{\mathrm{BP}}$ {angle bisector theorem}

$\therefore\ \frac{\mathrm{AP}_1}{\mathrm{BP}_1} = k$, and so P_1 is a fixed point independent of the choice of P.

Likewise, $\frac{\mathrm{AP}_2}{\mathrm{BP}_2} = \frac{\mathrm{AP}}{\mathrm{BP}} = k$, and so P_2 is a fixed point independent of the choice of P.

Since P_1 and P_2 are fixed points and $\mathrm{P}_1\widehat{\mathrm{P}}\mathrm{P}_2$ is a right angle, $[\mathrm{P}_1\mathrm{P}_2]$ subtends a right angle at P as P moves.

$\therefore$ P traces out a circle, with centre the midpoint of $[\mathrm{P}_1\mathrm{P}_2]$.

Corollary:

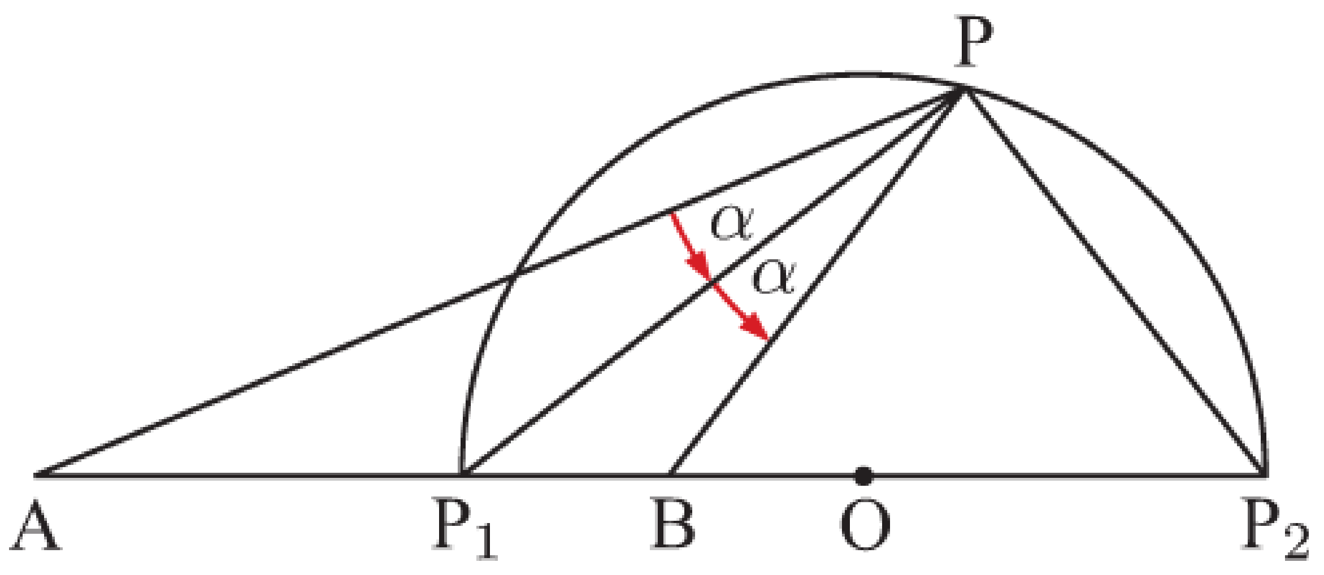
If the resulting Apollonius' circle has centre O and radius r, then $r^2 = \mathrm{OA.OB}$.

CONVERSE OF APOLLONIUS' CIRCLE THEOREM

Suppose a circle has fixed diameter $[\mathrm{P}_1\mathrm{P}_2]$, and point P moves anywhere on the circle. If two fixed points, A on $[\mathrm{P}_2\mathrm{P}_1]$ produced, and B on $[\mathrm{P}_1\mathrm{P}_2]$, can be found such that $\frac{\mathrm{AP}_1}{\mathrm{BP}_1} = \frac{\mathrm{AP}_2}{\mathrm{BP}_2}$, then $\frac{\mathrm{PA}}{\mathrm{PB}}$ is a positive constant.

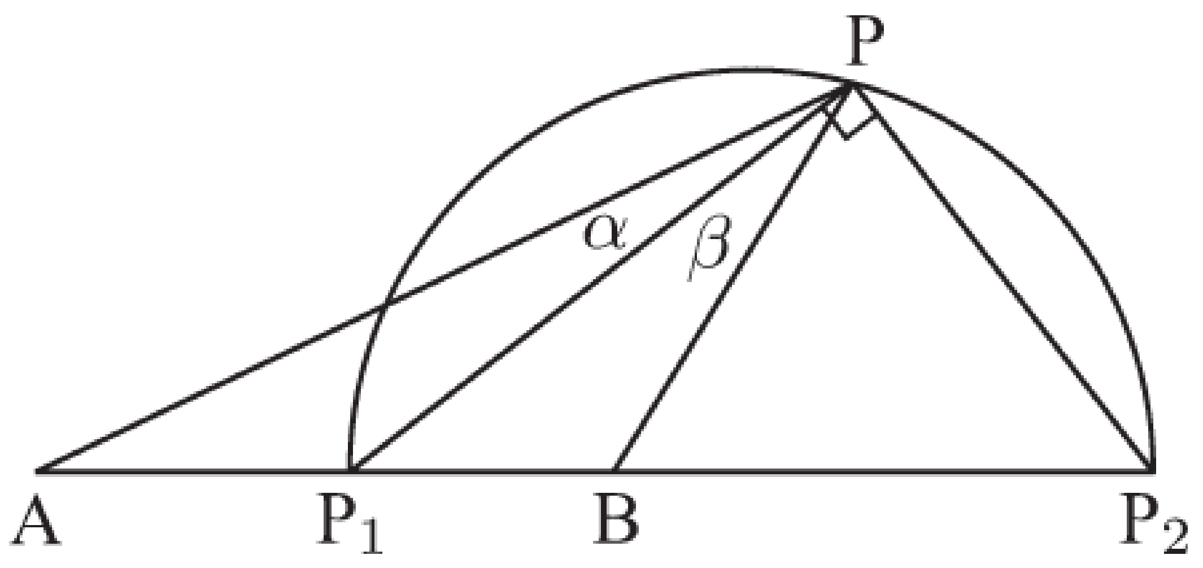
EXERCISE 2I

1. Prove the corollary of Apollonius' circle theorem.

2. 

In the given diagram, $OB = 3$ cm, $AP_1 = 4$ cm, and P_1P bisects $A\widehat{P}B$. Find the radius of the Apollonius' circle.

3. Suppose P has coordinates (x, y), A is $(-2, 0)$, and B is $(4, 0)$.
   - **a** Find the Cartesian equation connecting x and y if:
     - **i** $\dfrac{PA}{PB} = 1$
     - **ii** $\dfrac{PA}{PB} = \frac{1}{2}$
     - **iii** $\dfrac{PA}{PB} = 2$
     - **iv** $\dfrac{PA}{PB} = 3$
   - **b** Explain why the condition $k \neq 1$ was given in Apollonius' circle theorem.
   - **c** Explain why the Cartesian equation of a circle has the form $x^2 + y^2 + dx + ey + f = 0$ where d, e, and f are constants.

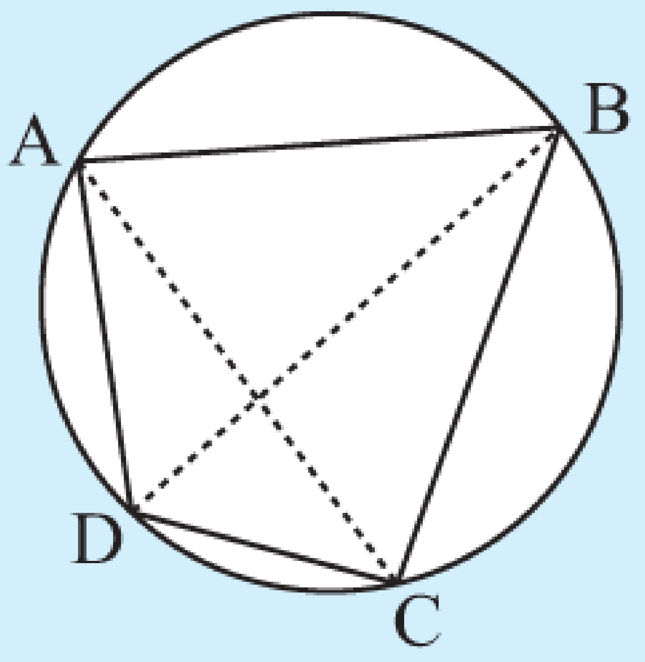
4. 

The points A, P_1, B, P_2 lie on a line, $\dfrac{AP_1}{BP_1} = \dfrac{AP_2}{BP_2}$, and $P_1\widehat{P}P_2$ is a right angle.
   - **a** Prove that $\alpha = \beta$.
   - **b** Hence, prove the converse of Apollonius' circle theorem.

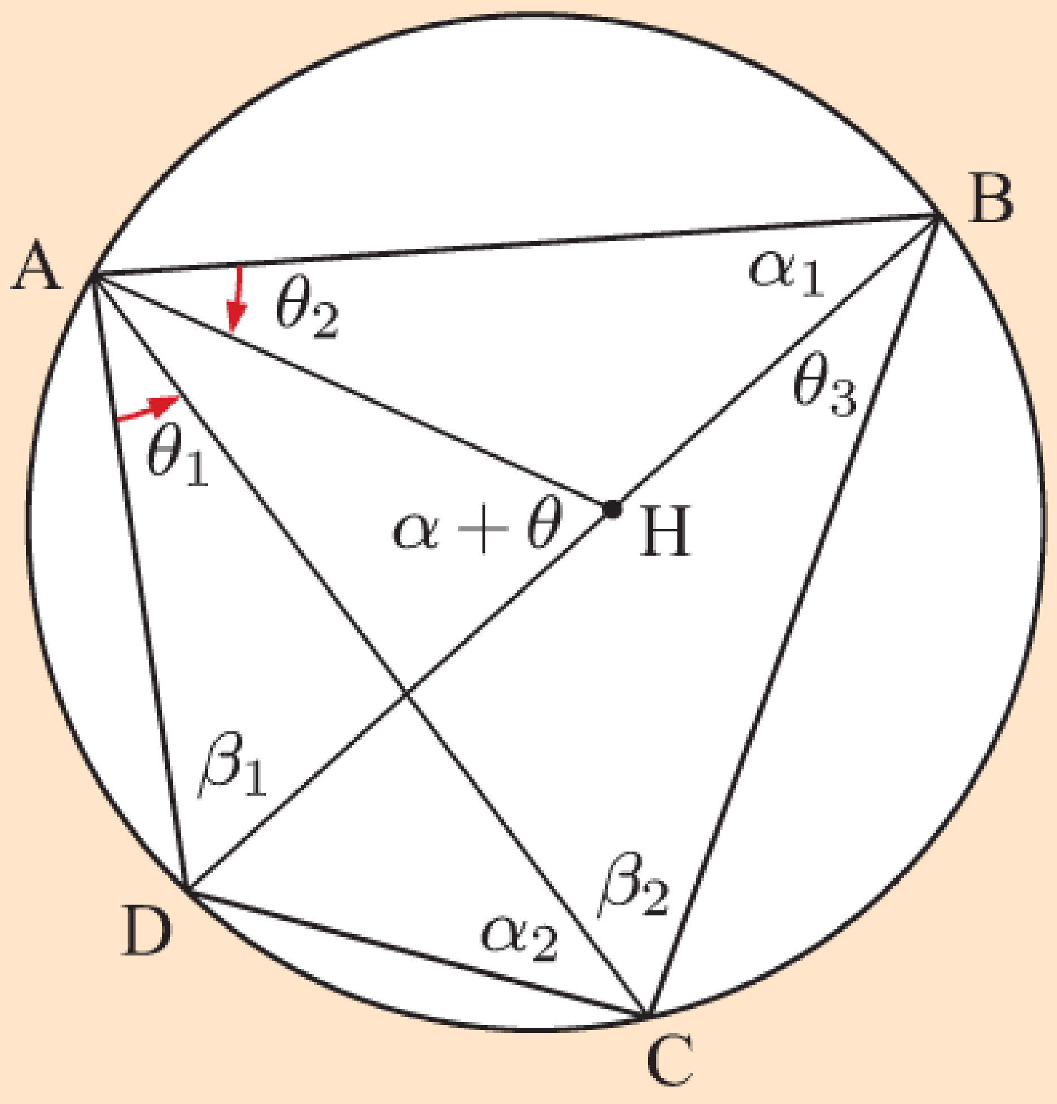
J PTOLEMY'S THEOREM FOR CYCLIC QUADRILATERALS

If a quadrilateral is cyclic, then the sum of the products of the lengths of the two pairs of opposite sides is equal to the product of the diagonals.

$$AB.CD + BC.DA = AC.BD$$



Proof:



We first draw [AH], where H lies on [DB] such that $\theta_1 = \theta_2$ as shown.

Now in $\triangle$s ABH and ACD:

- $\theta_1 = \theta_2$ {construction}
- $\alpha_1 = \alpha_2$ {angles subtended by the same arc}

The triangles are equiangular and therefore similar.

$$\therefore \quad \frac{AB}{AC} = \frac{BH}{CD}$$

$$\therefore \quad BH = \frac{AB.CD}{AC} \quad .... (1)$$

Also, in $\triangle$s ADH and ACB:

- $A\widehat{H}D = \alpha+\theta$ {exterior angle of $\triangle$ABH} and $\theta_3 = \theta_1$ {angles subtended by the same arc}

  $\therefore \quad A\widehat{H}D = A\widehat{B}C = \alpha+\theta$
- $\beta_1 = \beta_2$ {angles subtended by the same arc}

The triangles are equiangular and therefore similar.

$$\therefore \quad \frac{HD}{BC} = \frac{DA}{AC}$$

$$\therefore \quad HD = \frac{BC.DA}{AC} \quad .... (2)$$

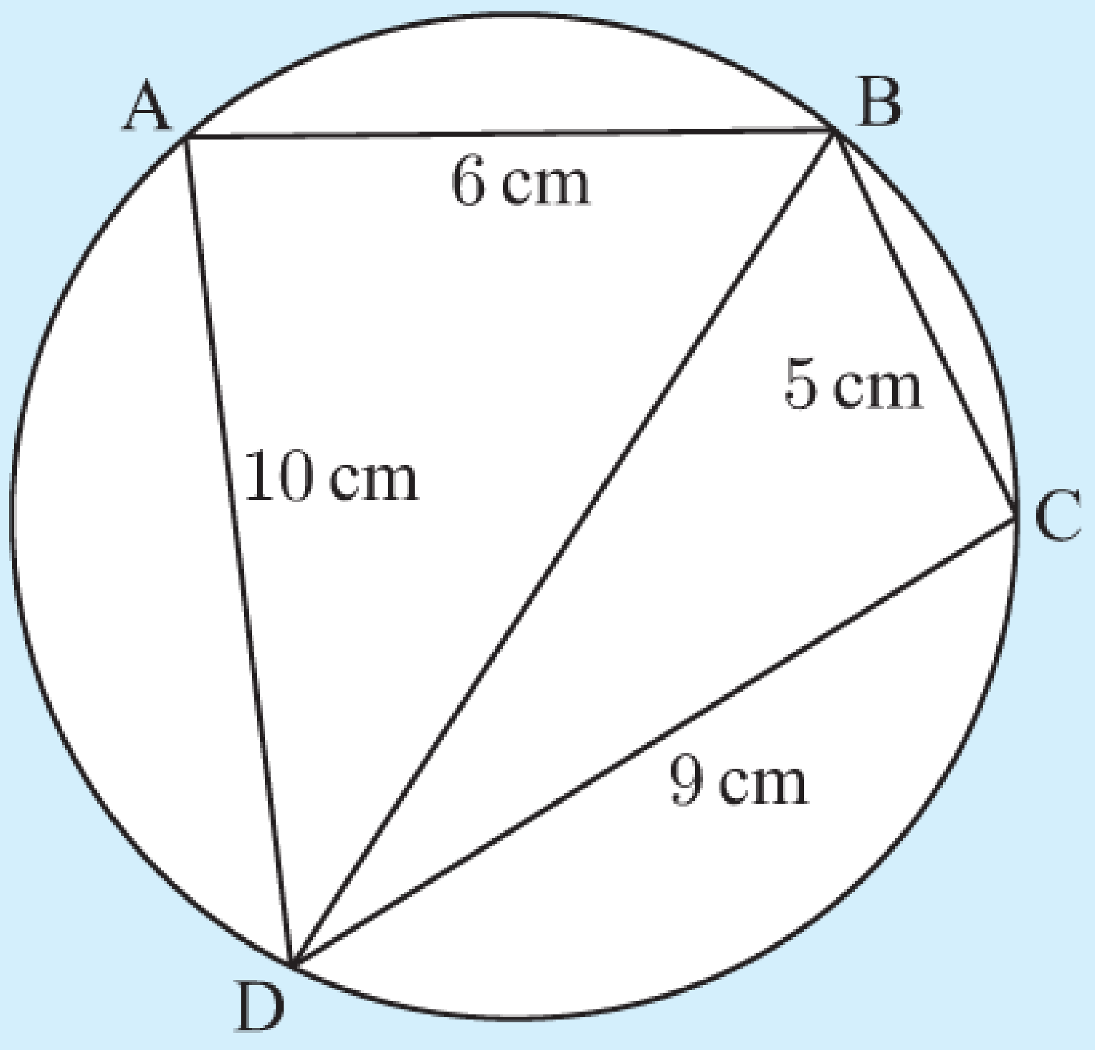
Using (1) and (2), $BD = BH + HD = \dfrac{AB.CD}{AC} + \dfrac{BC.DA}{AC}$

$$\therefore \quad BD = \frac{AB.CD + BC.DA}{AC}$$

Hence, $AB.CD + BC.DA = AC.BD$

Example 14

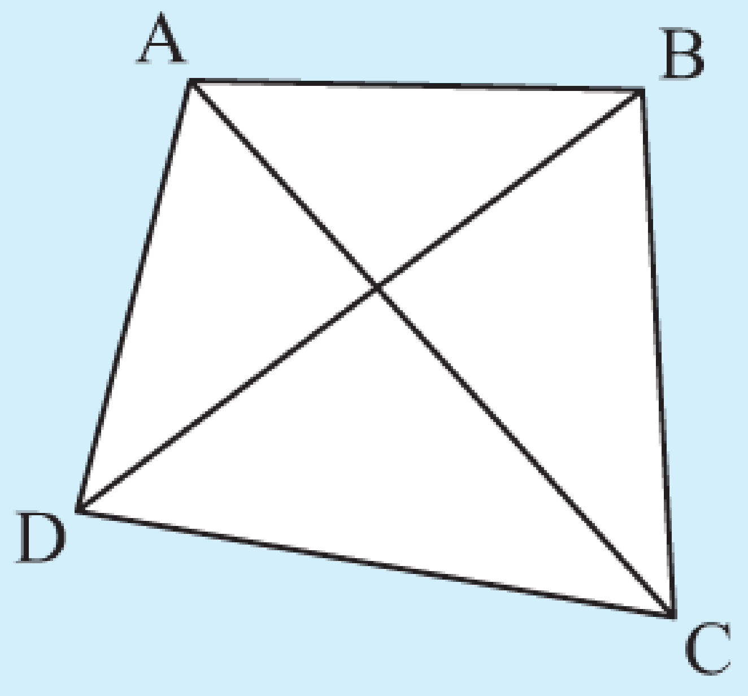
Find AC given that [BD] has length 12 cm.



$$\begin{aligned} AB.CD + BC.DA &= AC.BD \quad \{\text{Ptolemy's theorem}\} \\ \therefore\ 6 \times 9 + 5 \times 10 &= AC \times 12 \\ \therefore\ 104 &= AC \times 12 \\ \therefore\ AC &= 8\tfrac{2}{3} \text{ cm} \end{aligned}$$

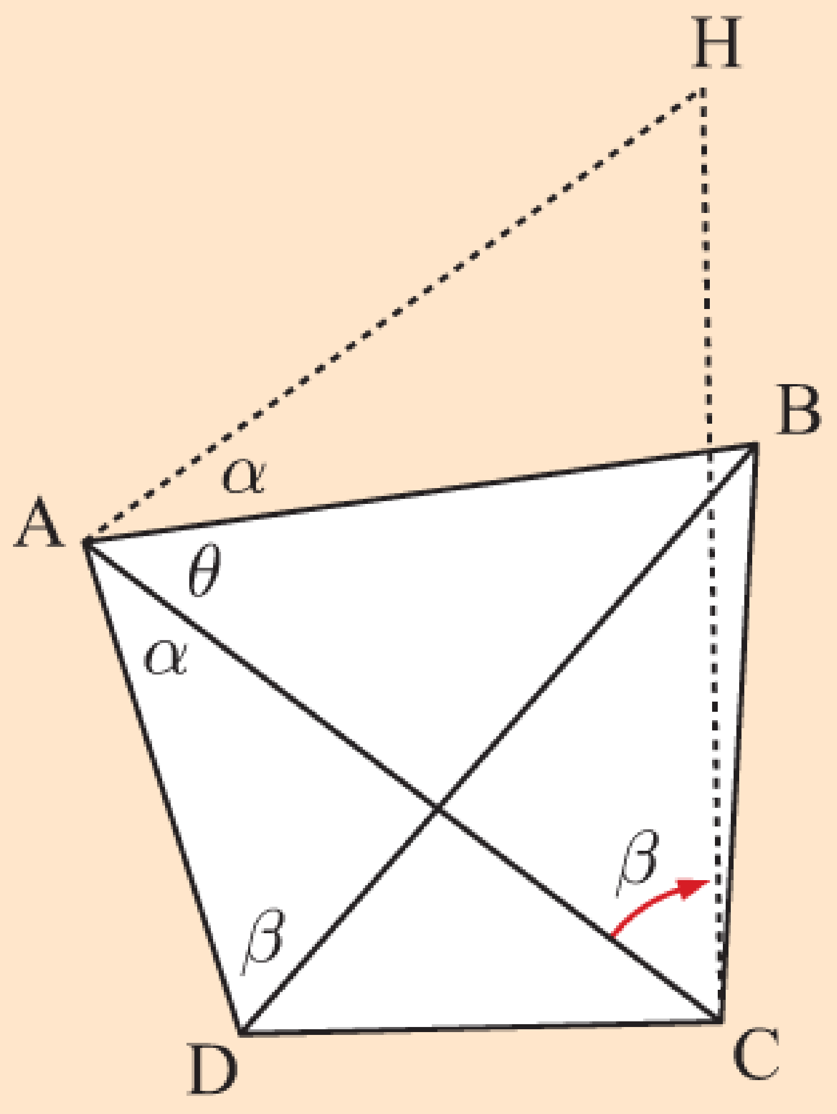
THE CONVERSE FOR PTOLEMY'S THEOREM FOR CYCLIC QUADRILATERALS

If the product of the lengths of the diagonals of a quadrilateral equals the sum of the products of the lengths of its pairs of opposite sides, then the quadrilateral is a cyclic quadrilateral.



If $AB.CD + BC.DA = AC.BD$
then ABCD is a cyclic quadrilateral.

Proof:



Let $C\widehat{A}D = \alpha$, $A\widehat{D}B = \beta$, and $B\widehat{A}C = \theta$.

Choose a point H on the same side of AC as B, such that $B\widehat{A}H = \alpha$ and $A\widehat{C}H = \beta$.

Now $C\widehat{A}H = D\widehat{A}B = \alpha + \theta$
and $A\widehat{C}H = A\widehat{D}B = \beta$

$\therefore$ $\triangle$s ACH and ADB are equiangular and therefore similar.

$$\therefore\ \underbrace{\frac{CH}{DB} = \frac{AC}{AD}}_{(1)} \underbrace{= \frac{AH}{AB}}_{(2)}$$

Rearranging (1), $CH = \dfrac{AC.DB}{AD}$ .... (3)

Given (2) and that $B\widehat{A}H = D\widehat{A}C = \alpha$, in $\triangle$s BAH and DAC we have two sides in the same ratio and the included angle between them equal.

$\therefore$ $\triangle$s BAH and DAC are similar.

$$\therefore \quad \frac{BH}{CD} = \frac{AB}{AD}$$

$$\therefore \quad BH = \frac{AB.CD}{AD} \quad .... \ (4)$$

Using (3) and (4),

$$\begin{aligned} CH - BH &= \frac{AC.DB}{AD} - \frac{AB.CD}{AD} \\ &= \frac{AC.BD - AB.CD}{AD} \\ &= \frac{BC.AD}{AD} \quad \{\text{by the assumption of the converse}\} \\ &= BC \end{aligned}$$

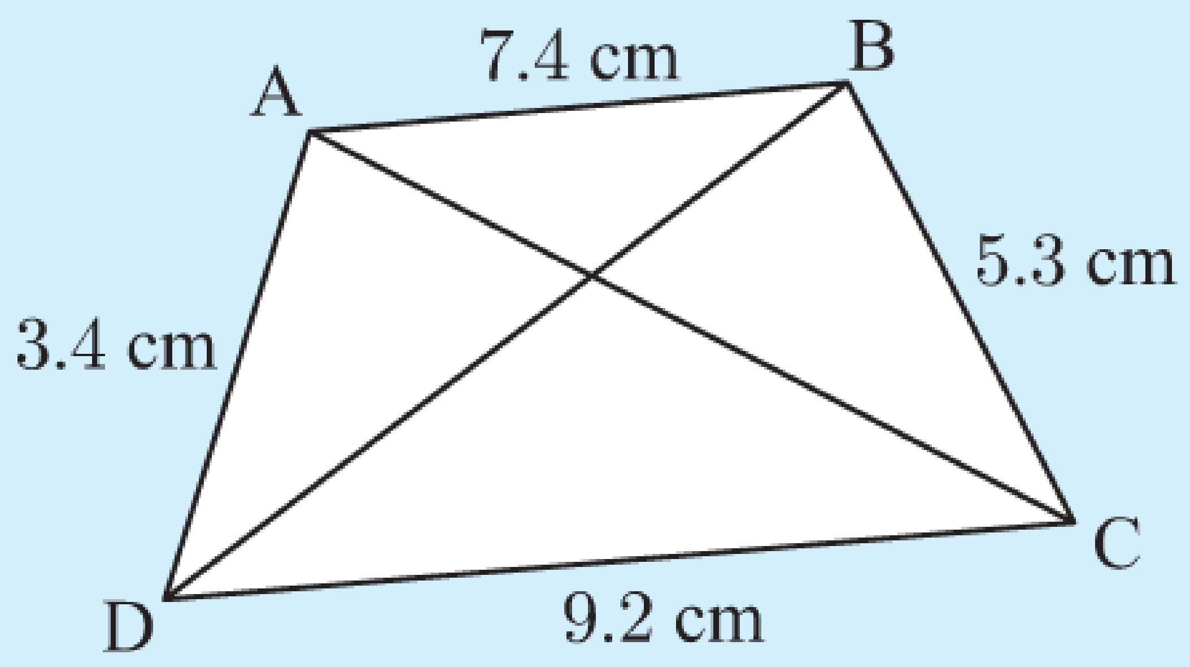
$\therefore$ $BC + BH = CH$, which can only be true if B, C, and H are collinear.

$\therefore$ $A\widehat{C}B = \beta$

$\therefore$ $A\widehat{C}B = A\widehat{D}B$ are equal angles subtended by [AB].

$\therefore$ ABCD is a cyclic quadrilateral.

Example 15



In the given quadrilateral ABCD, AC = 10.5 cm and BD = 8.2 cm.

Is ABCD a cyclic quadrilateral?

$$\begin{aligned} &AB.CD + BC.DA \\ &= 7.4 \times 9.2 + 5.3 \times 3.4 \\ &= 86.1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} &AC.BD \\ &= 10.5 \times 8.2 \\ &= 86.1 \end{aligned}$$

Hence, by the converse of Ptolemy's theorem, ABCD is a cyclic quadrilateral.

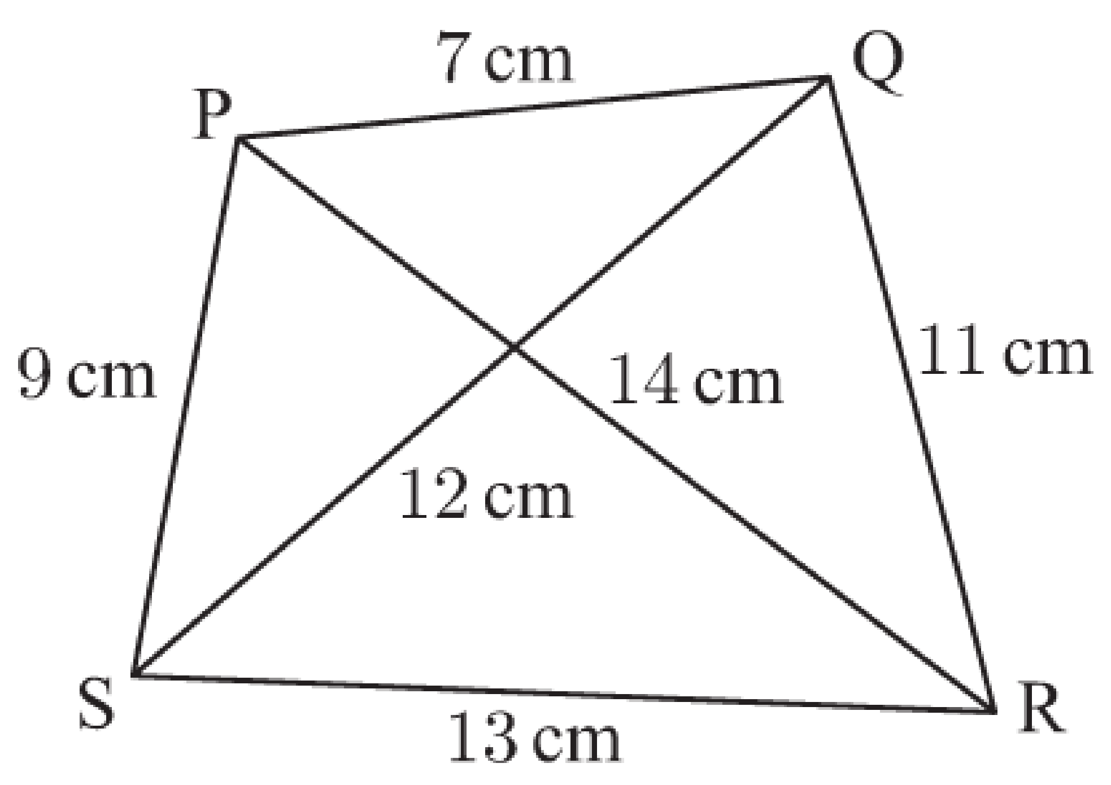
EXERCISE 2J

1 The side lengths of a cyclic quadrilateral, in clockwise order, are 6 cm, 9 cm, 7 cm, and 11 cm. If one diagonal is approximately 12.0 cm long, find the length of the other diagonal.

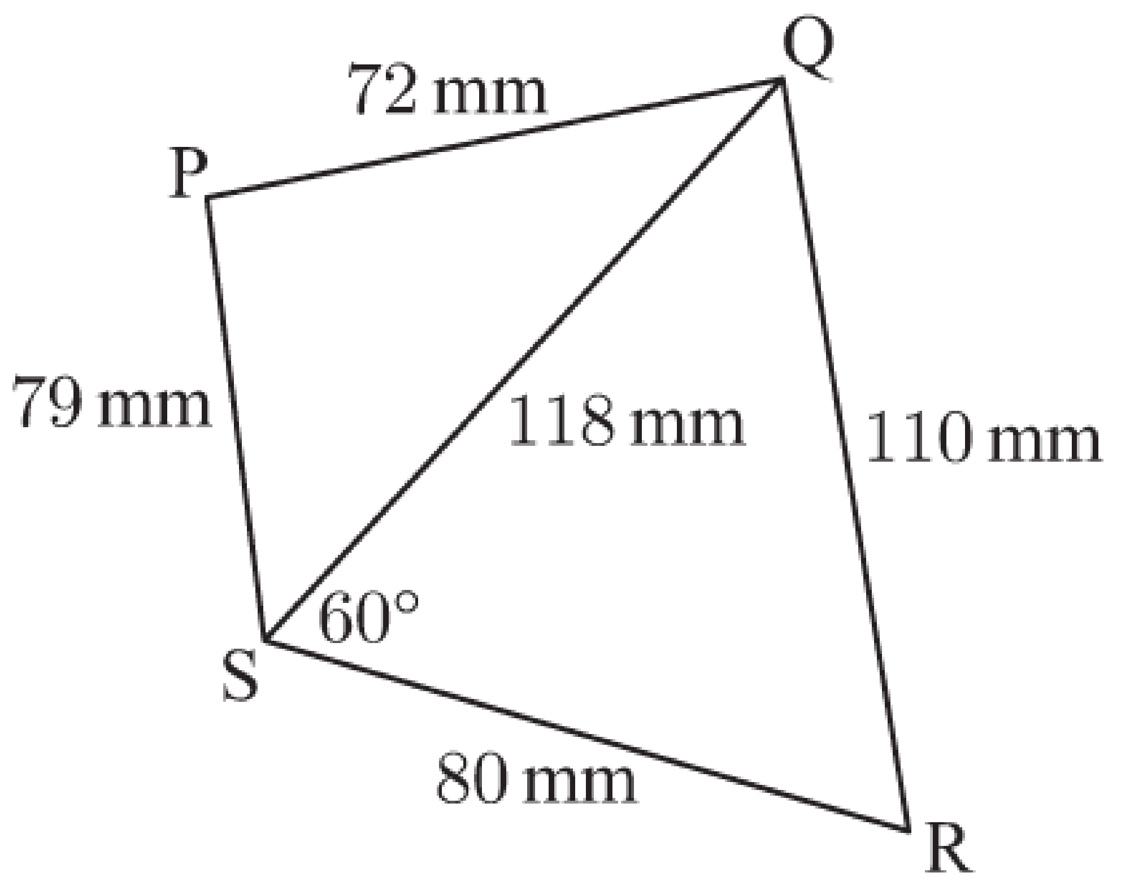
2 Three consecutive sides of a cyclic quadrilateral have lengths 6 cm, 5 cm, and 11 cm. Its diagonals have approximate lengths 10.1 cm and 9.54 cm. Find the length of the fourth side of the cyclic quadrilateral.

3 In each figure, determine whether PQRS is a cyclic quadrilateral:

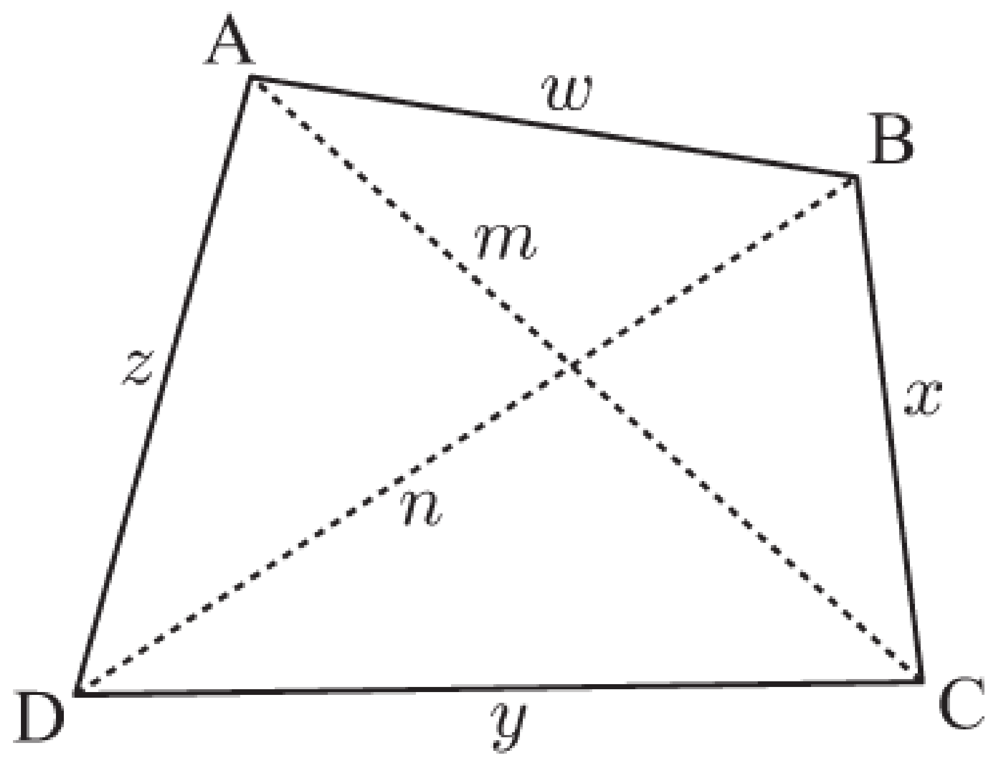
a


P
Q
R
S
7 cm
9 cm
14 cm
11 cm
12 cm
13 cm

b


P
Q
R
S
72 mm
79 mm
118 mm
110 mm
60°
80 mm

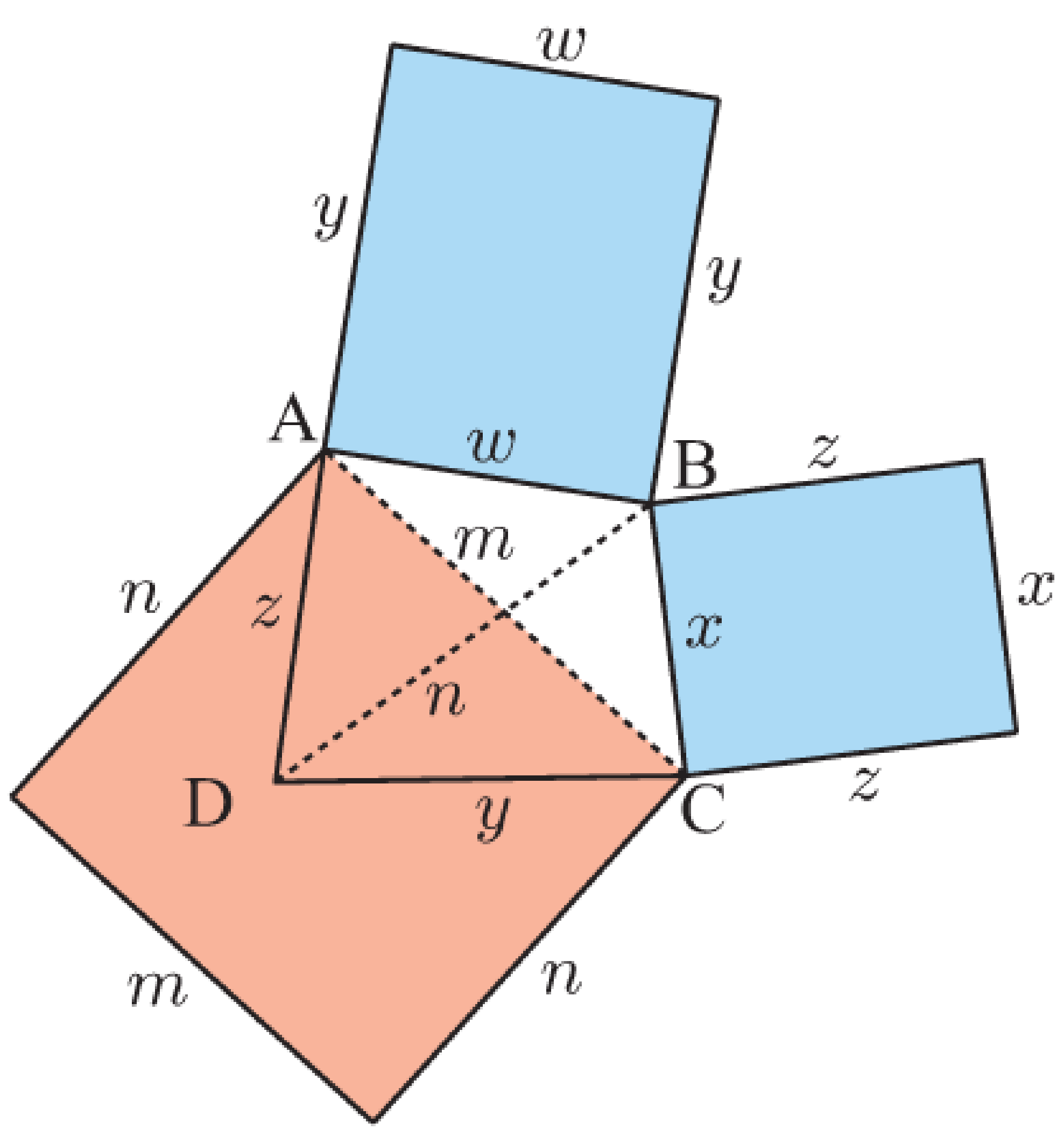
4 Consider a cyclic quadrilateral ABCD with the dimensions given. Diagonal [AC] has length m, and diagonal [BD] has length n.


A
B
C
D
w
m
z
x
n
y

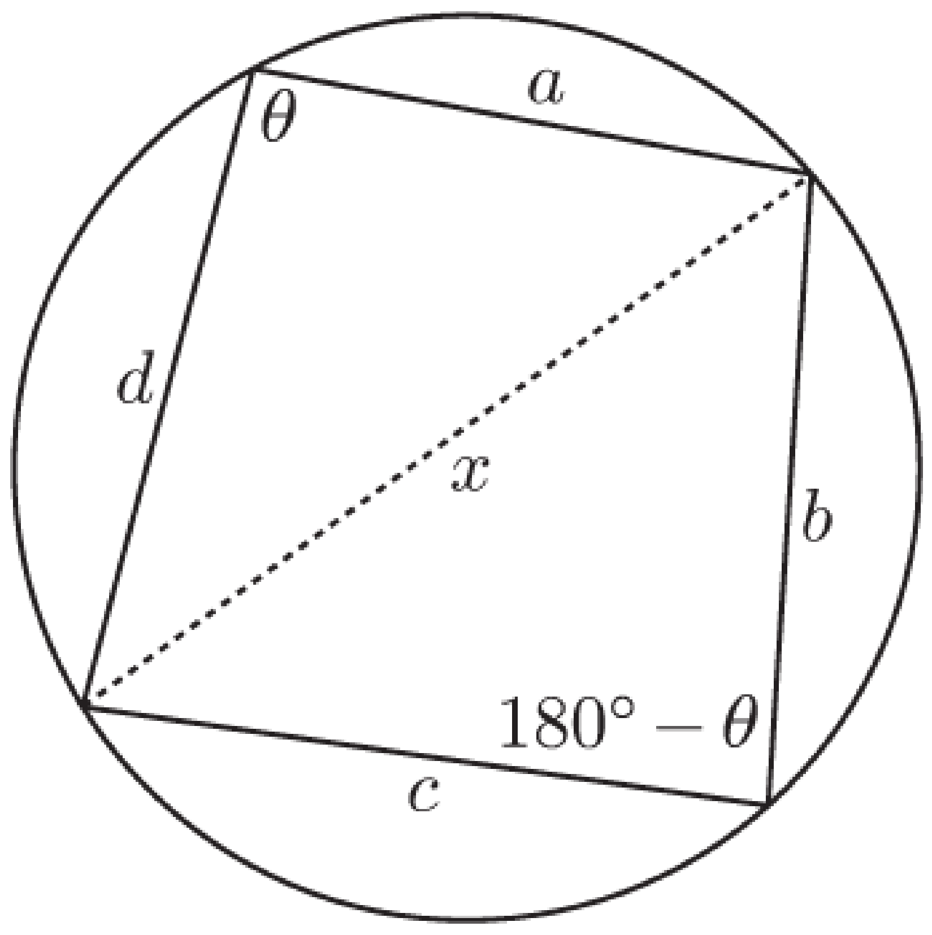
a Write down an equation connecting the variables.

b Suppose we construct rectangles on sides [AB], [BC], and [AC], with widths y, z, and n respectively.
What can be deduced about the shaded areas?

c Suppose ABCD is a rectangle.
What formula does Ptolemy's theorem give in this case?


w
y
y
A
w
B
z
m
n
z
x
x
n
D
y
C
z
m
n

5

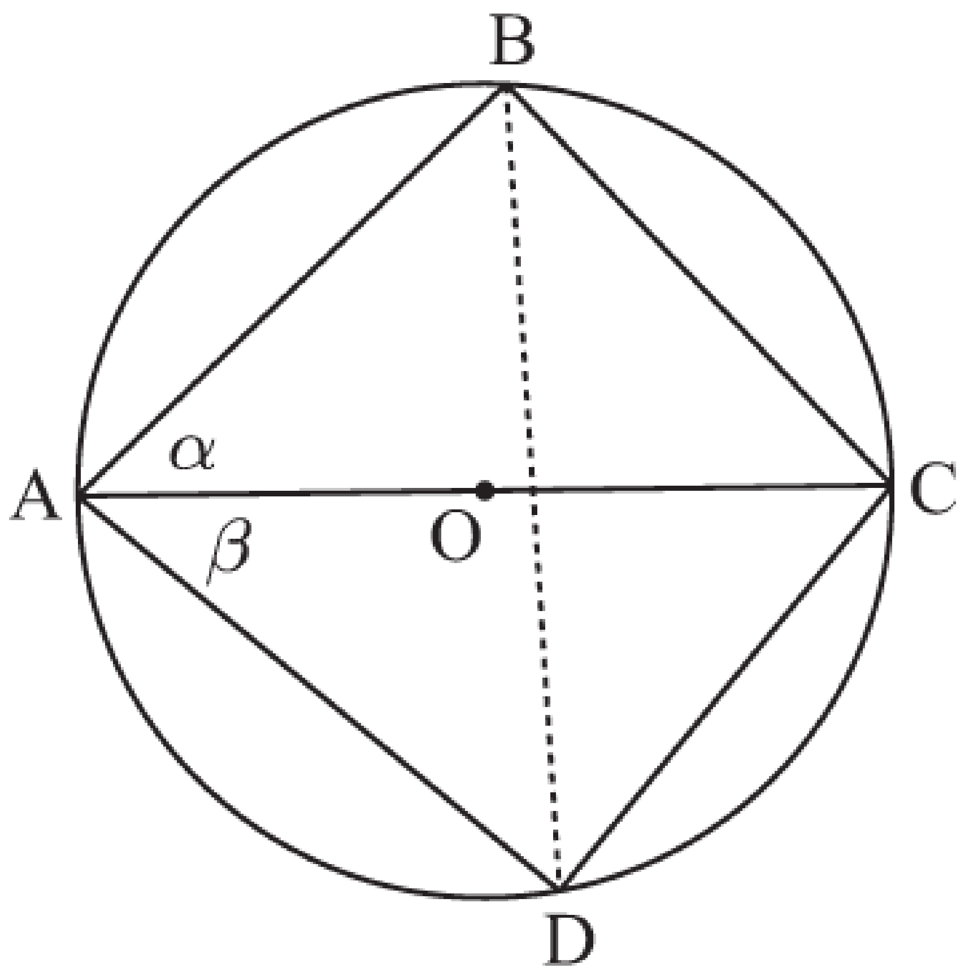

a
θ
d
x
b
$180° - \theta$
c

a Use the given figure and the Cosine Rule to deduce that $x^2 = \dfrac{(ac+bd)(ab+cd)}{(bc+ad)}$.

b If the other diagonal has length y units, show that $y^2 = \dfrac{(ac+bd)(ad+bc)}{ab+cd}$.

c Hence, prove Ptolemy's theorem.

6


B
A
α
β
O
C
D

[AC] is a diameter of a circle with centre O and radius 1 unit.

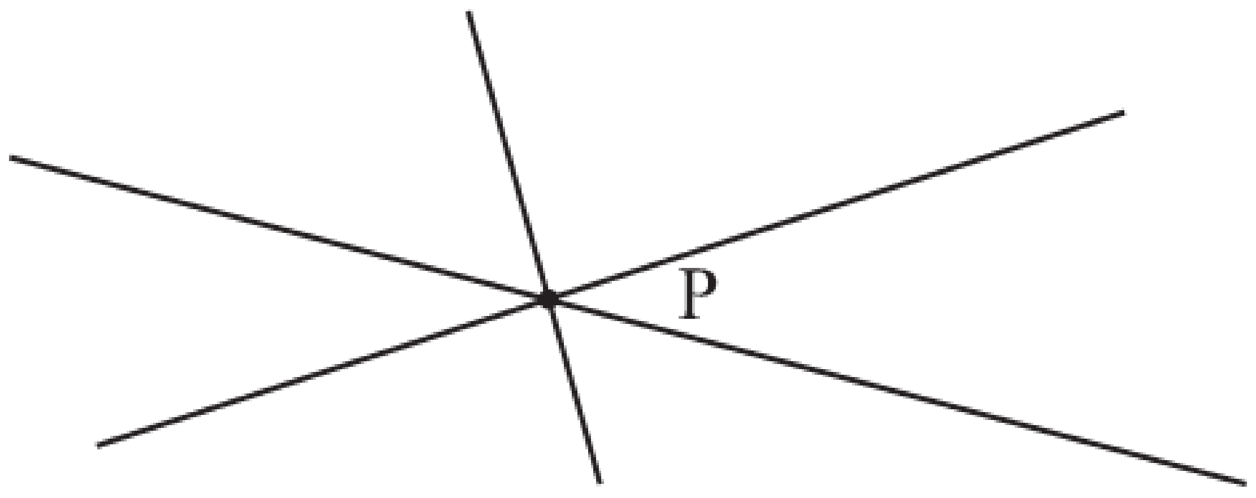
$B\widehat{A}C = \alpha$ and $D\widehat{A}C = \beta$.

Use Ptolemy's theorem to prove the addition formula $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

K THEOREMS OF CEVA AND MENELAUS

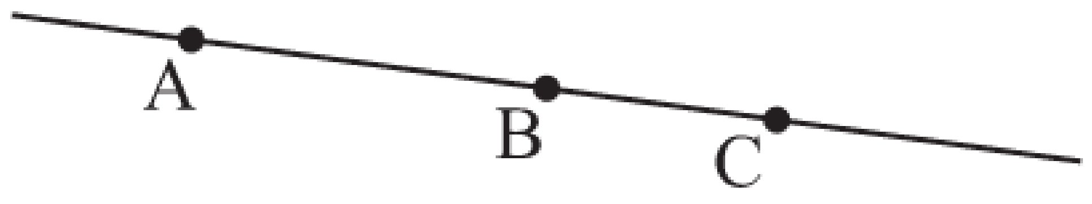
We have seen previously that:

- three or more lines are **concurrent** if they intersect at a common point



These lines are concurrent at P.

- three or more points are **collinear** if one straight line passes through all of them.

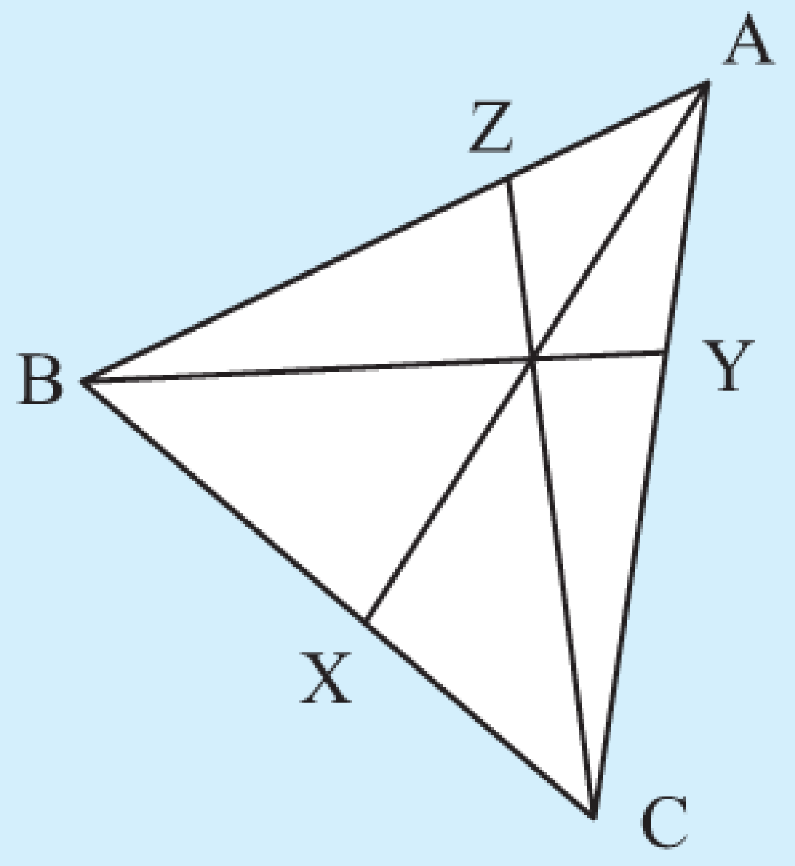


Points A, B, and C are collinear.

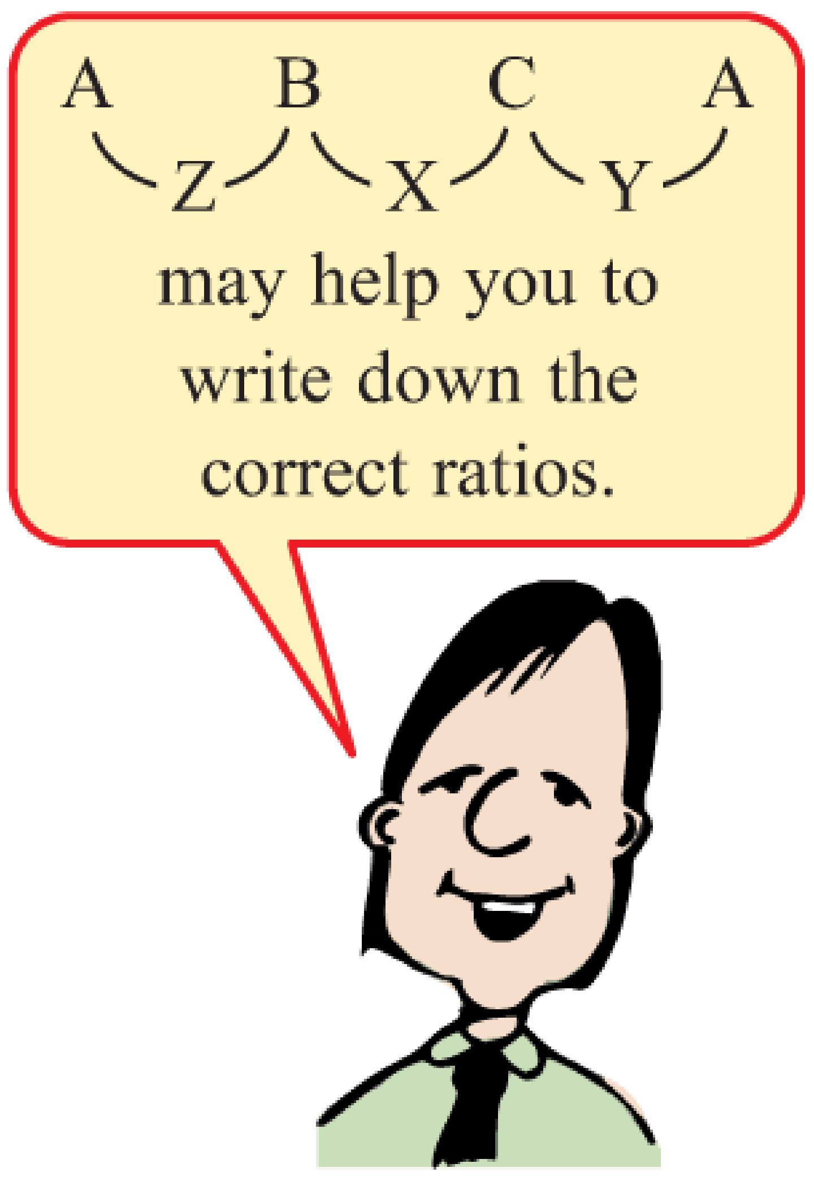
The converse theorems of **Ceva** and **Menelaus** enable us to establish concurrency and collinearity, respectively.

CEVA'S THEOREM

Any three concurrent lines drawn from the vertices of a triangle divide the sides (produced if necessary) so that the product of their respective ratios is unity.



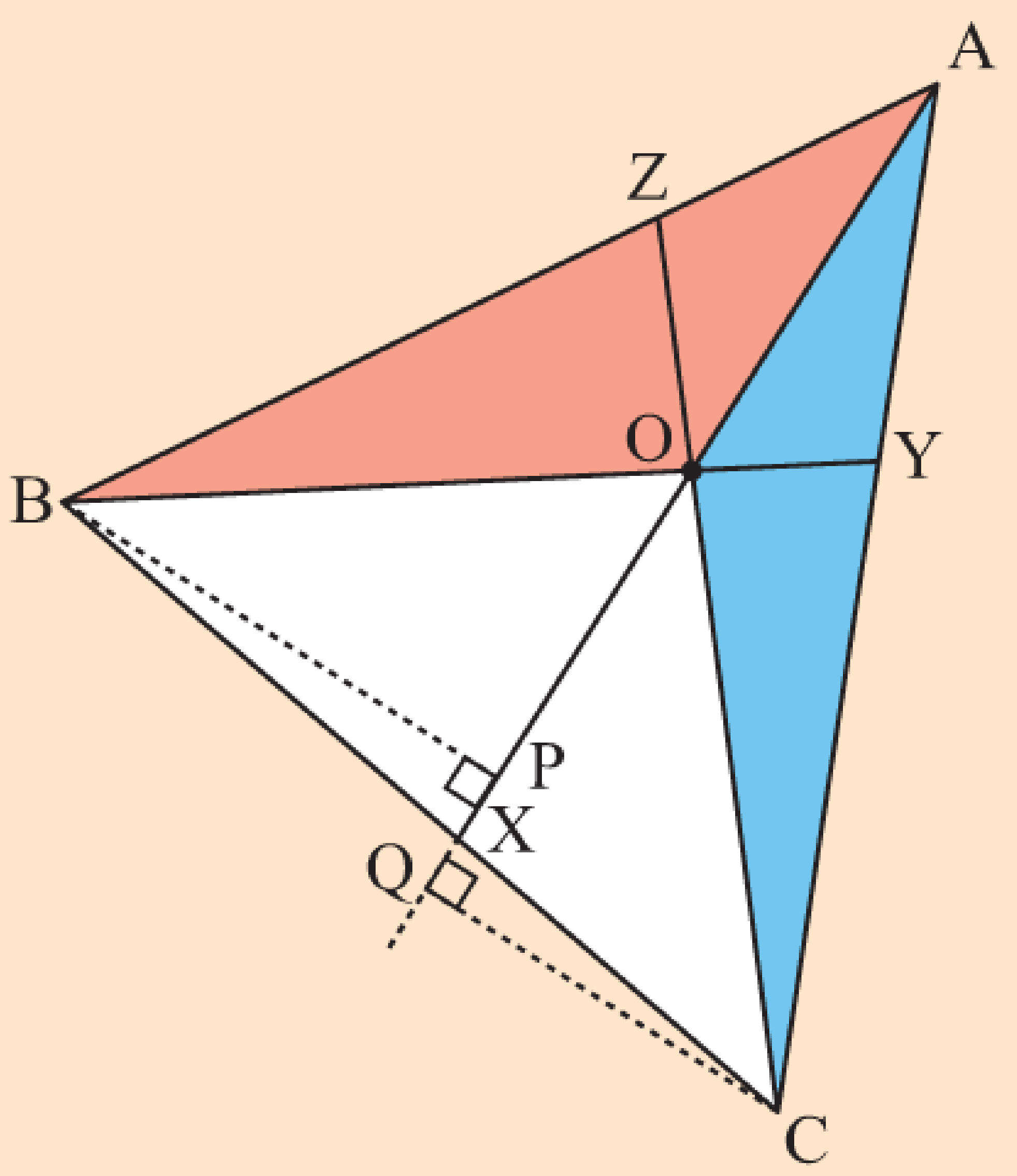
$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=1$$



may help you to write down the correct ratios.

Proof of Ceva's theorem:

We use the theorem that if two triangles have the same base, then the ratio of their areas is the same as the ratio of their altitudes.



In $\triangle ABC$, [AX], [BY], and [CZ] intersect at O.

We draw altitudes [BP] for $\triangle AOB$ and [CQ] for $\triangle AOC$.

In $\triangle$s BXP and CXQ:

- $B\widehat{P}X = C\widehat{Q}X$
- $B\widehat{X}P = C\widehat{X}Q$ {vertically opposite angles}

The triangles are equiangular and $\therefore$ similar.

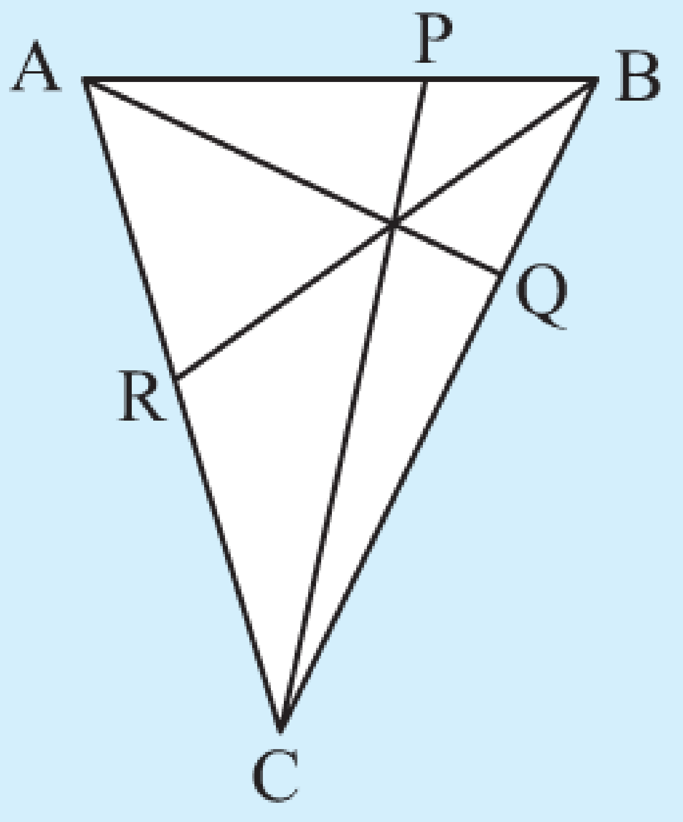
$$\therefore \quad \frac{BX}{CX}=\frac{BP}{CQ}=\frac{\text{area of }\triangle AOB}{\text{area of }\triangle AOC} \quad \dots\dots (1)$$

{as $\triangle$s have common base [AO]}

Likewise, $\dfrac{CY}{AY} = \dfrac{\text{area of } \triangle BOC}{\text{area of } \triangle BOA}$ .... (2) and $\dfrac{AZ}{BZ} = \dfrac{\text{area of } \triangle AOC}{\text{area of } \triangle BOC}$ .... (3)

Multiplying (1), (2), and (3) gives $\dfrac{AZ}{ZB}.\dfrac{BX}{XC}.\dfrac{CY}{YA} = \dfrac{\cancel{\text{area of } \triangle AOC}}{\cancel{\text{area of } \triangle BOC}} \times \dfrac{\cancel{\text{area of } \triangle AOB}}{\cancel{\text{area of } \triangle AOC}} \times \dfrac{\cancel{\text{area of } \triangle BOC}}{\cancel{\text{area of } \triangle AOB}} = 1$

Example 16



P divides [AB] in the ratio 2 : 1 and
Q divides [BC] in the ratio 3 : 7.

Find the ratio in which R divides [CA].

P divides [AB] in the ratio 2 : 1 $\Rightarrow$ AP : PB = 2 : 1

$$\therefore \quad \frac{AP}{PB} = \frac{2}{1}$$

Q divides [BC] in the ratio 3 : 7 $\Rightarrow$ $\dfrac{BQ}{QC} = \dfrac{3}{7}$

But $\dfrac{AP}{PB}.\dfrac{BQ}{QC}.\dfrac{CR}{RA} = 1$ {Ceva's theorem}

$$\therefore \quad \frac{2}{1} \times \frac{3}{7} \times \frac{CR}{RA} = 1$$

$$\therefore \quad \frac{CR}{RA} = \frac{7}{6}$$

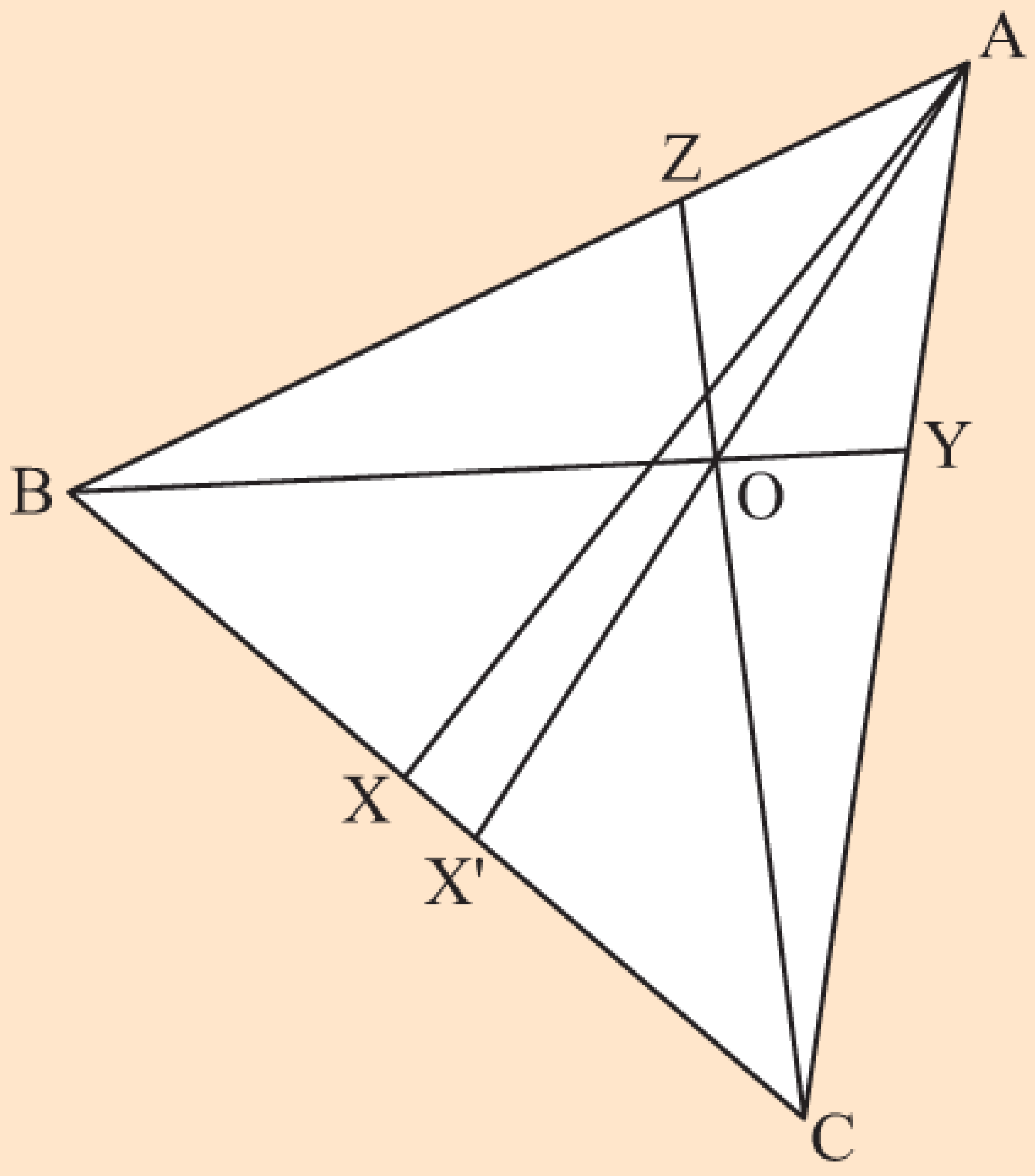
$\therefore$ R divides [CA] in the ratio 7 : 6.

If P **divides** [AB] in the ratio $r : s$ then AP : PB = $r : s$.

THE CONVERSE OF CEVA'S THEOREM

If three lines are drawn from the vertices of a triangle to cut the opposite sides (or sides produced) such that the product of their respective ratios is unity, then the three lines are concurrent.

Proof:



Let [BY] and [CZ] meet at O.

Suppose [AO] produced meets [BC] at point X′.

$\therefore$ $\dfrac{BX'}{X'C}.\dfrac{CY}{YA}.\dfrac{AZ}{ZB} = 1$ {Ceva's theorem}

But $\dfrac{BX}{XC}.\dfrac{CY}{YA}.\dfrac{AZ}{ZB} = 1$ {given}

$$\therefore \quad \frac{BX'}{X'C} = \frac{BX}{XC}$$

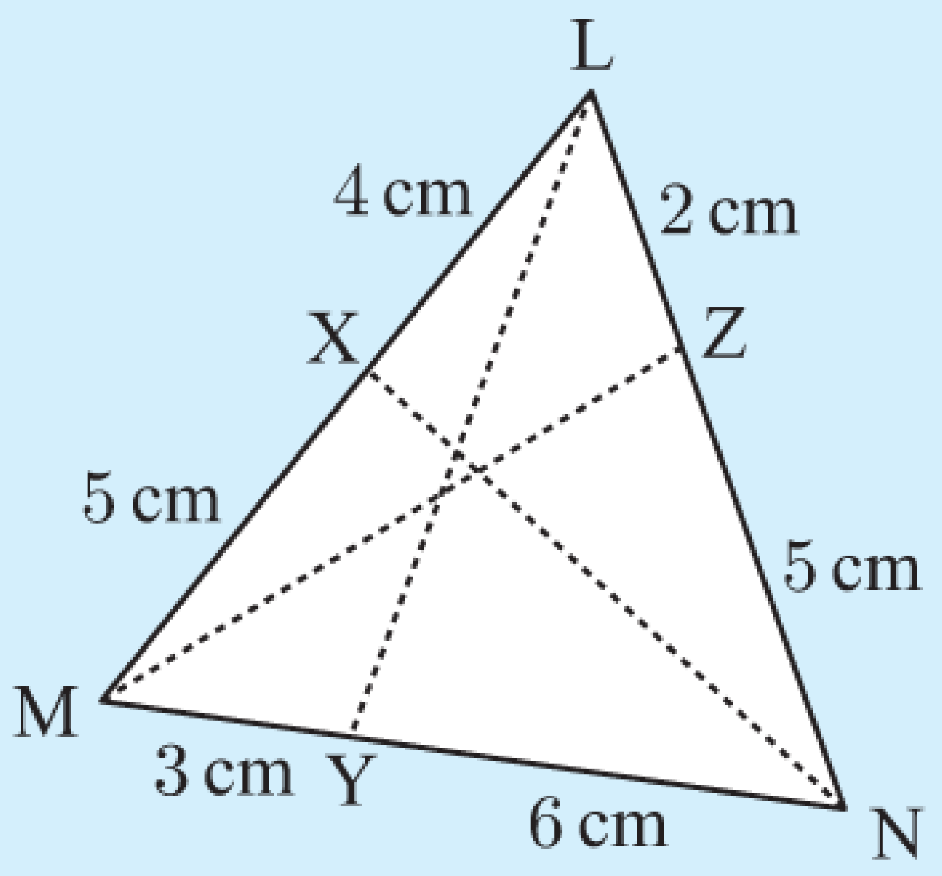
$\therefore$ X and X′ coincide
{as B, X, X′, and C are collinear}

$\therefore$ [AX], [BY], and [CZ] are concurrent.

Example 17

LMN is a triangle. X is on [LM], Y is on [MN], and Z is on [NL]. LX = 4 cm, MY = 3 cm, NZ = 5 cm, YN = 6 cm, ZL = 2 cm, and XM = 5 cm.

Prove that [MZ], [NX], and [LY] are concurrent.



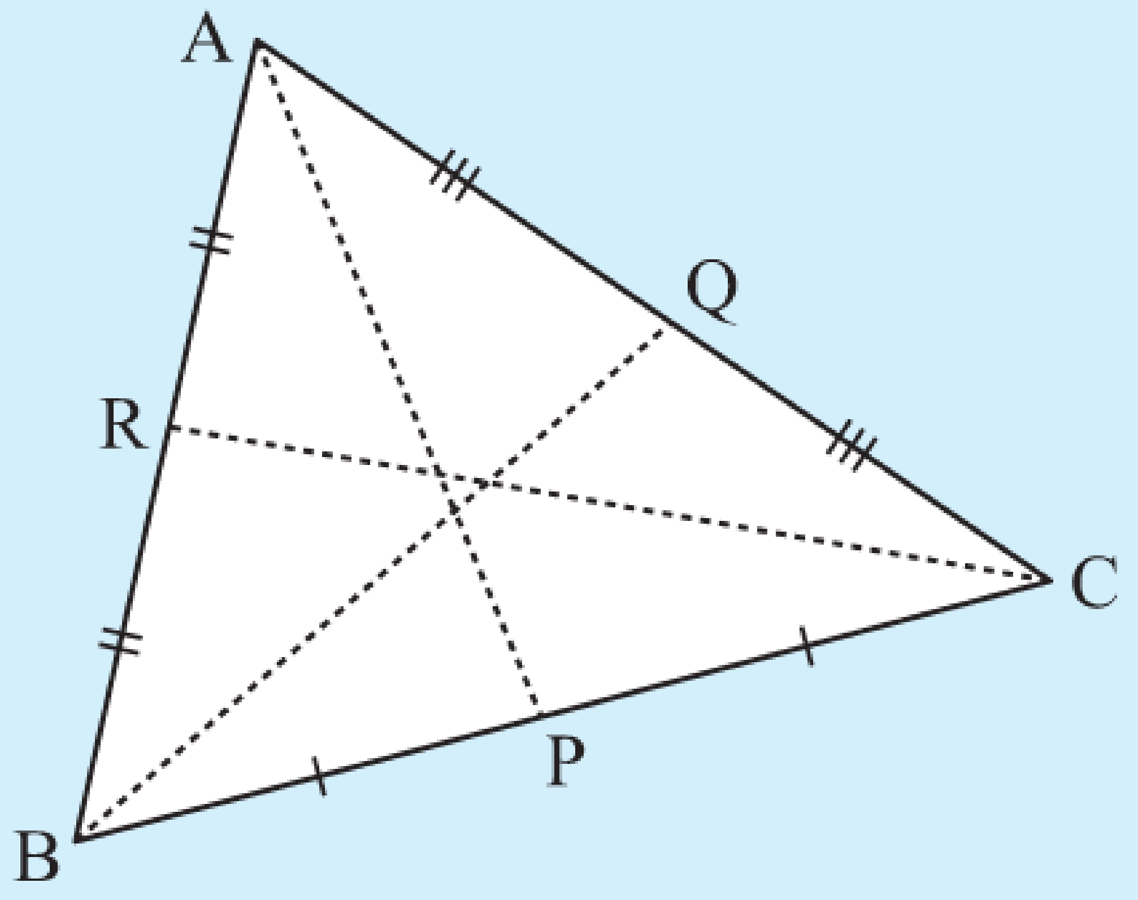
$$\frac{LX}{XM}\cdot\frac{MY}{YN}\cdot\frac{NZ}{ZL}$$
$$= \tfrac{4}{5} \times \tfrac{3}{6} \times \tfrac{5}{2}$$
$$= 1$$

$\therefore$ [MZ], [NX], and [LY] are concurrent.

{converse of Ceva's theorem}

Example 18

Use the converse of Ceva's theorem to prove that the medians of a triangle are concurrent.



Let the medians of $\triangle ABC$ be [AP], [BQ], and [CR] respectively.

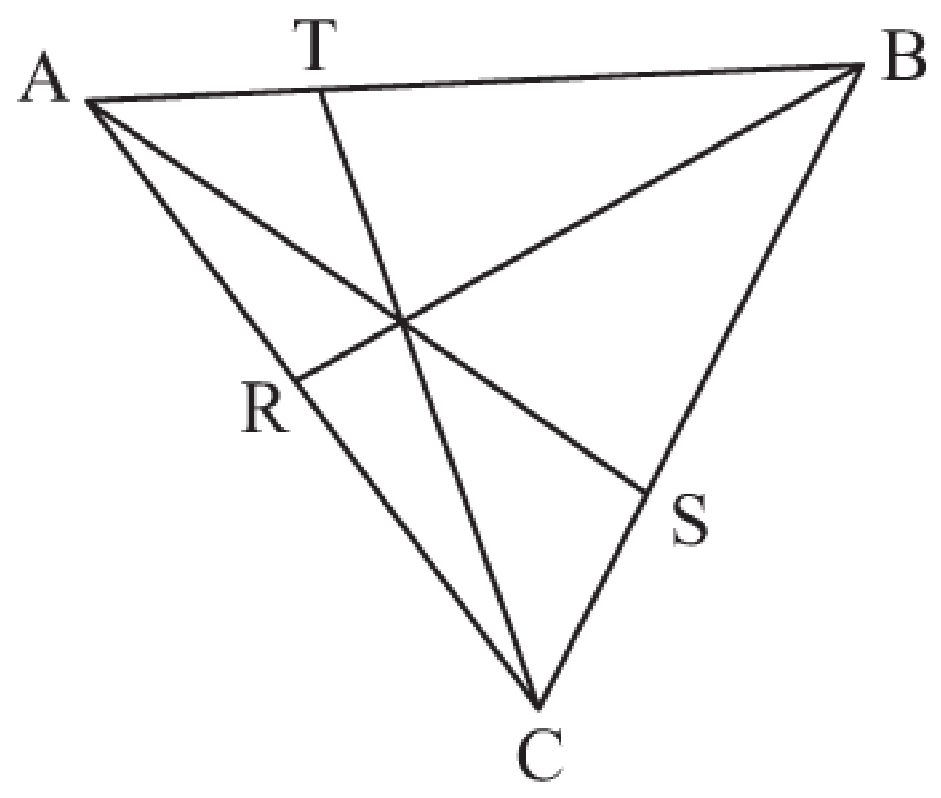
Now $\dfrac{AR}{RB}\cdot\dfrac{BP}{PC}\cdot\dfrac{CQ}{QA} = 1 \times 1 \times 1 = 1$

$\Rightarrow$ [AP], [BQ], and [CR] are concurrent.

{converse of Ceva's theorem}

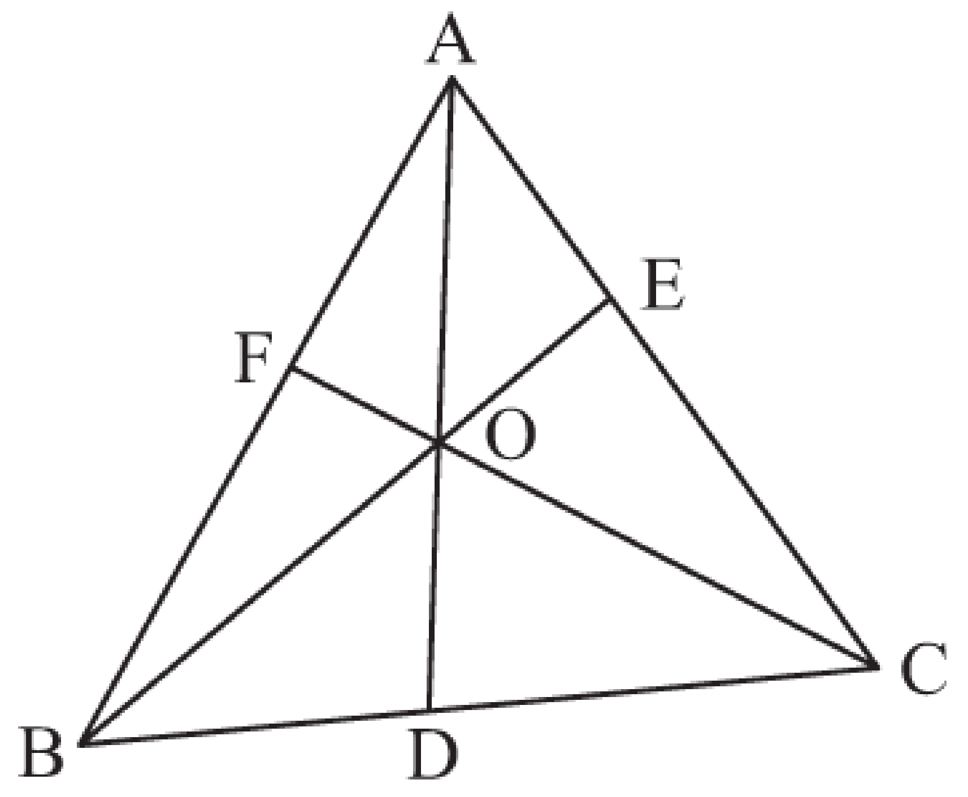
EXERCISE 2K.1

1



T divides [AB] in the ratio 3 : 7.
S divides [BC] in the ratio 5 : 3.
Find the ratio in which R divides [AC].

2



In $\triangle ABC$, D lies on [BC] such that $BD = \frac{1}{2}BC$.

E lies on [AC] such that $CE = \frac{2}{3}CA$.

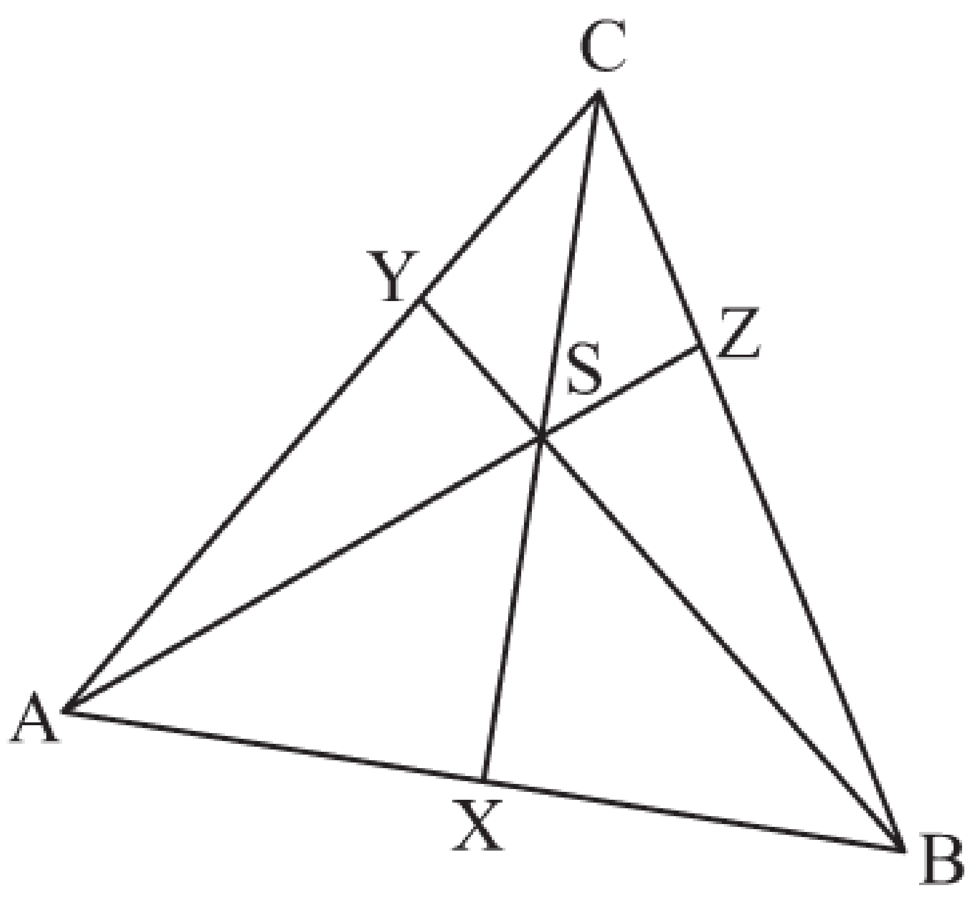
[BE] and [AD] intersect at O, and [CO] produced meets [AB] at F.

Find:

a AF : FB

b area of $\triangle AOB$: area of $\triangle BOC$.

3



In the diagram, BZ : ZC = 2 : 1 and AY : YC = 3 : 2.

a Find the ratio in which X divides [AB].

b Find the ratio in which S divides [AZ].

4 P, Q, and R lie on sides [AB], [BC], and [CA] of triangle ABC respectively, such that $AP = \frac{2}{3}AB$, $BQ = \frac{3}{4}BC$, and $CR = \frac{1}{7}CA$. Prove that [AQ], [BR], and [CP] are concurrent.

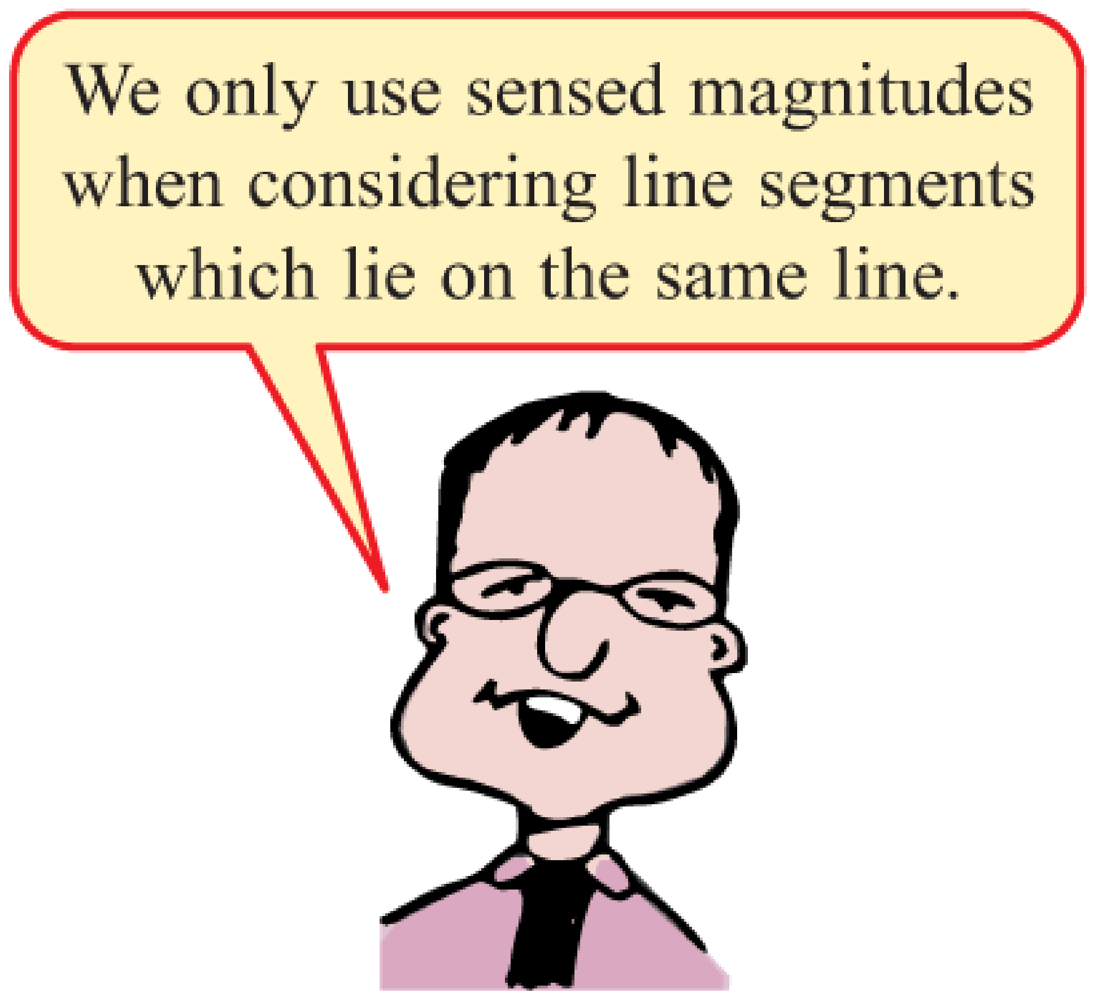
5 Use the converse of Ceva's theorem to prove that the angle bisectors of a triangle are concurrent.

6 The inscribed circle of triangle PQR has tangents [QR], [RP], and [PQ] which touch the circle at A, B, and C, respectively. Prove that [PA], [QB], and [RC] are concurrent.

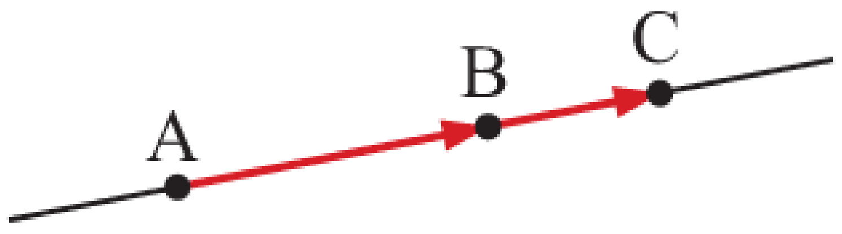
7 Use the converse of Ceva's theorem to prove that the altitudes from the three vertices of a triangle are concurrent.

MENELAUS' THEOREM

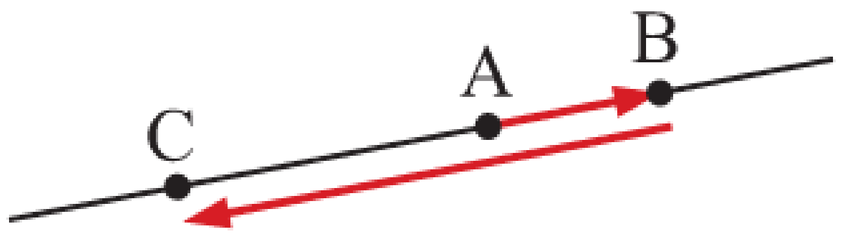
So far, when we have considered ratios of lengths of line segments, we have used the length or magnitude only. To state Menelaus' theorem, we need to use **sensed magnitudes**, which means that a ratio is taken to be positive or negative depending on whether the line segments are written as vectors with the same direction.


We only use sensed magnitudes when considering line segments which lie on the same line.

For example:

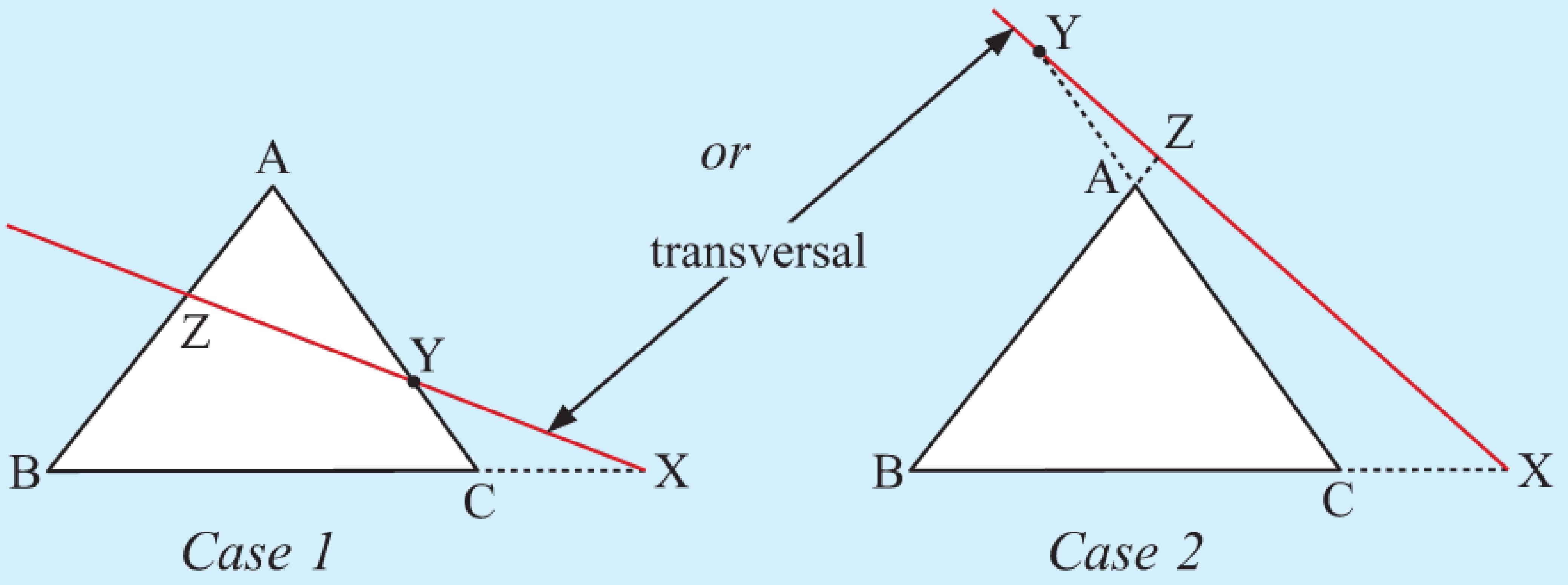
- 

  $\frac{AB}{BC}$ is positive, since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same direction.

- 

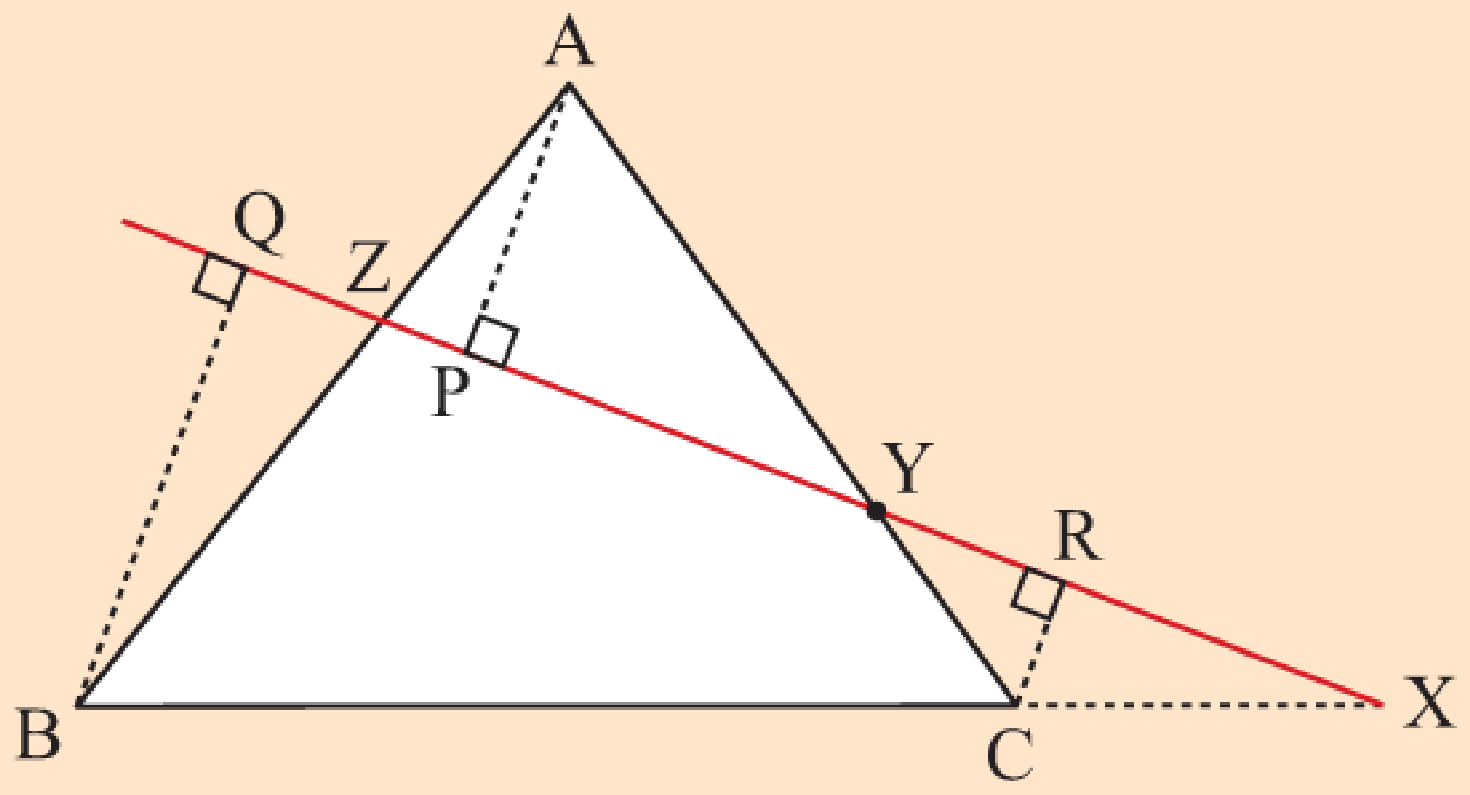
  $\frac{AB}{BC}$ is negative, since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are opposite in direction.

If a transversal is drawn to cut the sides of a triangle (produced if necessary), then using sensed magnitudes, the product of the ratios of alternate segments is minus one.



$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$$

Proof: (for *Case 1*)



We draw perpendiculars from A, B, and C to the transversal.

$\triangle$s BQX and CRX are similar $\Rightarrow \dfrac{BX}{CX} = \dfrac{BQ}{CR}$

$\therefore \dfrac{BX}{XC} = -\dfrac{BQ}{CR}$

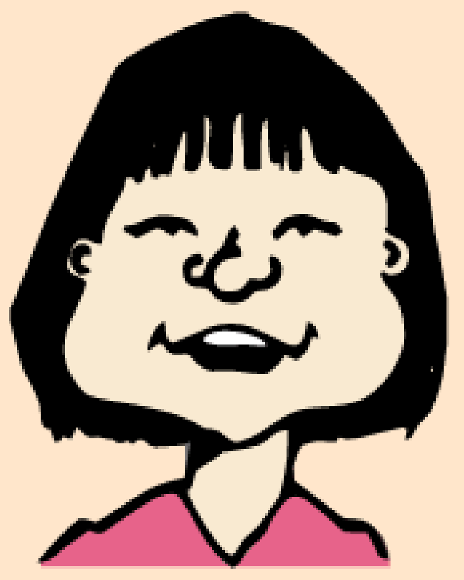
$\triangle$s CYR and AYP are similar $\Rightarrow \dfrac{CY}{AY} = \dfrac{CR}{AP}$

$\therefore \dfrac{CY}{YA} = \dfrac{CR}{PA}$

$\triangle$s BQZ and APZ are similar $\Rightarrow \dfrac{AZ}{BZ} = \dfrac{AP}{BQ}$

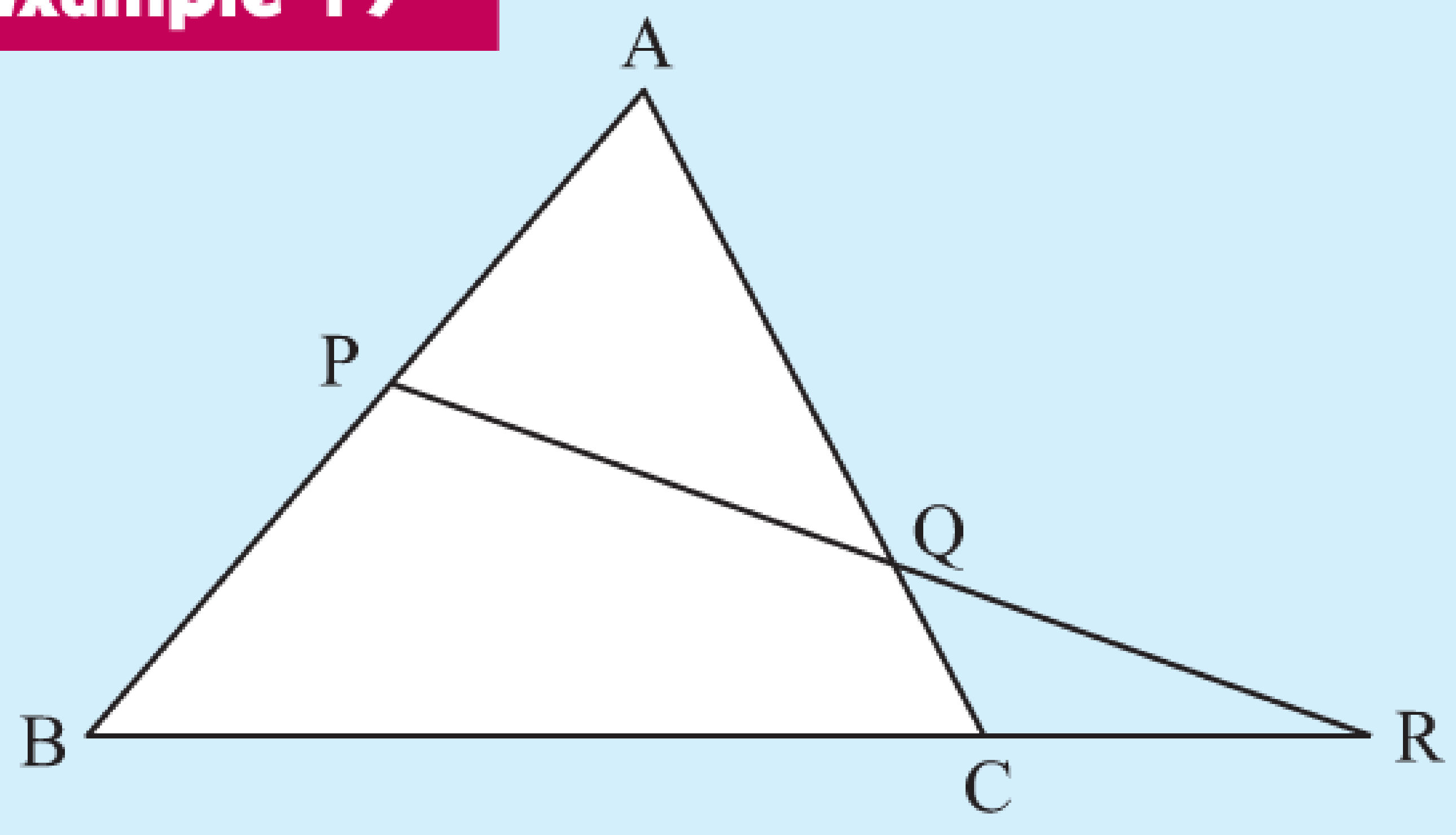
$\therefore \dfrac{AZ}{ZB} = \dfrac{PA}{BQ}$

Thus
$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = -\frac{BQ}{CR}\cdot\frac{CR}{PA}\cdot\frac{PA}{BQ}$$
$$= -1$$



You should show that this proof also holds for *Case 2*.

Example 19



P divides [AB] in the ratio 2 : 3, and Q divides [AC] in the ratio 5 : 2.

In what ratio does R divide [BC]?

Since PQR is a transversal of $\triangle$ABC, $\dfrac{AP}{PB}\cdot\dfrac{BR}{RC}\cdot\dfrac{CQ}{QA} = -1$

Now $\dfrac{AP}{PB} = \dfrac{2}{3}$ and $\dfrac{AQ}{QC} = \dfrac{5}{2}$

$$\therefore \frac{2}{3}\times\frac{BR}{RC}\times\frac{2}{5} = -1 \quad \left\{\frac{CQ}{QA} = \frac{-QC}{-AQ} = \frac{QC}{AQ} = \left(\frac{AQ}{QC}\right)^{-1}\right\}$$

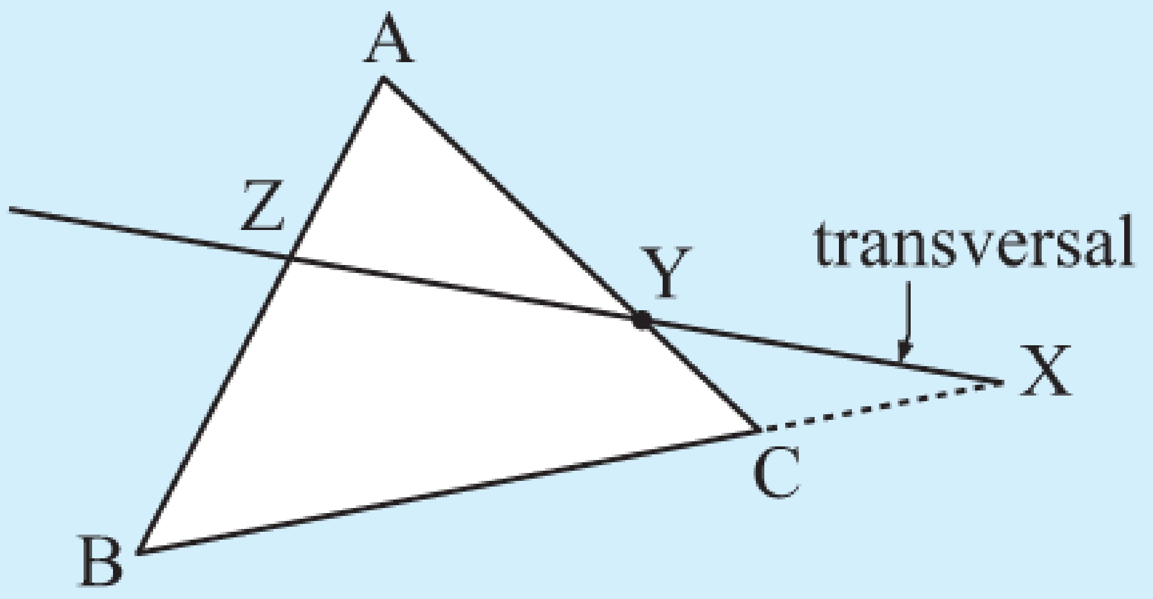
$$\therefore \frac{BR}{RC} = -\frac{15}{4}$$

$\therefore$ R divides [BC] externally in the ratio 15 : 4.

Since $\dfrac{BR}{RC} < 0$, R does not lie on [BC], but rather on [BC] produced.

THE CONVERSE OF MENELAUS' THEOREM

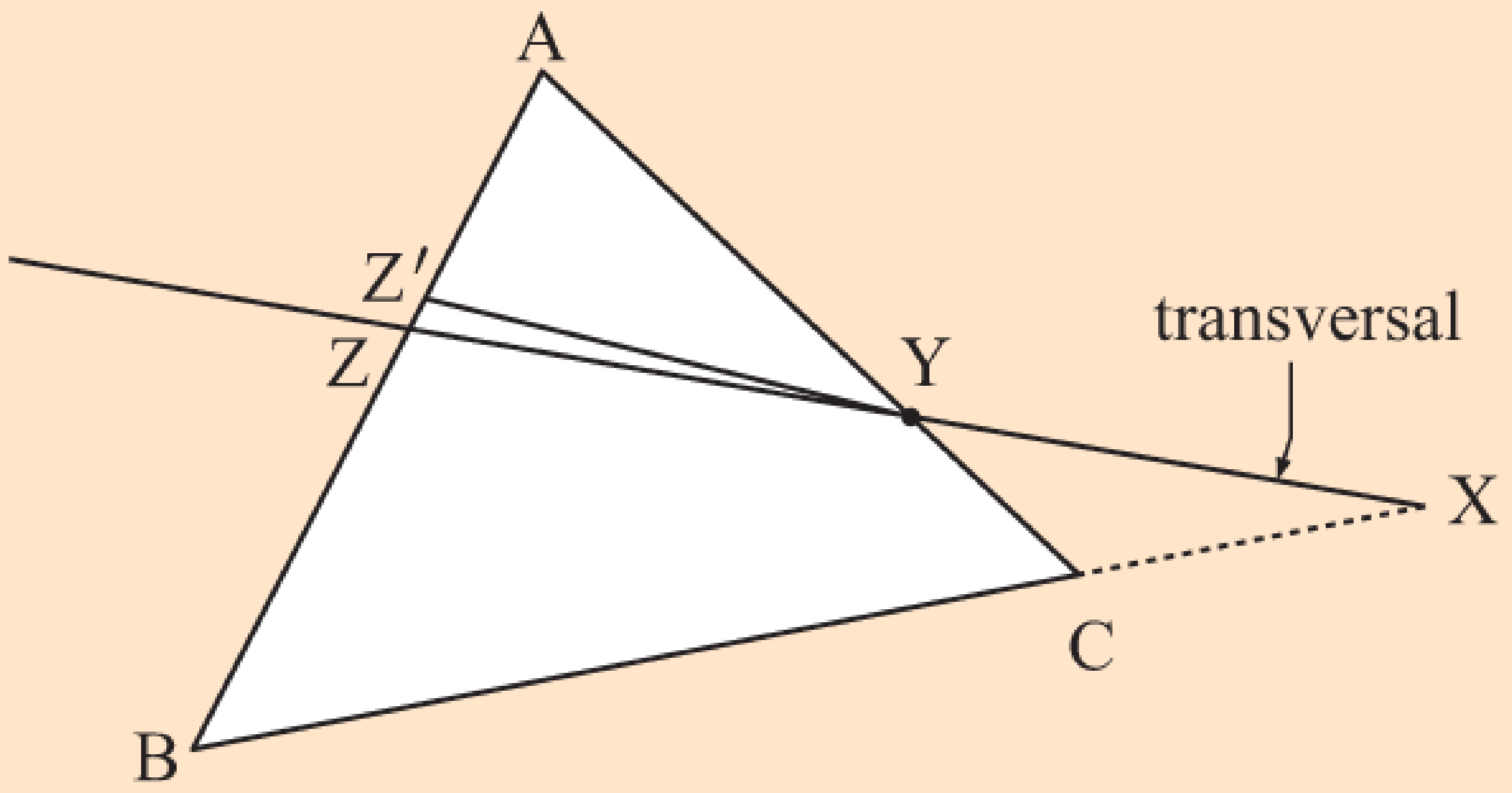
If three points on two sides of a triangle and the other side produced (or on all three sides produced) are such that the product of the ratios of alternate segments is equal to minus one, then the three points are collinear.



If $\frac{BX}{XC}.\frac{CY}{YA}.\frac{AZ}{ZB} = -1$,

then X, Y, and Z are collinear.

Proof: (for the illustrated case)



Let XYZ′ be a straight line.

$\therefore$ $\frac{BX}{XC}.\frac{CY}{YA}.\frac{AZ'}{Z'B} = -1$ {Menelaus' theorem}

But $\frac{BX}{XC}.\frac{CY}{YA}.\frac{AZ}{ZB} = -1$

$\therefore$ $\frac{AZ'}{Z'B} = \frac{AZ}{ZB}$

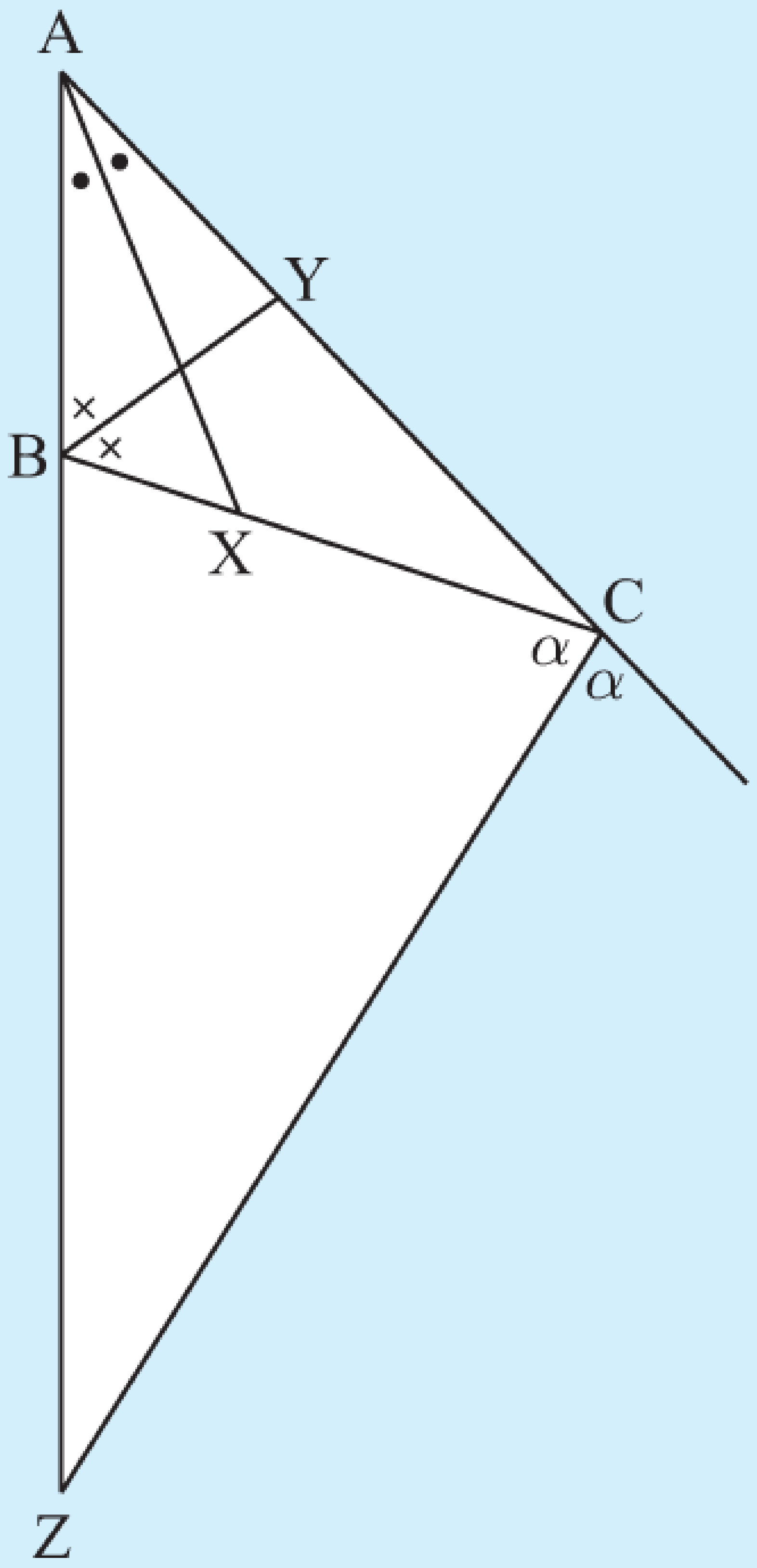
{A, Z′, Z, and B are collinear}

$\therefore$ Z′ and Z coincide.

$\therefore$ X, Y, and Z are collinear.

Example 20

In a triangle two angles are bisected internally, and the third angle is bisected externally. Prove that the points where the angle bisectors meet the triangle's sides are collinear.



Let the triangle be ABC, and the internal angle bisectors at A and B meet [BC] and [AC] at X and Y respectively. Let the external angle at C be bisected by [CZ] where Z lies on [AB] produced.

By the angle bisector theorem, as [AX] bisects $B\widehat{A}C$,

$$\frac{AB}{AC} = \frac{BX}{XC} \Rightarrow \frac{BX}{CX} = -\frac{AB}{AC} \quad .... (1)$$

Likewise, as [BY] bisects $A\widehat{B}C$,

$$\frac{BA}{BC} = \frac{AY}{CY} \Rightarrow \frac{CY}{AY} = \frac{BC}{BA}$$

$$\Rightarrow \frac{CY}{YA} = \frac{BC}{AB} \quad .... (2)$$

Also, as [CZ] bisects the external angle,

$$-\frac{CA}{CB} = \frac{AZ}{BZ} \Rightarrow \frac{AZ}{ZB} = \frac{CA}{CB}$$

$$\Rightarrow \frac{AZ}{ZB} = \frac{AC}{BC} \quad .... (3)$$

From (1), (2), and (3), $\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB} = -\frac{AB}{AC}\cdot\frac{BC}{AB}\cdot\frac{AC}{BC}$

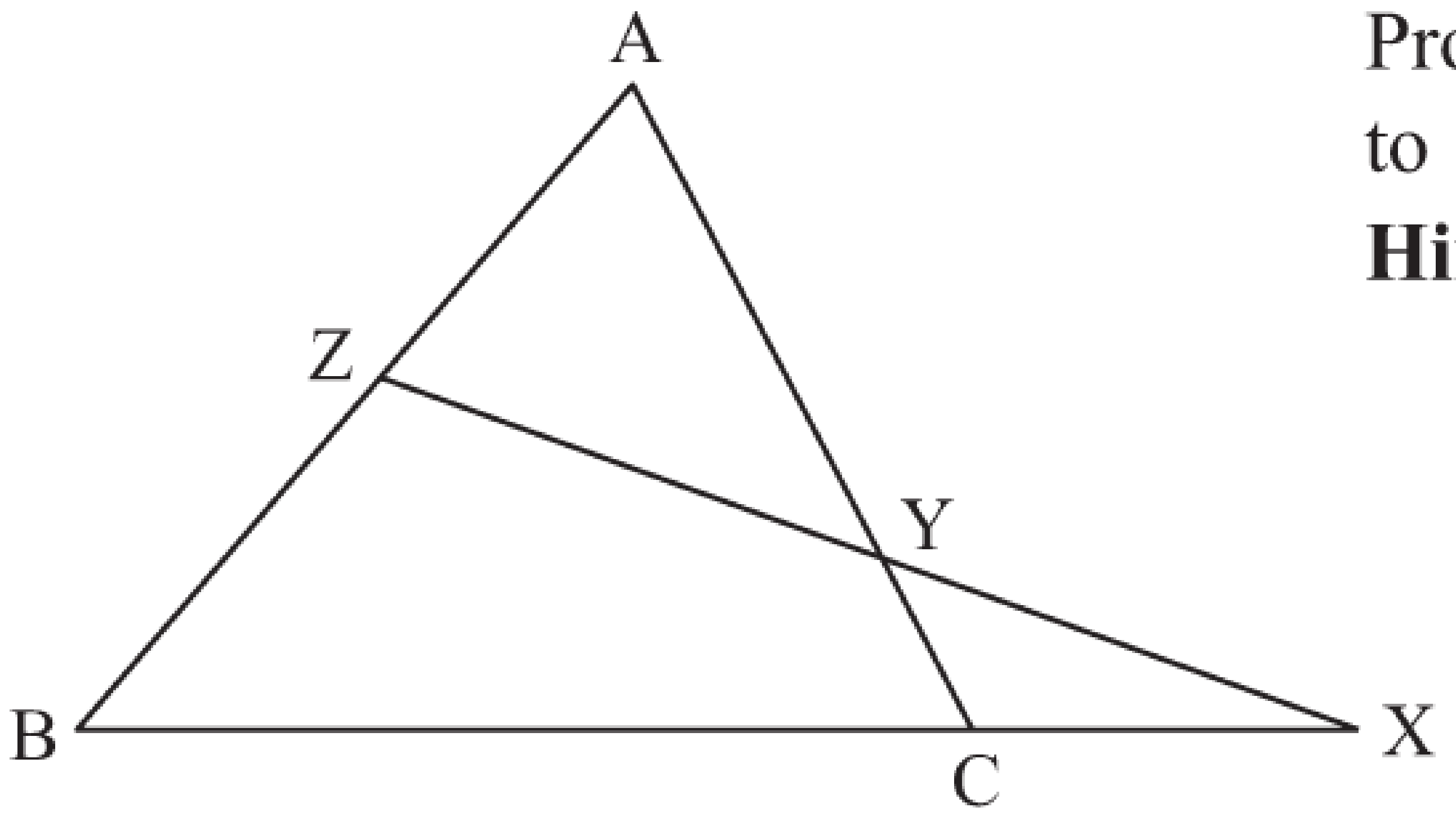
$= -1$

$\Rightarrow$ X, Y, and Z are collinear. {converse of Menelaus' theorem}

EXERCISE 2K.2

1 Transversal XYZ of triangle ABC cuts [BC], [CA], and [AB] produced, at X, Y, and Z respectively. If BX : XC = 3 : 5 and AY : YC = 2 : 1, find the ratio in which Z divides [AB].

2



Prove Menelaus' theorem by constructing [AW] parallel to [XB], to meet the transversal at W.

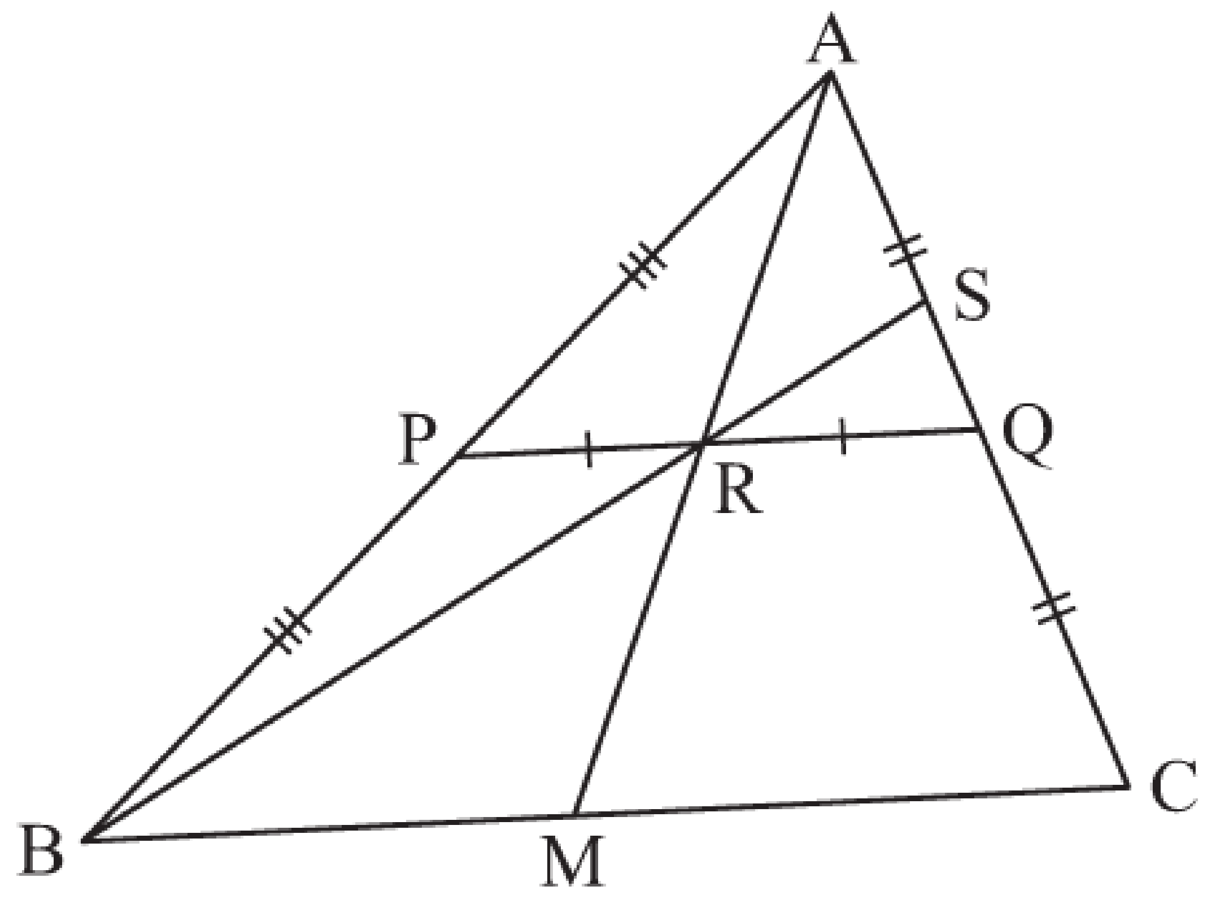
Hint: Look for similar triangles.

3 ABC is a triangle in which D divides [BC] in the ratio 2 : 3, and E divides [CA] in the ratio 5 : 4. Find the ratio in which [BE] divides [AD].

4 In the figure alongside, P and Q are the midpoints of sides [AB] and [AC] respectively. R is the midpoint of [PQ].

[BR] produced meets [AC] at S, and [AR] produced meets [BC] at M.

- **a** Show that M is the midpoint of [BC].
- **b** Find the ratio in which S divides [AC].
- **c** Find the ratio in which R divides [BS].

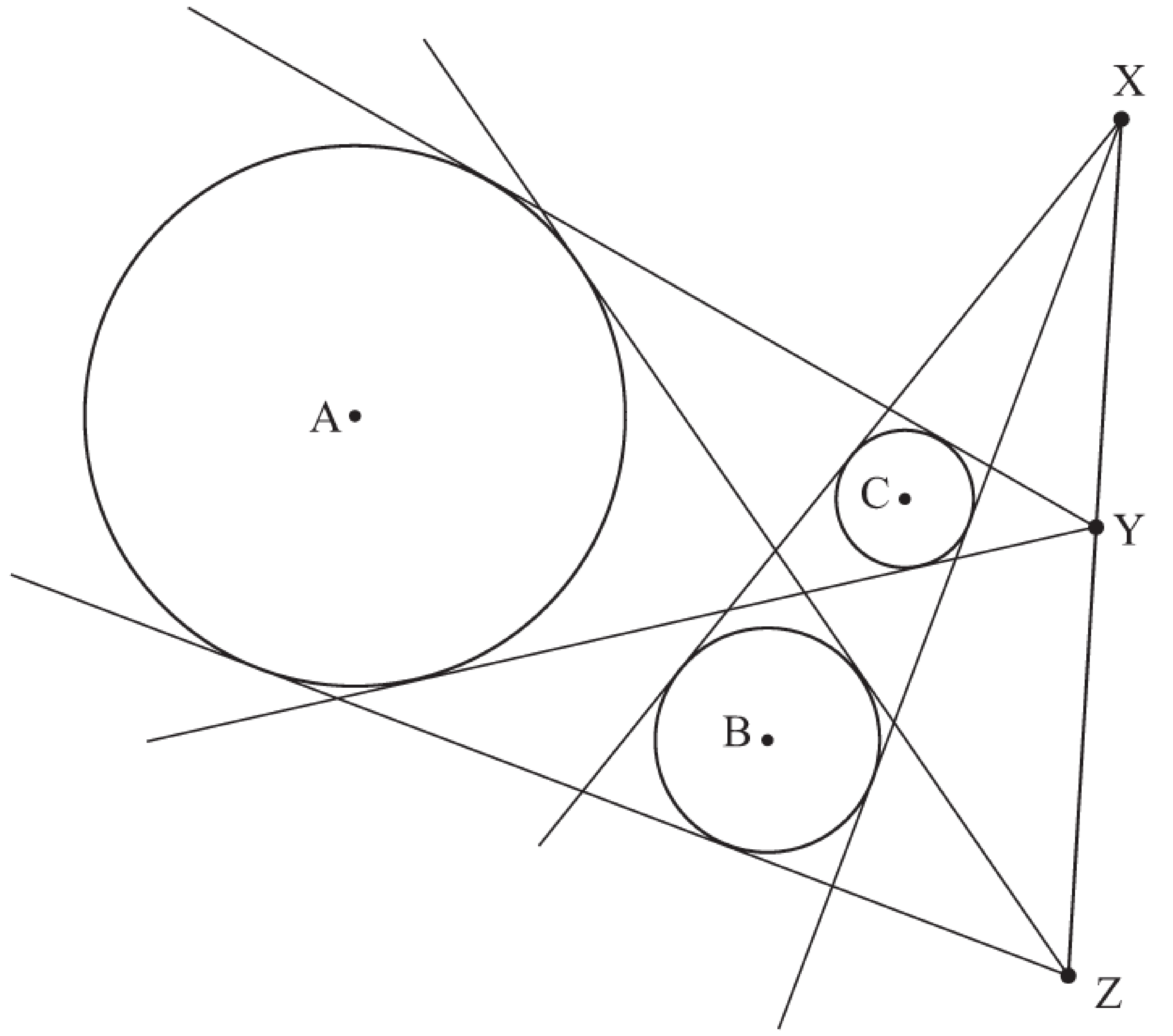


5 Common external tangents are drawn for the three pairs of illustrated circles.

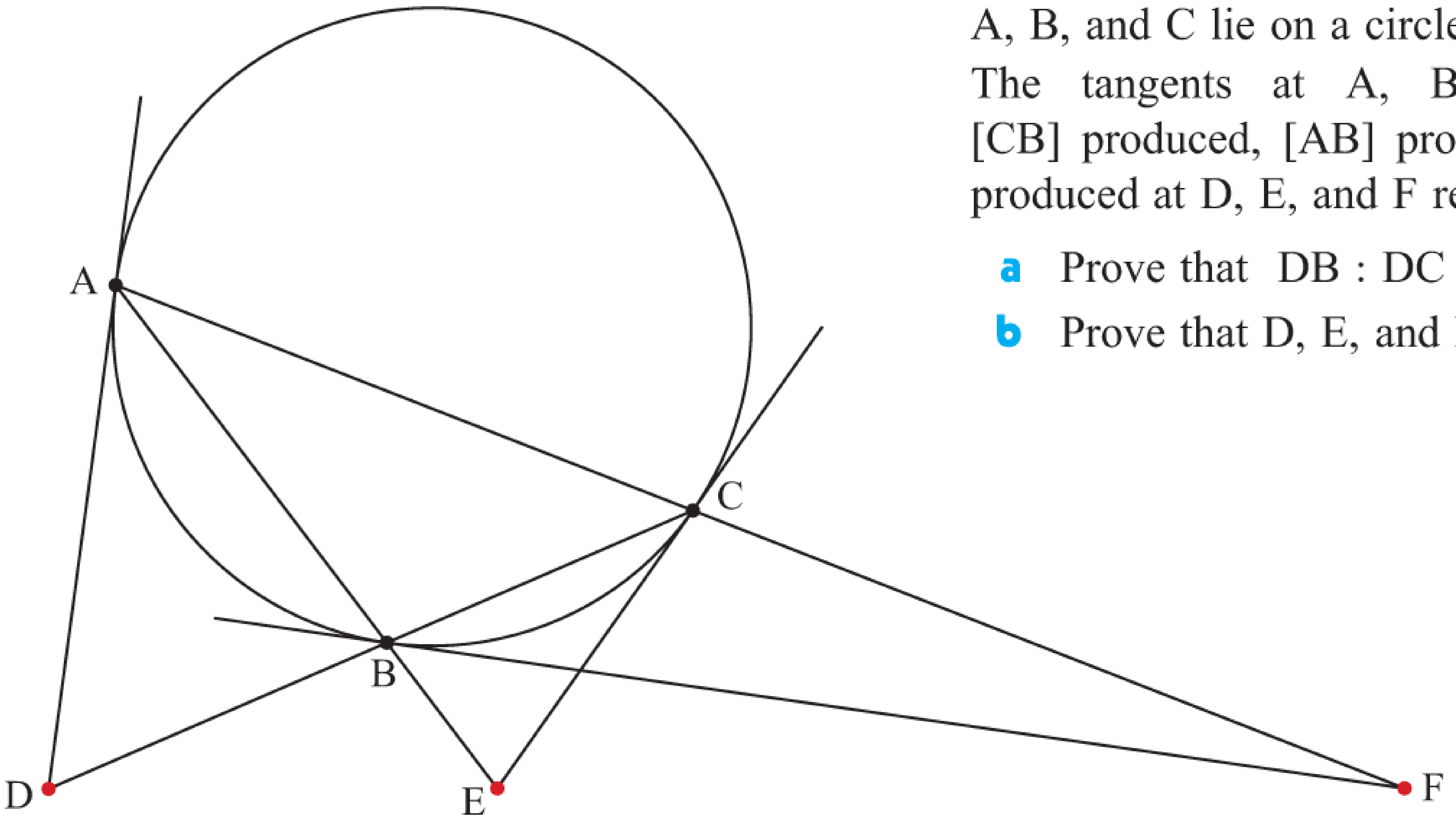
The circles have different radii a, b, and c units.

Use the converse of Menelaus' theorem to prove that X, Y, and Z are collinear.

DEMO



6



A, B, and C lie on a circle.

The tangents at A, B, and C meet [CB] produced, [AB] produced, and [AC] produced at D, E, and F respectively.

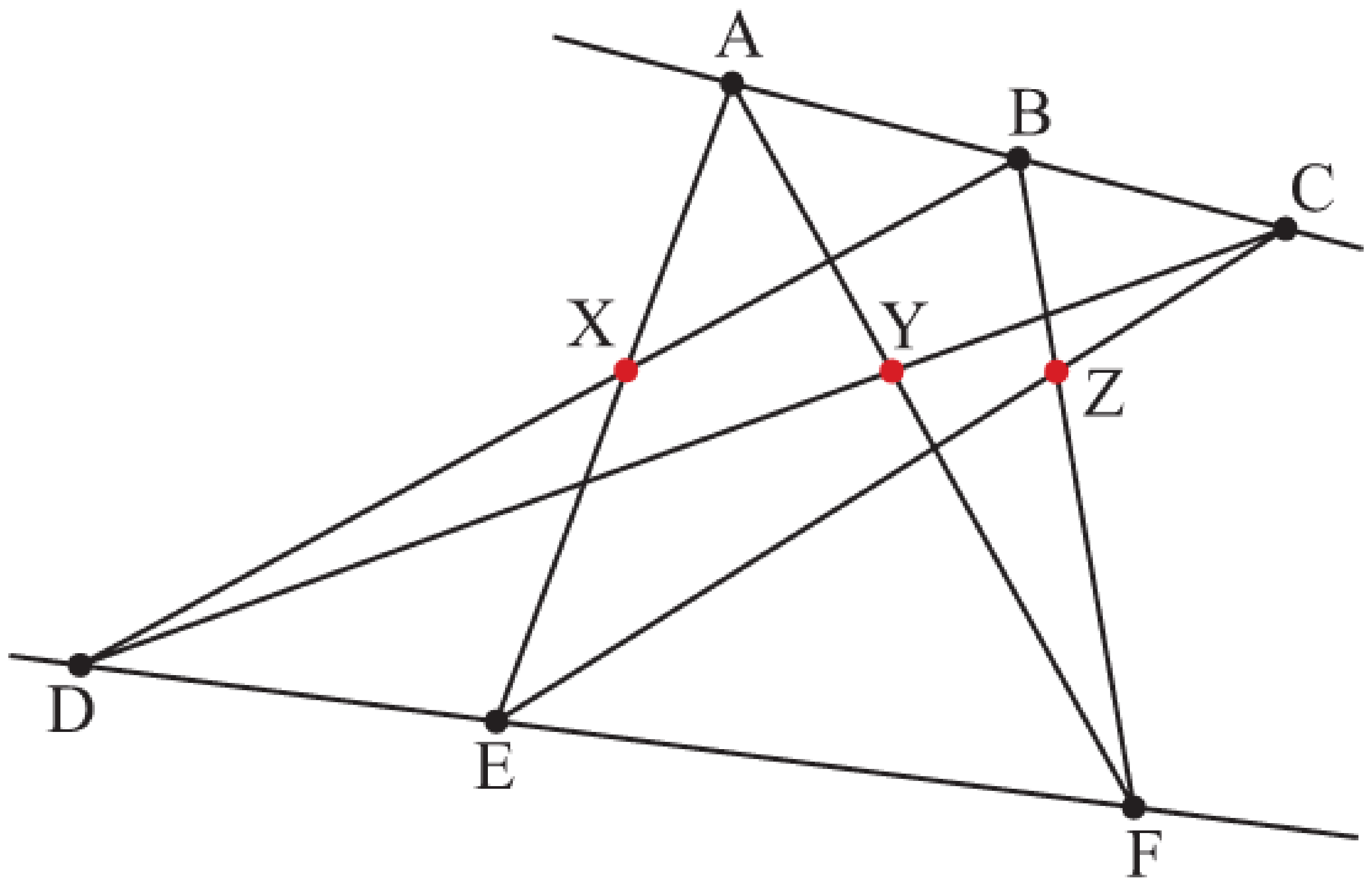
- **a** Prove that DB : DC = $AB^2 : AC^2$.
- **b** Prove that D, E, and F are collinear.

7 Consider two lines. One line contains the distinct points A, B, and C. The other line contains the distinct points D, E, and F.

Suppose [AE] and [BD] meet at X, [AF] and [CD] meet at Y, and [BF] and [CE] meet at Z.

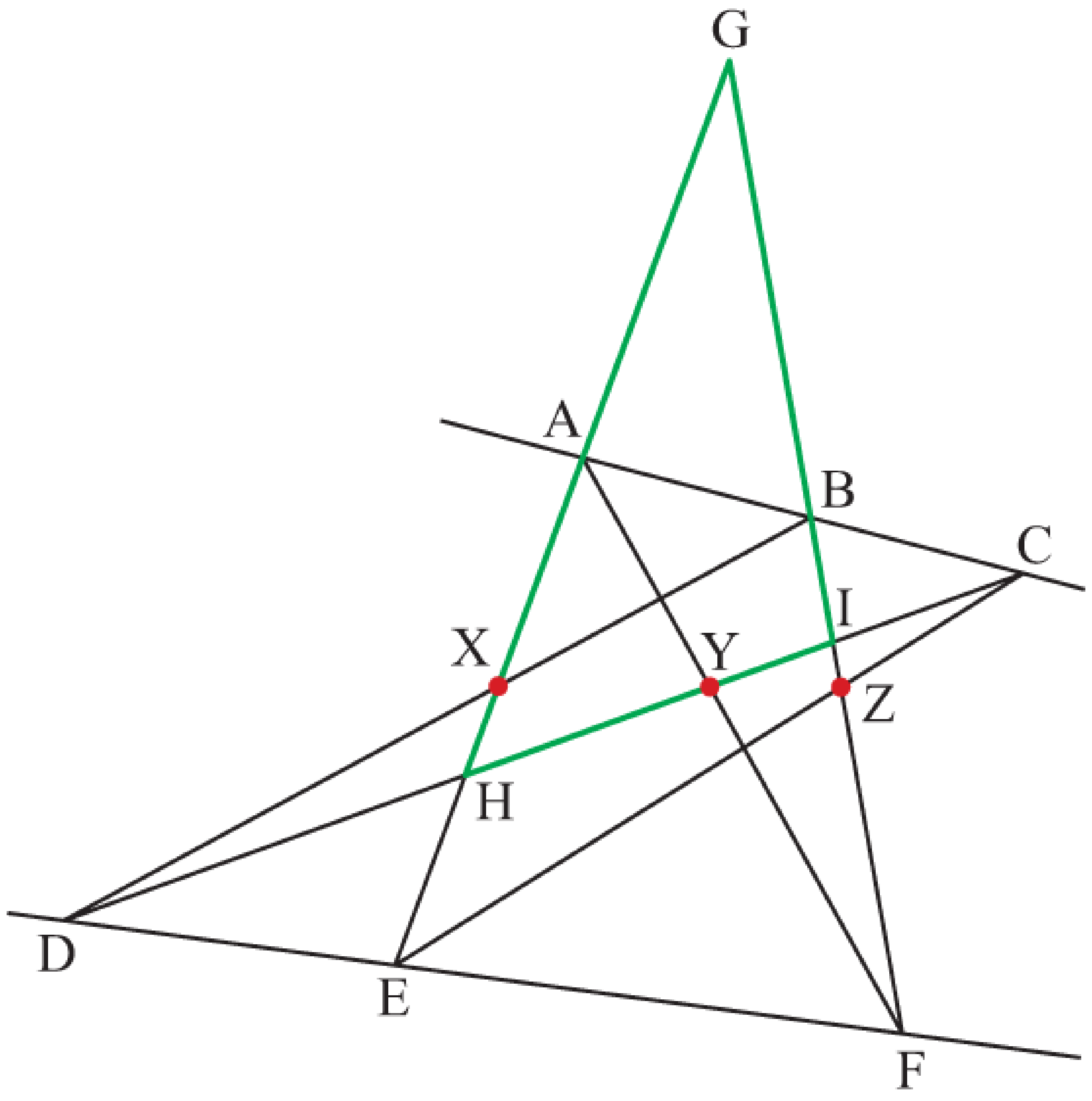
Pappus of Alexandria discovered that X, Y, and Z are always collinear.

Prove Pappus' theorem.



Hint: Produce [EA] and [FB] to meet at G. Let [DC] intersect [GF] at I. Apply Menelaus' theorem to each of the five transversals of triangle GHI.

GEOMETRY PACKAGE



L THE EQUATION OF A LOCUS

A **locus** is a set of points satisfying a particular equation, relation, or set of conditions.

The plural of locus is **loci**.

If $P(x, y)$ represents any point in a locus, the Cartesian equation connecting x and y is the **equation of the locus**.

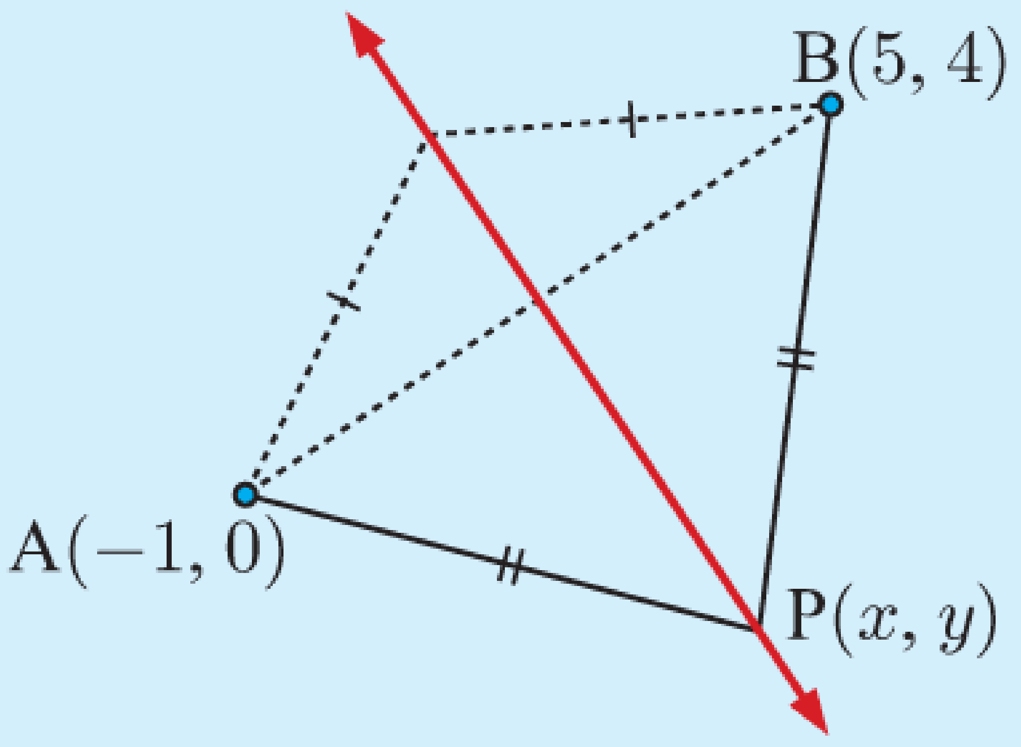
Example 21

Consider the locus of all points which are equidistant from $A(-1, 0)$ and $B(5, 4)$.

a Find the equation of the locus. **b** Describe the locus.

a

$$\begin{aligned} AP &= BP, \\ \therefore\ AP^2 &= BP^2 \\ \therefore\ (x+1)^2 + y^2 &= (x-5)^2 + (y-4)^2 \\ \therefore\ \cancel{x^2} + 2x + 1 + \cancel{y^2} &= \cancel{x^2} - 10x + 25 + \cancel{y^2} - 8y + 16 \\ \therefore\ 12x + 8y &= 40 \\ \therefore\ 3x + 2y &= 10 \end{aligned}$$



b The locus is the line $3x + 2y = 10$, which is the perpendicular bisector of [AB].

Check: The gradient of [AB] $= \dfrac{4-0}{5-(-1)} = \frac{4}{6} = \frac{2}{3}$

$\therefore$ gradient of perpendicular is $-\frac{3}{2}$.

The midpoint of [AB] is $\left(\dfrac{5+(-1)}{2}, \dfrac{4+0}{2}\right)$ or $(2, 2)$,

so the perpendicular bisector of [AB] is $3x + 2y = 3(2) + 2(2)$

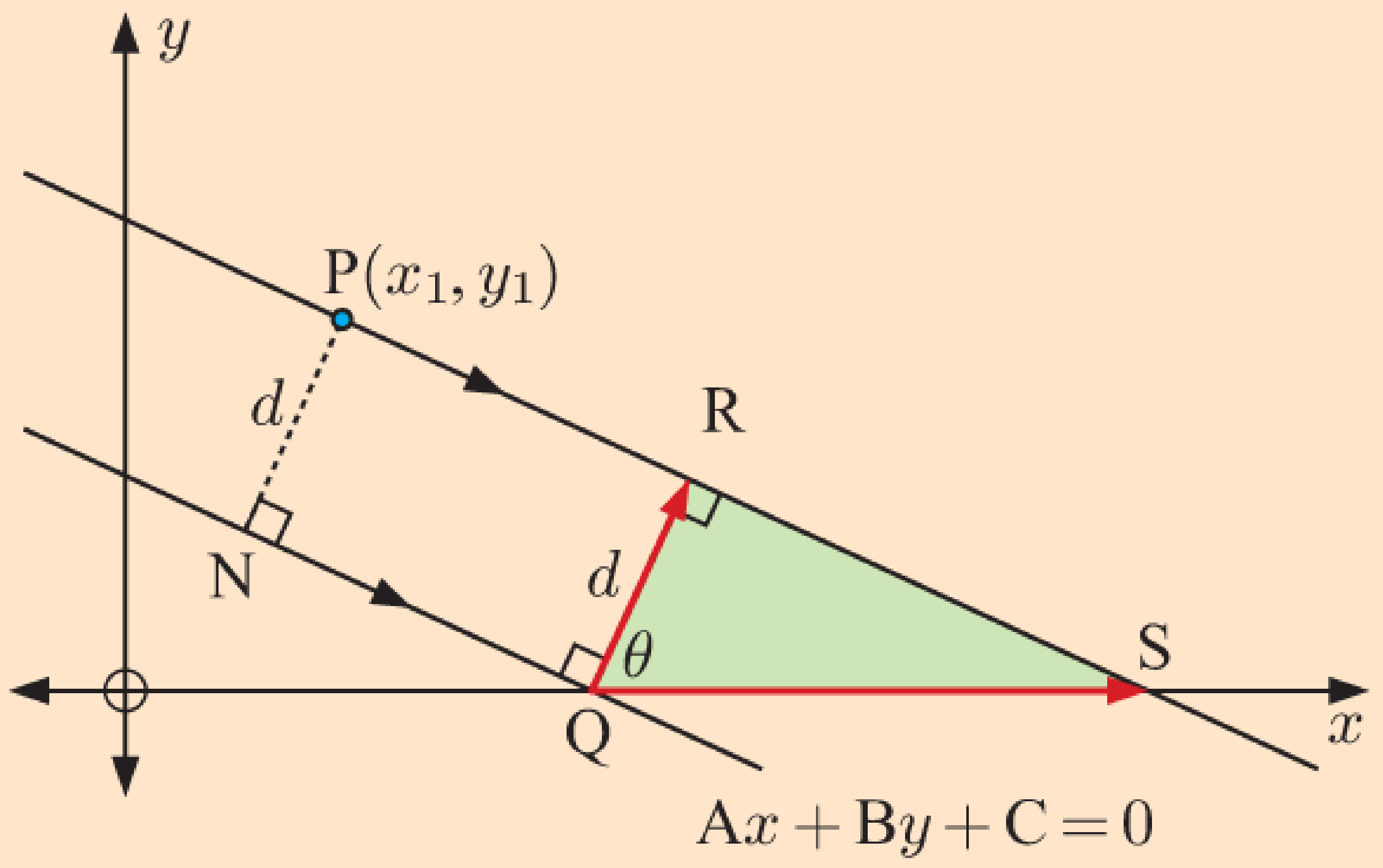
which is $3x + 2y = 10$.

In this Section we will be more concerned with finding the Cartesian equation of a locus rather than describing its nature. The nature of a locus is discussed in some sections which follow.

A useful result from finding equations of loci is the **distance from a point to a line formula**. This formula is:

The distance from the point (x_1, y_1) to $Ax + By + C = 0$ is $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

Proof:



We draw a line through $P(x_1, y_1)$ parallel to the given line.

This line has equation $Ax + By = Ax_1 + By_1$. We label points as shown.

So, Q is $\left(-\frac{C}{A}, 0\right)$ and S is $\left(\frac{Ax_1+By_1}{A}, 0\right)$

$$\therefore \quad \overrightarrow{QS} = \begin{pmatrix} \frac{Ax_1+By_1+C}{A} \\ 0 \end{pmatrix}$$

Now $Ax + By + C = 0$ has gradient $-\frac{A}{B}$

$\therefore$ [QR] has gradient $\frac{B}{A}$

$\therefore \quad \overrightarrow{QR} = \begin{pmatrix} kA \\ kB \end{pmatrix}$ for some constant k.

In $\triangle$QRS, let $R\widehat{Q}S = \theta$.

$$\therefore \quad \cos\theta = \frac{d}{|\overrightarrow{QS}|} \quad \{\text{NQRP is a rectangle}\}$$

$$\begin{aligned} \therefore \quad d &= |\overrightarrow{QS}| \cos\theta \\ &= \frac{|\overrightarrow{QS}||\overrightarrow{QR}|\cos\theta}{|\overrightarrow{QR}|} \\ &= \frac{|\overrightarrow{QS} \bullet \overrightarrow{QR}|}{\sqrt{k^2A^2 + k^2B^2}} \\ &= \frac{|k(Ax_1+By_1+C)|}{\sqrt{k^2(A^2+B^2)}} \\ &= \frac{|k||Ax_1+By_1+C|}{|k|\sqrt{A^2+B^2}} \\ &= \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}} \end{aligned}$$

Example 22

Find the distance from $(2, 3)$ to the line with equation $y = \frac{2}{3}x + 1$.

$$\begin{aligned} & y = \tfrac{2}{3}x + 1 \\ \Rightarrow \quad & \tfrac{2}{3}x - y + 1 = 0 \\ \Rightarrow \quad & 2x - 3y + 3 = 0 \end{aligned}$$

$$\therefore \quad d = \frac{|2(2) - 3(3) + 3|}{\sqrt{2^2 + (-3)^2}} = \frac{|-2|}{\sqrt{13}} = \frac{2}{\sqrt{13}} \text{ units}$$

Example 23

Find the equation of the locus of all points which are parallel to the line $2x + y = 11$ and $2\sqrt{5}$ units from it.

$2x + y = 11 \;\Rightarrow\; 2x + y - 11 = 0$

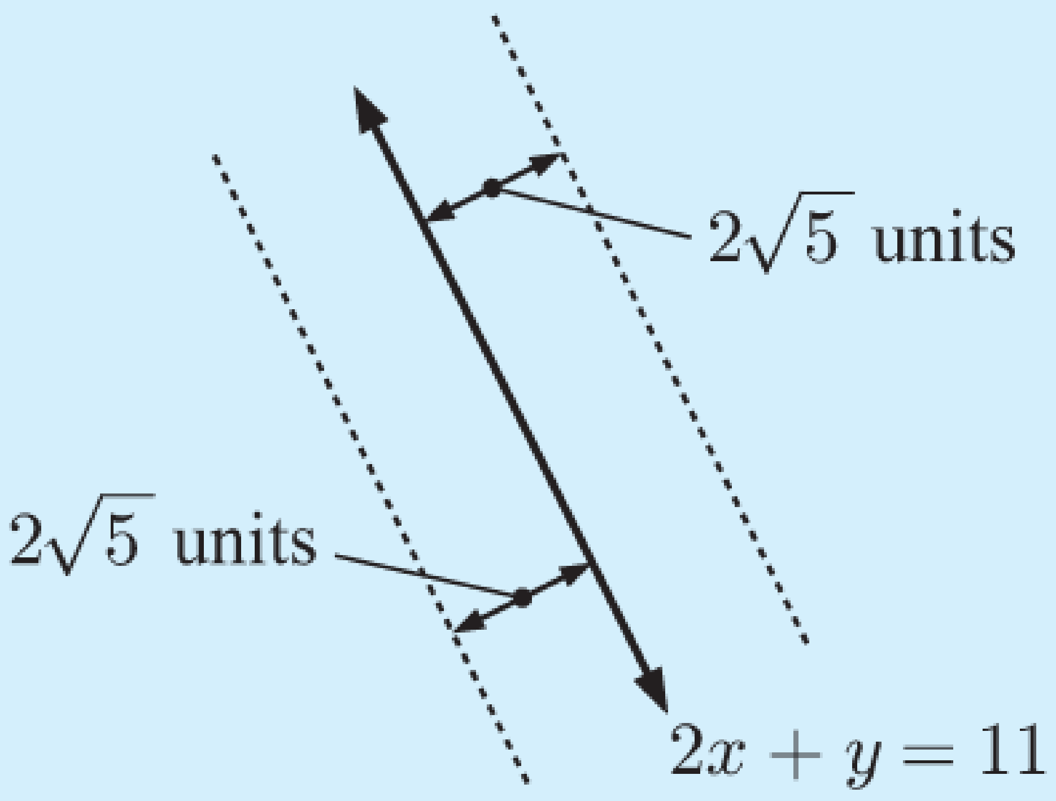
Let $\mathrm{P}(x, y)$ be a point in the locus

$$\therefore \frac{|\,2x + y - 11\,|}{\sqrt{2^2 + 1^2}} = 2\sqrt{5}$$

$$\therefore |\,2x + y - 11\,| = 10$$

$$\therefore 2x + y - 11 = \pm 10$$

$$\therefore 2x + y = 1 \quad \text{or} \quad 2x + y = 21$$



$\therefore$ the locus includes all points on the two lines $2x + y = 1$ and $2x + y = 21$ which are both parallel to the given line.

Example 24

Suppose A is $(0, 3)$ and B is $(0, -3)$. Find the Cartesian equation of the locus of $\mathrm{P}(x, y)$ such that $\mathrm{AP} + \mathrm{BP} = 8$ units.

$$\mathrm{AP} + \mathrm{BP} = 8$$

$$\therefore \sqrt{(x-0)^2 + (y-3)^2} + \sqrt{(x-0)^2 + (y+3)^2} = 8$$

$$\therefore \sqrt{x^2 + y^2 - 6y + 9} = 8 - \sqrt{x^2 + y^2 + 6y + 9}$$

$$\therefore x^2 + y^2 - 6y + 9 = (8 - \sqrt{x^2 + y^2 + 6y + 9})^2 \qquad \{\text{squaring both sides}\}$$

$$\therefore \cancel{x^2} + \cancel{y^2} - 6y + \cancel{9} = 64 - 16\sqrt{x^2 + y^2 + 6y + 9} + \cancel{x^2} + \cancel{y^2} + 6y + \cancel{9}$$

$$\therefore 16\sqrt{x^2 + y^2 + 6y + 9} = 12y + 64$$

$$\therefore 4\sqrt{x^2 + y^2 + 6y + 9} = 3y + 16$$

$$\therefore 16(x^2 + y^2 + 6y + 9) = 9y^2 + 96y + 256$$

$$\therefore 16x^2 + 16y^2 + \cancel{96y} + 144 = 9y^2 + \cancel{96y} + 256$$

$$16x^2 + 7y^2 = 112 \quad \left(\text{or } \frac{x^2}{7} + \frac{y^2}{16} = 1\right)$$

EXERCISE 2L

1 Find the distance from:

- **a** $(3, 2)$ to $2x + 5y + 6 = 0$
- **b** $(-1, 4)$ to $4x - 3y = 4$
- **c** $(2, -1)$ to $y = 3x - 2$
- **d** $(-1, -3)$ to $mx + y = 5$, $m \in \mathbb{R}$.

2 Find the distance between the parallel lines:

- **a** $3x + 2y = 5$ and $3x + 2y + 1 = 0$
- **b** $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$

3 Find the value of k if:

- **a** the distance from $(k, -3)$ to $3x - 2y + 6 = 0$ is $\sqrt{13}$ units
- **b** $\mathrm{A}(1, -2)$ is equidistant from $x + y = k$ and $x - y + 7 = 0$.

4 Find the equation of the locus of all points which are parallel to the line $x - y = 4$ and are $2\sqrt{2}$ units from it.

5 Find the locus of all points $P(x, y)$ which are equidistant from $N(-1, 8)$ and $S(5, 4)$.

6 Suppose A is $(-3, 0)$ and B is $(3, 0)$. Find the locus of all points $P(x, y)$ such that $\widehat{APB}$ is a right angle.

7 Find the Cartesian equation of the locus of $P(x, y)$ if:

- **a** P is the same distance from $(2, 1)$ as it is from $2x - y = 5$
- **b** P is equidistant from the lines $3x - 4y = 3$ and $5x - 12y = 4$.

8
- **a** $A(-1, 0)$ and $B(3, 0)$ are given points. $P(x, y)$ moves such that $\dfrac{AP}{BP} = 2$.
  - **i** Find the Cartesian equation of the locus of P.
  - **ii** Describe the locus of P. Give reasons for your answer.
- **b** Repeat **a** for the case where $\dfrac{AP}{BP} = \frac{1}{2}$.

Example 25

The distance from the point $P(x, y)$ to $A(-1, 3)$ is a half of the distance from P to the line $x + 2y = 7$. Find the Cartesian equation of the locus of P.

$$AP = \tfrac{1}{2} \times \text{P's distance from } x + 2y - 7 = 0$$

$$\therefore \sqrt{(x+1)^2 + (y-3)^2} = \frac{1}{2}\,\frac{|\,x + 2y - 7\,|}{\sqrt{1+4}}$$

$$\therefore (x+1)^2 + (y-3)^2 = \frac{(x+2y-7)^2}{20} \qquad \{\text{squaring both sides}\}$$

$$\therefore 20(x^2 + 2x + 1 + y^2 - 6y + 9) = x^2 + 4y^2 + 49 + 4xy - 28y - 14x$$

$$\{(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac\}$$

$$\therefore 20x^2 + 20y^2 + 40x - 120y + 200 = x^2 + 4y^2 + 49 + 4xy - 28y - 14x$$

$$\therefore 19x^2 - 4xy + 16y^2 + 54x - 92y + 151 = 0$$

9 Find the Cartesian equation of the locus of $R(x, y)$ if R's distance from $A(3, 0)$ is:

- **a** equal to its distance from the line $x = -3$
- **b** half its distance from the line $x = 12$
- **c** 1.5 times its distance from $x = \frac{4}{3}$.

10 Suppose A is $(2, 0)$ and B is $(-2, 0)$. Find the Cartesian equation of the locus of $Q(x, y)$ such that:

- **a** $AQ + BQ = 6$
- **b** $AQ - BQ = 2$.

M THE COORDINATE GEOMETRY OF CIRCLES

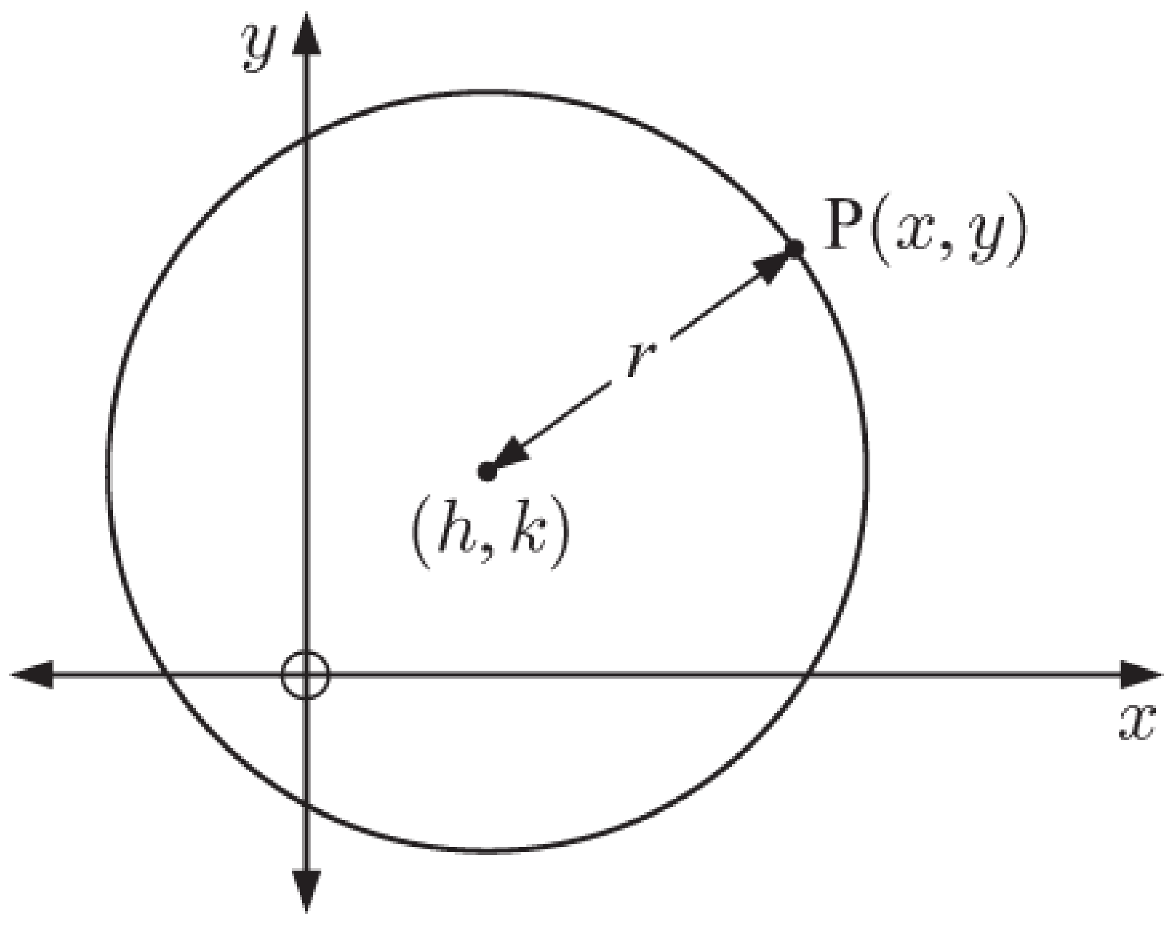
A **circle** is the set of all points which are equidistant from a point called its centre.

THE CENTRE-RADIUS FORM OF THE EQUATION OF A CIRCLE

The equation of a circle with centre (h, k) and radius r is

$$(x-h)^2+(y-k)^2=r^2.$$

The proof is a simple application of the *distance formula*.



Example 26

Find the equation of a circle with centre $(2, -3)$ and radius $\sqrt{7}$ units.

The equation is $(x-2)^2+(y--3)^2=(\sqrt{7})^2$ $\{h=2,\ k=-3,\ r=\sqrt{7}\}$

which is $(x-2)^2+(y+3)^2=7.$

THE GENERAL FORM OF THE EQUATION OF A CIRCLE

If we expand and simplify $(x-2)^2+(y+3)^2=7,$

we obtain $x^2-4x+4+y^2+6y+9=7$

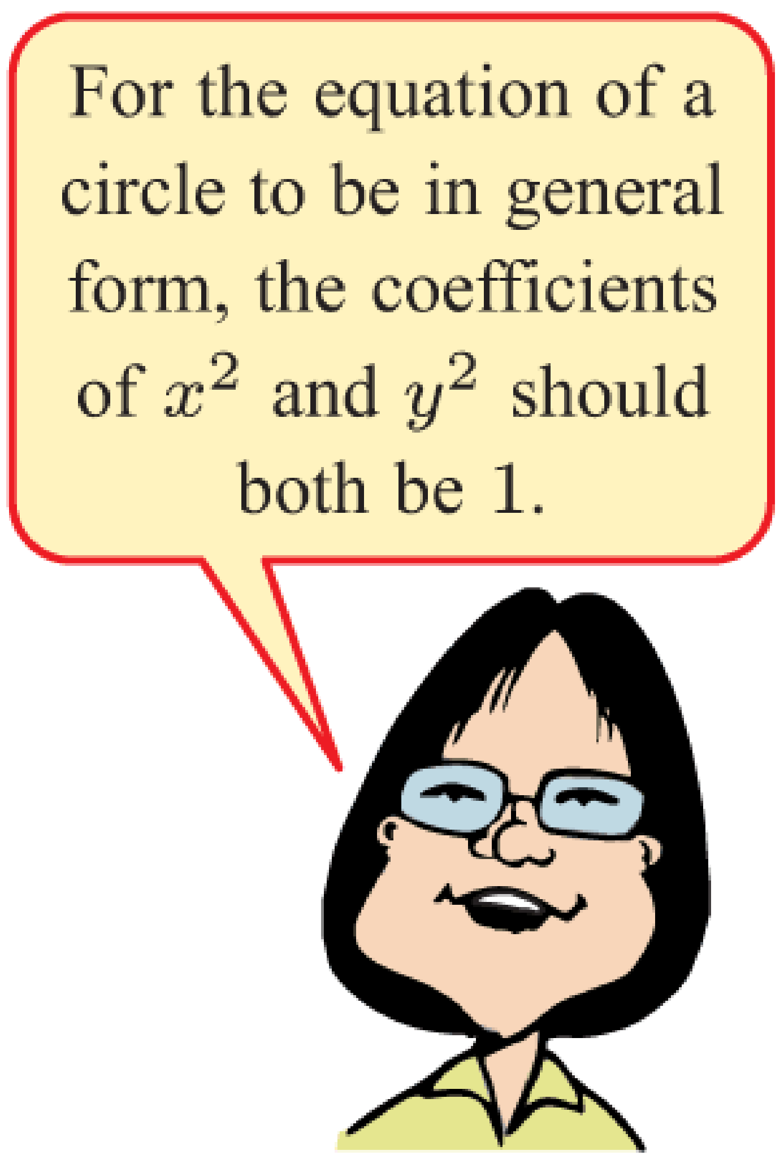
$\therefore\ x^2+y^2-4x+6y+6=0$

This equation is of the form $x^2+y^2+dx+ey+f=0$ with $d=-4,\ e=6,\ f=6.$

In fact, the equation of any circle can be put into this form.

The **general form** of the equation of a circle is $x^2+y^2+dx+ey+f=0.$

We are often given equations in general form and need to find the centre and radius of the circle. We can do this by 'completing the square' for both the x and y terms.


For the equation of a circle to be in general form, the coefficients of x^2 and y^2 should both be 1.

Example 27

Find the centre and radius of the circle with equation $x^2+y^2+6x-2y-6=0$.

$$\begin{aligned} x^2+y^2+6x-2y-6&=0\\ \therefore\ x^2+6x \quad +y^2-2y \quad &=6\\ \therefore\ x^2+6x+3^2+y^2-2y+1^2&=6+3^2+1^2 \quad \{\text{completing the squares}\}\\ \therefore\ (x+3)^2+(y-1)^2&=16=4^2 \end{aligned}$$

$\therefore$ the circle has centre $(-3, 1)$ and radius 4 units.

Example 28

The point $(m, 2)$ lies on the circle with equation $(x-2)^2+(y-5)^2=25$.
Find the possible values of m.

Since $(m, 2)$ lies on the circle, $(m-2)^2+(2-5)^2=25$

$$\begin{aligned} \therefore\ (m-2)^2+9&=25\\ \therefore\ (m-2)^2&=16\\ \therefore\ m-2&=\pm 4\\ \therefore\ m&=6 \text{ or } -2 \end{aligned}$$

EXERCISE 2M.1

1 Find the centre and radius of the circle with equation:

a $(x-2)^2+(y-3)^2=4$ **b** $x^2+(y+3)^2=9$ **c** $(x-2)^2+y^2=7$

2 Write down the equation of the circle with:

a centre $(2, 3)$ and radius 5 units **b** centre $(-2, 4)$ and radius 1 unit

c centre $(4, -1)$ and radius $\sqrt{3}$ units **d** centre $(-3, -1)$ and radius $\sqrt{11}$ units.

3 Find, in centre-radius form, the equation of the circle with the properties:

a centre $(3, -2)$ and touching the x-axis **b** centre $(-4, 3)$ and touching the y-axis

c centre $(5, 3)$ and passing through $(4, -1)$

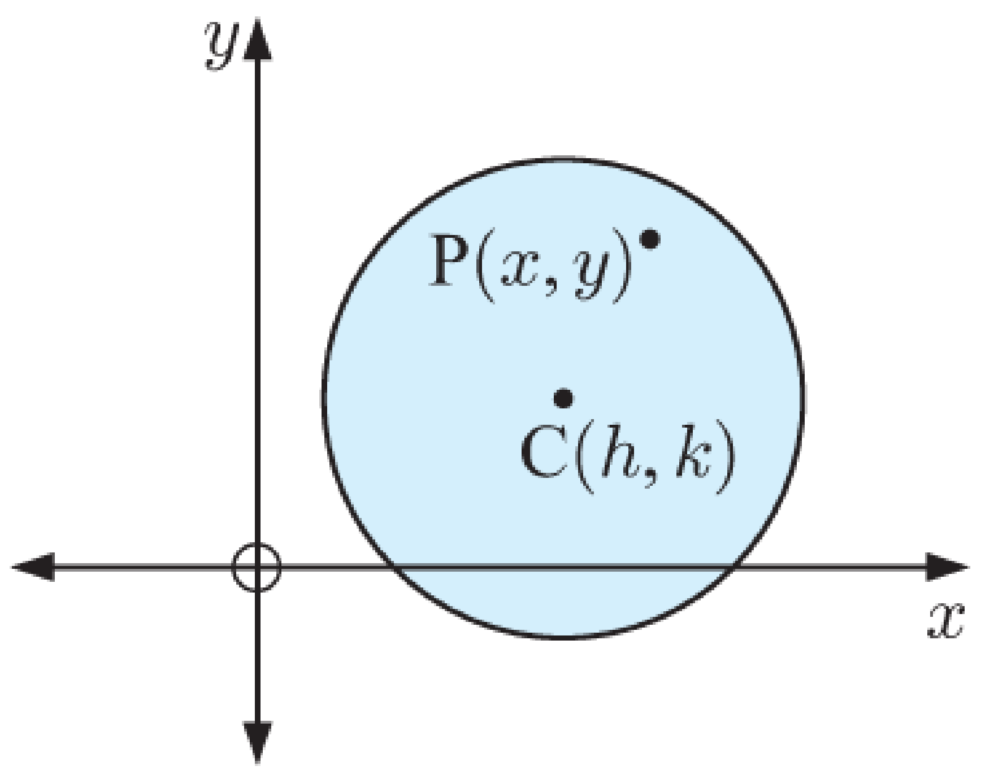
d $(-2, 3)$ and $(6, 1)$ are end-points of a diameter

e radius $\sqrt{7}$ and concentric with $(x+3)^2+(y-2)^2=5$.

4 Describe what the following equations represent on the Cartesian plane:

a $(x+2)^2+(y-7)^2=5$ **b** $(x+2)^2+(y-7)^2=0$ **c** $(x+2)^2+(y-7)^2=-5$

5 Consider the shaded region inside the circle, centre (h, k), radius r units.



a Let $P(x, y)$ be any point inside the circle.
Show that $(x-h)^2+(y-k)^2<r^2$.

b What region is defined by the inequality
$(x-h)^2+(y-k)^2>r^2$?

6 Without sketching the circle with equation $(x+2)^2+(y-3)^2=25$, determine whether the following points lie on the circle, inside the circle, or outside the circle:

a A(2, 0) **b** B(1, 1) **c** D(3, 0) **d** E(4, 1)

7 Find m given that:

a $(3, m)$ lies on the circle with equation $(x+1)^2+(y-2)^2=25$

b $(m, -2)$ lies on the circle with equation $(x+2)^2+(y-3)^2=36$

c $(3, -1)$ lies on the circle with equation $(x+4)^2+(y+m)^2=53$.

8 Find the centre and radius of the circle with equation:

a $x^2+y^2+6x-2y-3=0$ **b** $x^2+y^2-6x-2=0$

c $x^2+y^2+4y-1=0$ **d** $x^2+y^2+4x-8y+3=0$

e $x^2+y^2-4x-6y-3=0$ **f** $x^2+y^2-8x=0$

9 Find k given that:

a $x^2+y^2-12x+8y+k=0$ is a circle with radius 4 units

b $x^2+y^2+6x-4y=k$ is a circle with radius $\sqrt{11}$ units

c $x^2+y^2+4x-2y+k=0$ represents a circle.

10 In general form, a circle has equation $x^2+y^2+dx+ey+f=0$.

a Show that its centre is $\left(-\frac{d}{2}, -\frac{e}{2}\right)$ and its radius $r=\sqrt{\frac{d^2}{4}+\frac{e^2}{4}-f}$ where $d^2+e^2>4f$.

b Hence, find the centre and radius of the circle with equation $3x^2+3y^2+6x-9y+2=0$.

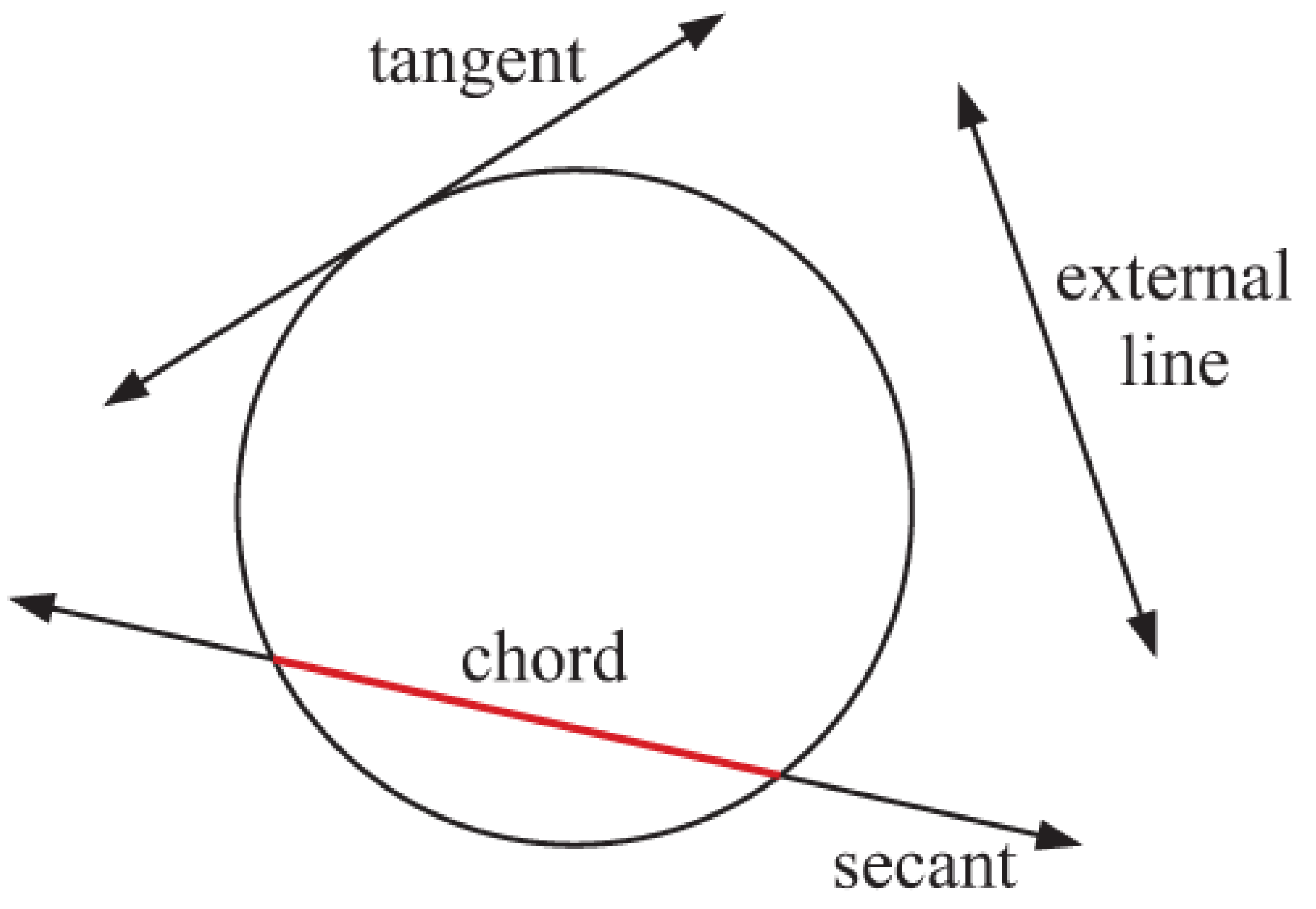
c Comment on the locus with equation $x^2+y^2+dx+ey+f=0$ in the case:

i $d^2+e^2=4f$ **ii** $d^2+e^2<4f$

TANGENTS TO A CIRCLE

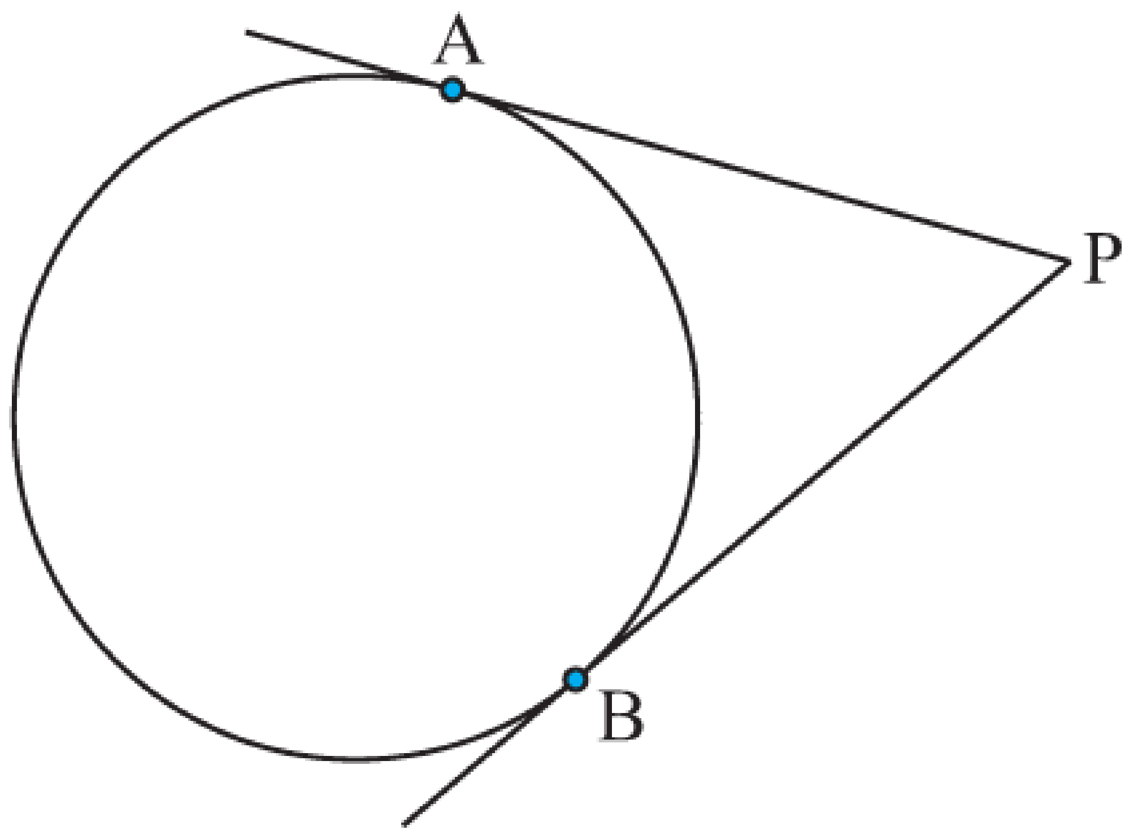
For a given circle in the plane, we can describe any line as:

- **external** if it does not meet the circle
- a **tangent** if it touches the circle at one point
- a **secant** if it cuts the circle at two points.



For any point P on the circle, there is a unique tangent through P called the **tangent at P**.

For any external point P there are exactly 2 tangents, called the **external tangents from P**. In the diagram, A and B are the two points of contact.



Example 29

Find the equation of the tangent to the circle with equation $x^2 + y^2 - 8x - 4 = 0$ at the point $\mathrm{P}(8, -2)$.

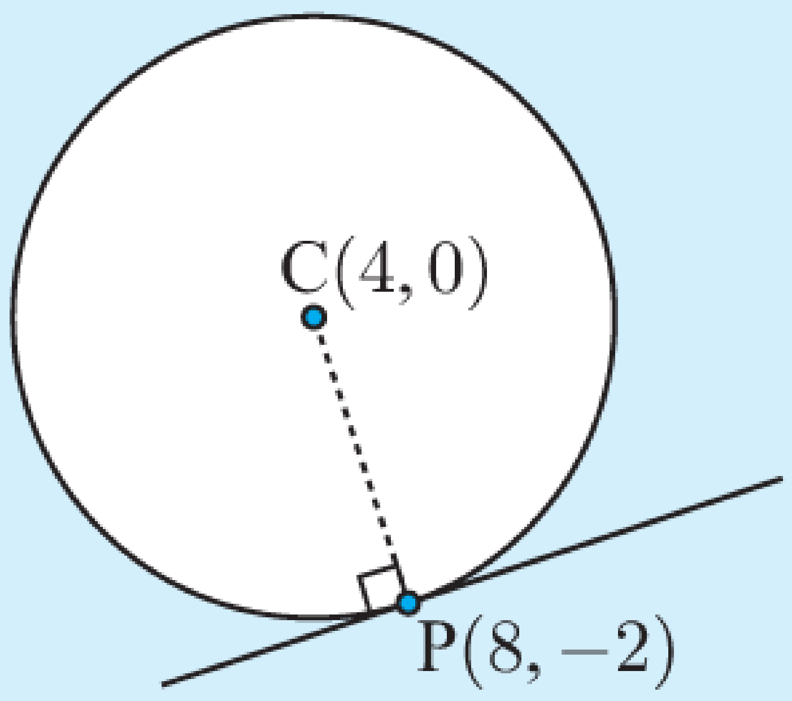
$$x^2 + y^2 - 8x - 4 = 0$$

$$\therefore \quad x^2 - 8x \qquad + y^2 = 4$$

$$\therefore \quad x^2 - 8x + 4^2 + y^2 = 4 + 4^2$$

$$\therefore \quad (x-4)^2 + y^2 = 20$$

$\therefore$ the circle has centre $(4, 0)$.



The gradient of [CP] is $\dfrac{-2-0}{8-4} = \dfrac{-2}{4} = -\frac{1}{2}$

$\therefore$ the gradient of the tangent is $\frac{2}{1}$

$\therefore$ the equation of the tangent is $2x - y = 2(8) - (-2)$

which is $2x - y = 18$.

Example 30

Find the equations of the tangents from the external point $\mathrm{P}(0, -4)$ to the circle with equation $x^2 + y^2 - 10x - 2y + 16 = 0$.

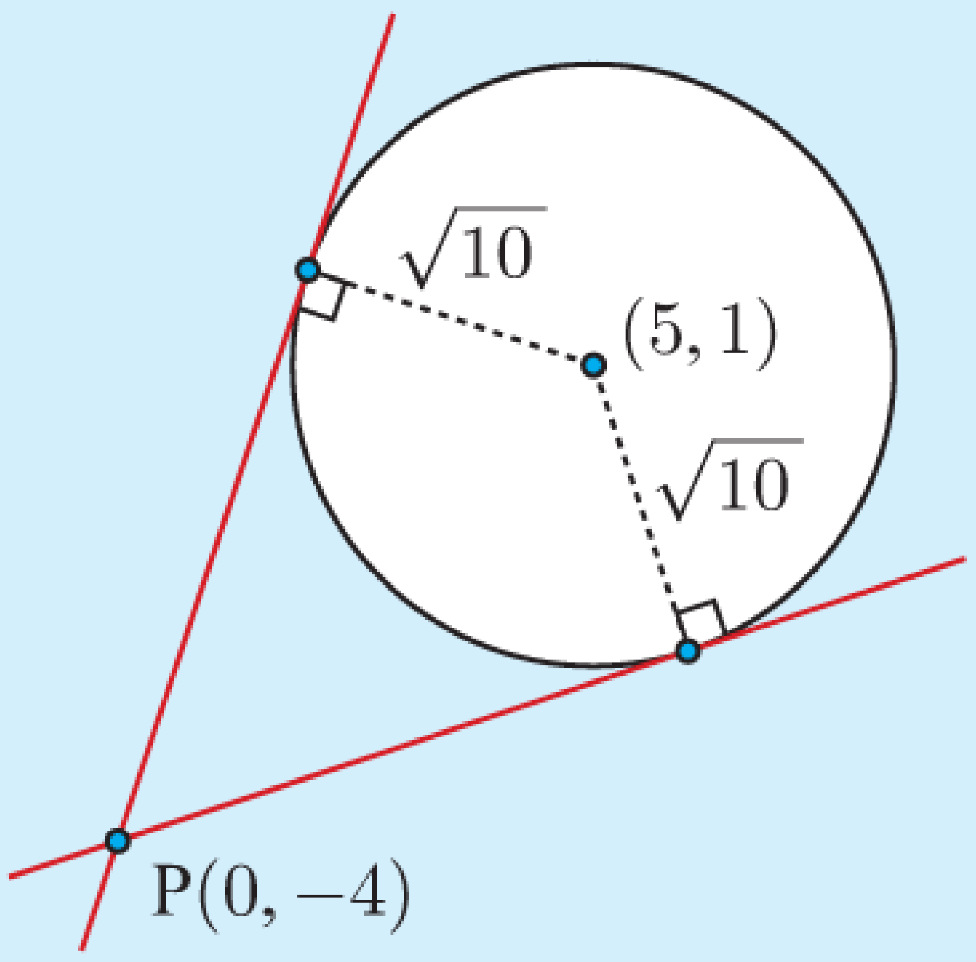
$$x^2 + y^2 - 10x - 2y + 16 = 0$$

$$\therefore \quad x^2 - 10x \qquad + y^2 - 2y \qquad = -16$$

$$\therefore \quad x^2 - 10x + 5^2 + y^2 - 2y + 1^2 = -16 + 5^2 + 1^2$$

$$\therefore \quad (x-5)^2 + (y-1)^2 = 10$$

which is a circle with centre $(5, 1)$ and radius $\sqrt{10}$ units.



Let m be the gradient of a tangent from P.

$\therefore$ it has equation $y = mx + c$ for some constant c.

But $(0, -4)$ lies on the tangent, so $c = -4$

$\therefore$ the equation is $y = mx - 4$ which is $mx - y - 4 = 0$.

The centre of the circle is $\sqrt{10}$ units from each tangent.

$$\therefore \quad \frac{|\, m(5) - (1) - 4 \,|}{\sqrt{m^2+1}} = \sqrt{10} \qquad \{\text{point to a line formula}\}$$

$$\therefore \quad |\, 5m - 5 \,| = \sqrt{10(m^2+1)}$$

$$\therefore \quad 25m^2 - 50m + 25 = 10m^2 + 10$$

$$\therefore \quad 15m^2 - 50m + 15 = 0$$

$$\therefore \quad 3m^2 - 10m + 3 = 0$$

$$\therefore \quad (m-3)(3m-1) = 0$$

$$\therefore \quad m = 3 \text{ or } \tfrac{1}{3}$$

$\therefore$ the tangents are $y = 3x - 4$ and $y = \frac{1}{3}x - 4$.

EXERCISE 2M.2

1 Find the equation of the tangent to the circle with equation:

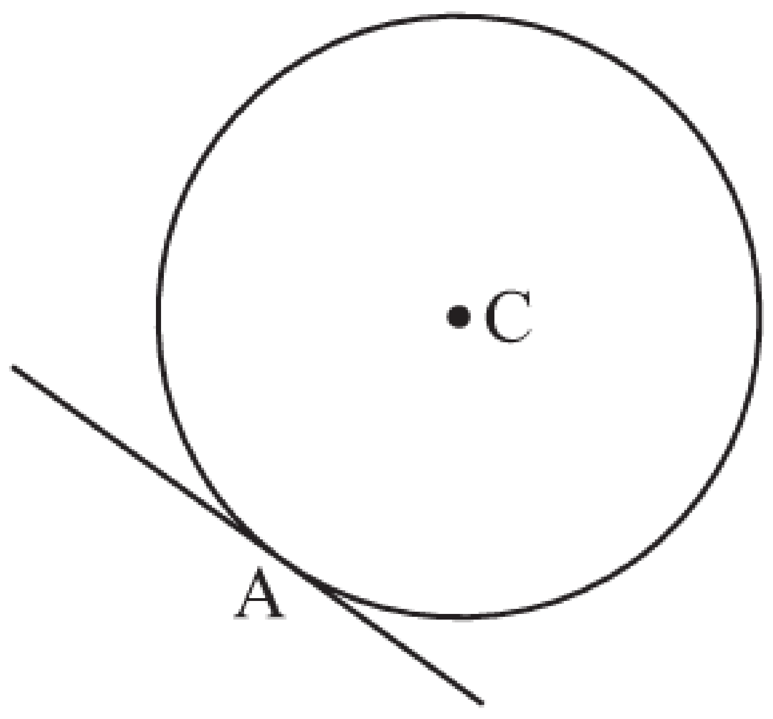
a $x^2 + y^2 + 6x - 10y + 17 = 0$ at the point P$(-2,\ 1)$

b $x^2 + y^2 + 6y = 16$ at the point P$(0,\ 2)$.

2 The boundary of a circular pond is defined by the equation $x^2 + y^2 - 24x - 16y + 111 = 0$.
A straight path meets the edge of the lake at grid reference A$(3,\ 4)$.

a Given that the grid units are metres, find the diameter of the circular pond.

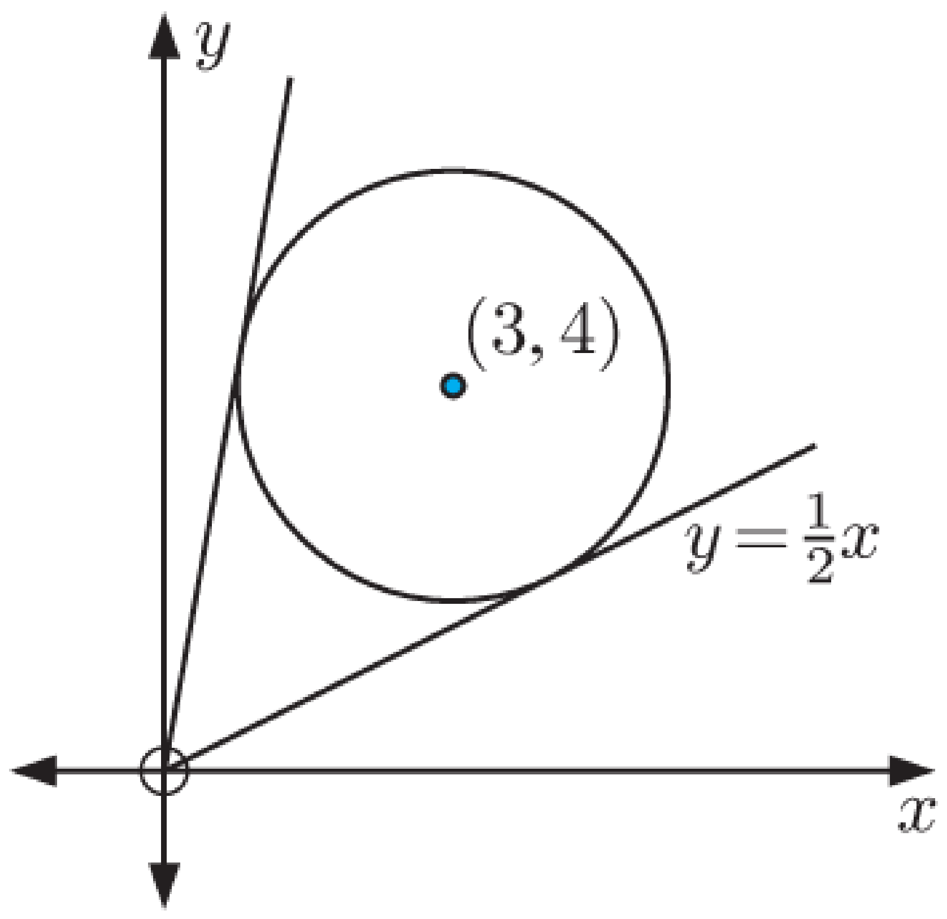
b Find the equation of the straight path.



3 A circle has centre $(2,\ 3)$ and radius 4 units. P$(8,\ 7)$ is external to the circle. Find the equations of the two tangents from P to the circle.

4 Find the equations of the two tangents from the origin O to the circle with centre $(4,\ 3)$ and radius 2 units.

5



A circle has centre $(3,\ 4)$. One tangent from the origin O has equation $y = \frac{1}{2}x$.
Find the equation of the other tangent.

6 A circle with centre $(3,\ -2)$ has a tangent with equation $3x - 4y + 8 = 0$.

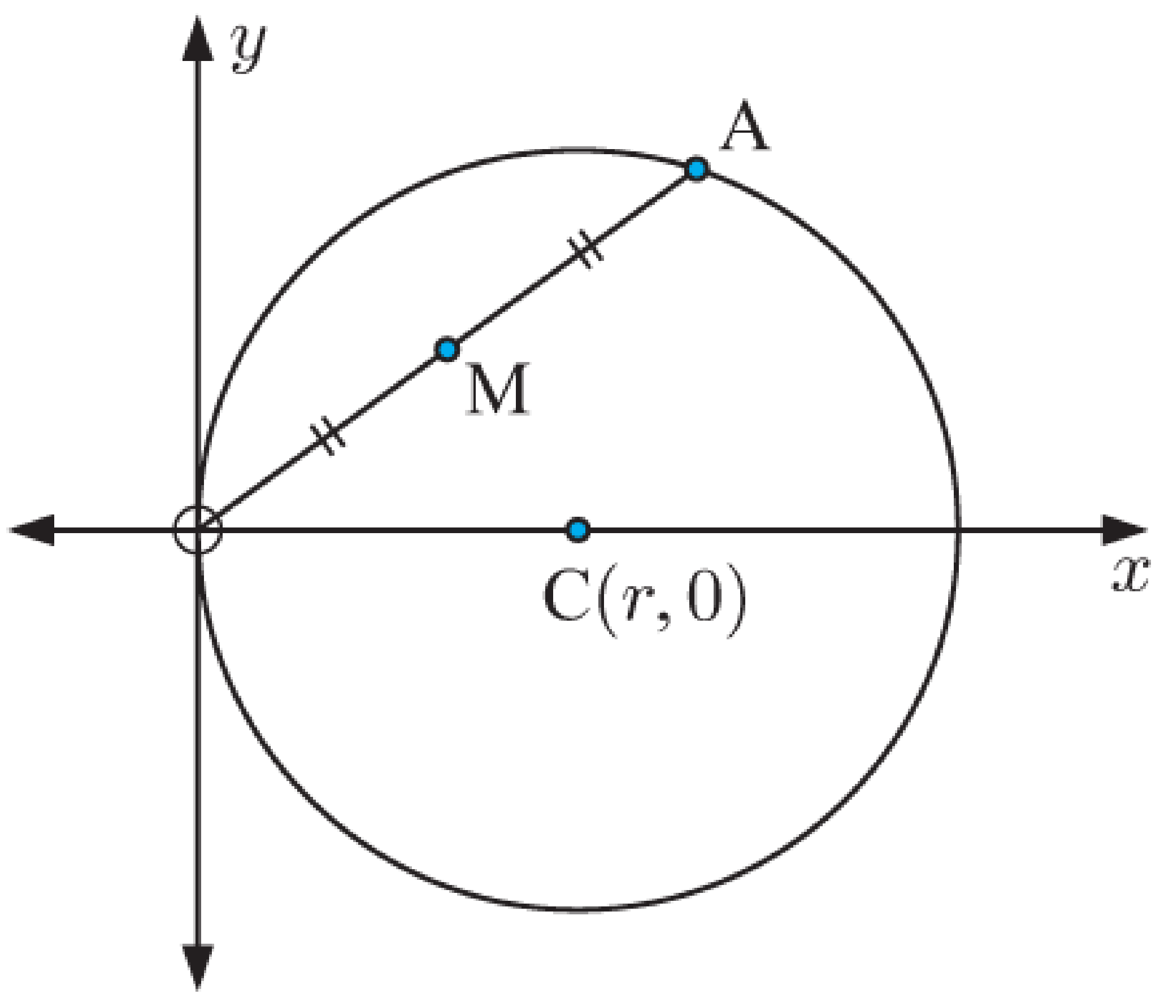
a Find the equation of the circle.

b Find the tangent's point of contact with the circle.

7 Consider the circle $x^2 + y^2 - 4x + 2y = 0$. Find the value(s) of k for which $3x + 4y = k$ is:

a a tangent **b** a secant **c** an external line.

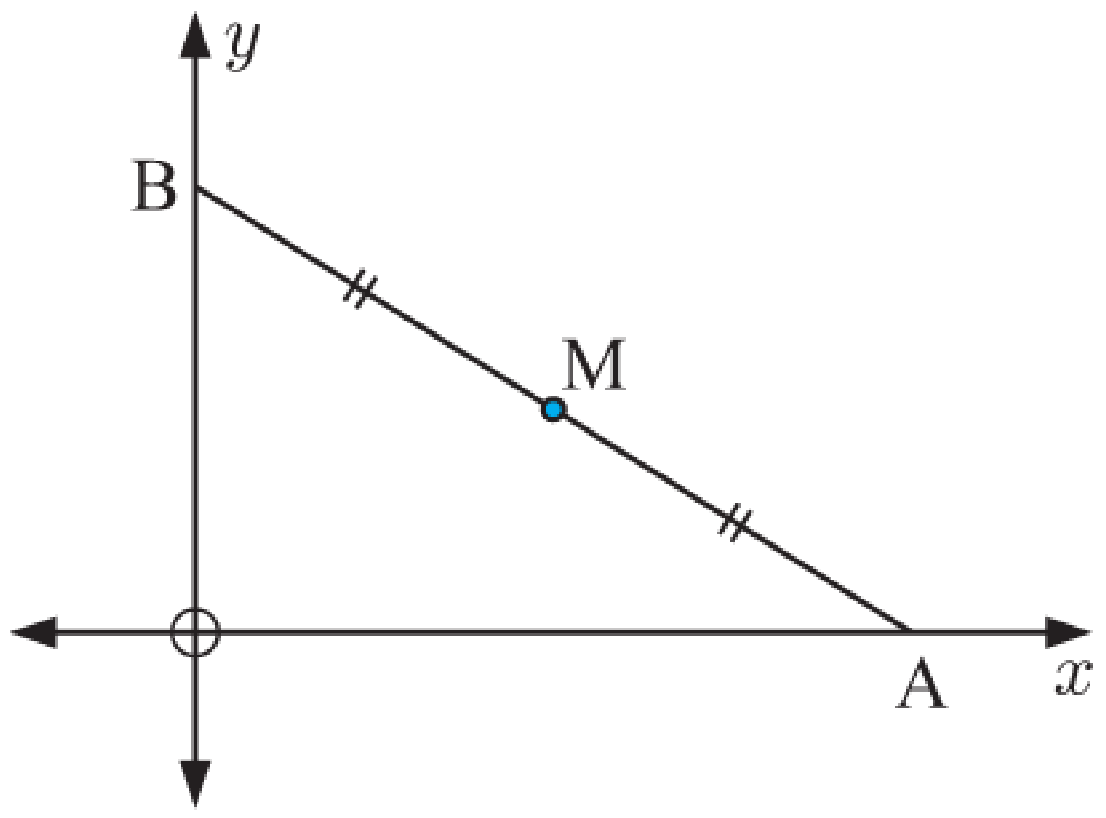
8



C$(r,\ 0)$ is the centre of a fixed circle with radius r.
A is a point which is free to move on the circle, and M is the midpoint of [OA].

a Find the Cartesian equation of the locus of M.

b Describe the locus of M.

9 

Line segment [AB] has fixed length p units. A can only move on the x-axis, and B can only move on the y-axis. M is the midpoint of [AB].

- **a** Find the Cartesian equation of the locus of M.
- **b** Describe the locus of M.

10 Suppose A is $(1, 0)$, B is $(5, 0)$, and k is a constant. $P(x, y)$ is a point such that $\frac{AP}{BP} = k$ for all positions of P. Find the equation and nature of the locus of P if:

- **a** $k = 3$
- **b** $k = \frac{1}{3}$
- **c** $k = 1$

11 Suppose A is $(2, 0)$ and B is $(6, 0)$. The point $P(x, y)$ moves such that $\frac{AP}{BP} = 2$ for all positions of P.

- **a** Deduce that P lies on a circle, and find the circle's centre and radius.
- **b** The circle in **a** cuts the x-axis at points P_1 and P_2, where P_2 is to the right of P_1. Deduce the coordinates of P_1 and P_2.
- **c** Show that $\frac{AP}{BP} = \frac{AP_1}{BP_1} = \frac{AP_2}{BP_2}$.
- **d** Hence, deduce that $[PP_1]$ bisects $A\widehat{P}B$, and $[PP_2]$ bisects the exterior angle $A\widehat{P}B$, for all positions of P.

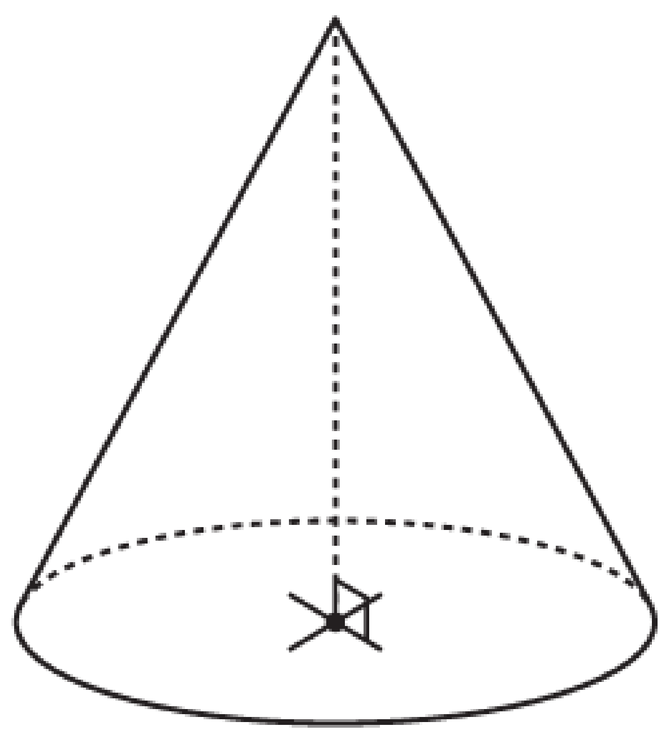
N CONIC SECTIONS

Consider a right-circular cone, which means the apex is directly above the centre of the base.

Suppose you have a second identical cone which you place upside-down on the first.

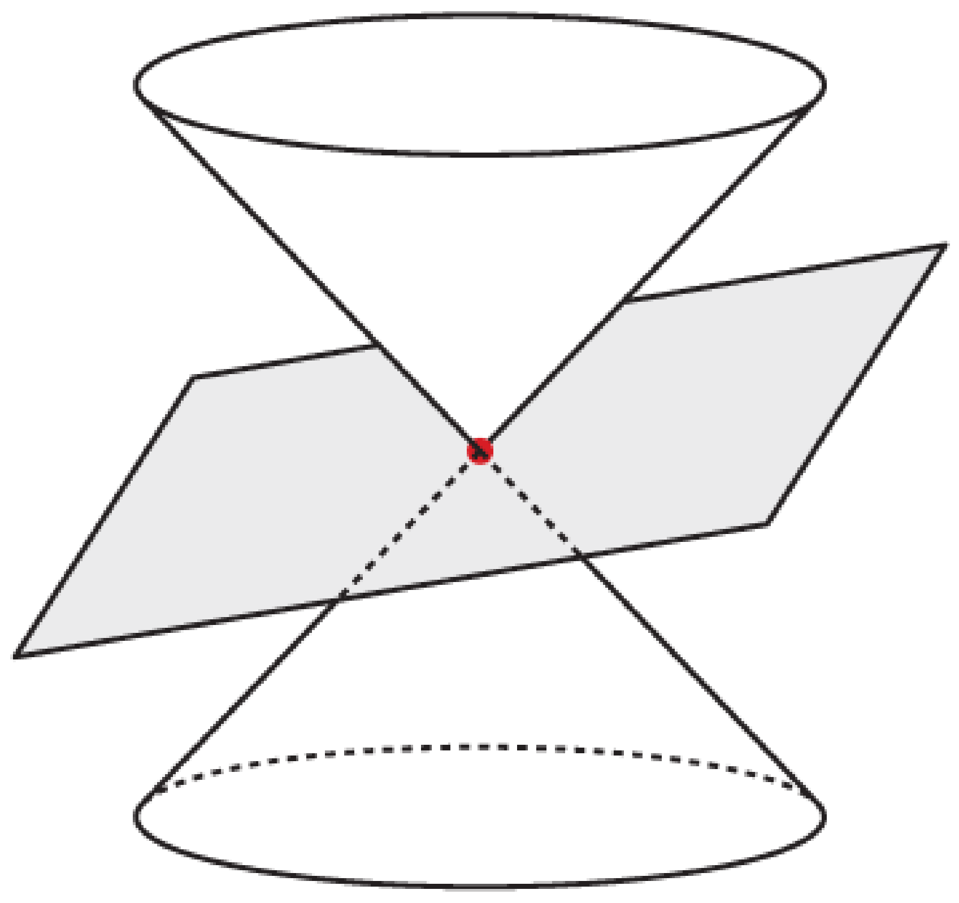
Now suppose the cones are infinitely tall.

We call the resulting shape a **double inverted right-circular cone**.

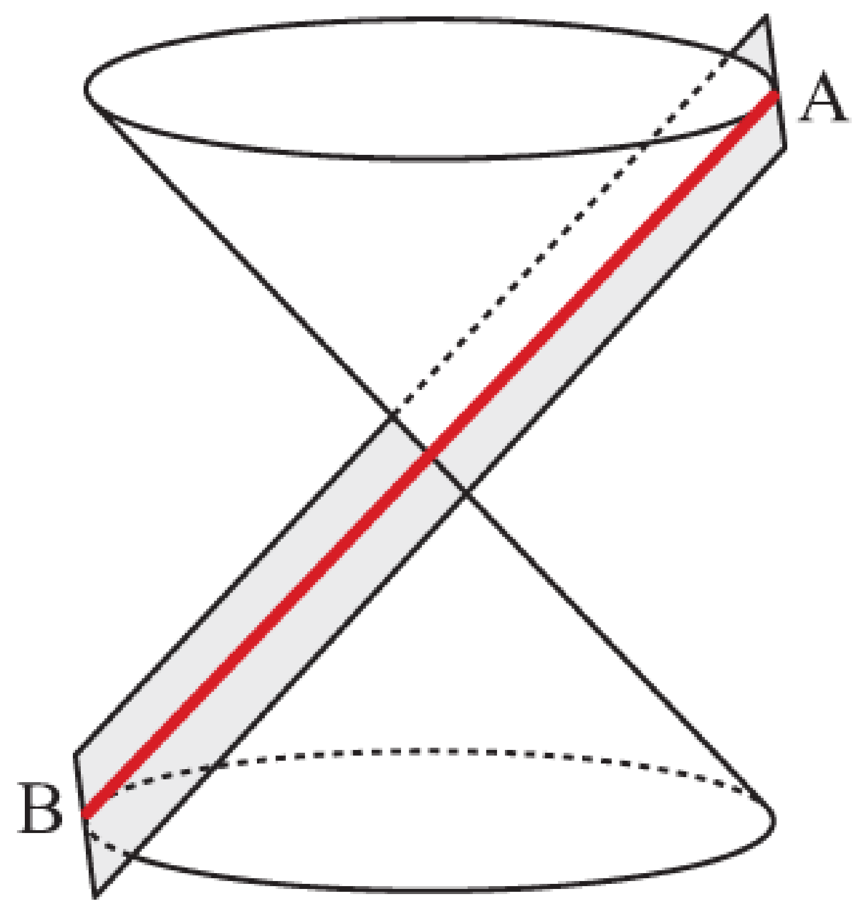


When a double inverted right-circular cone is cut by a plane, 7 possible intersections may result.

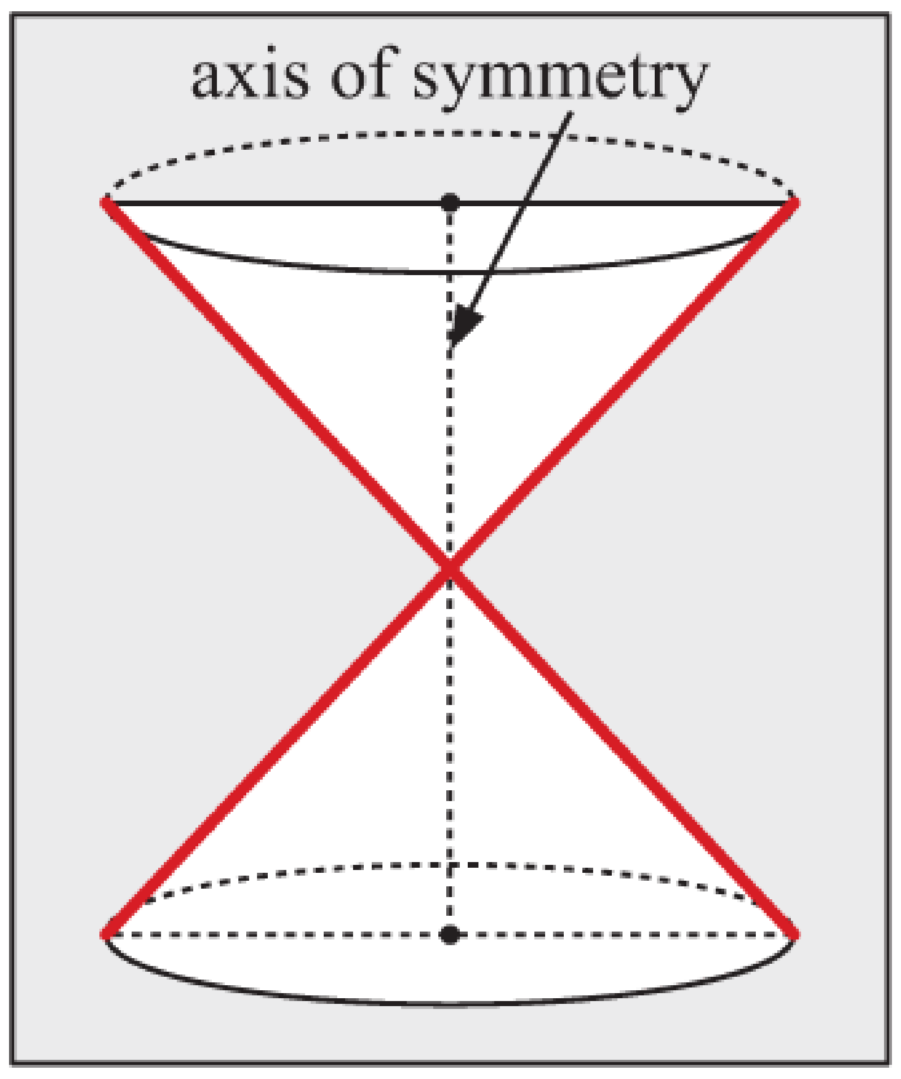
1 a **point** when the plane meets the double-cone where the apexes touch, and at no other points



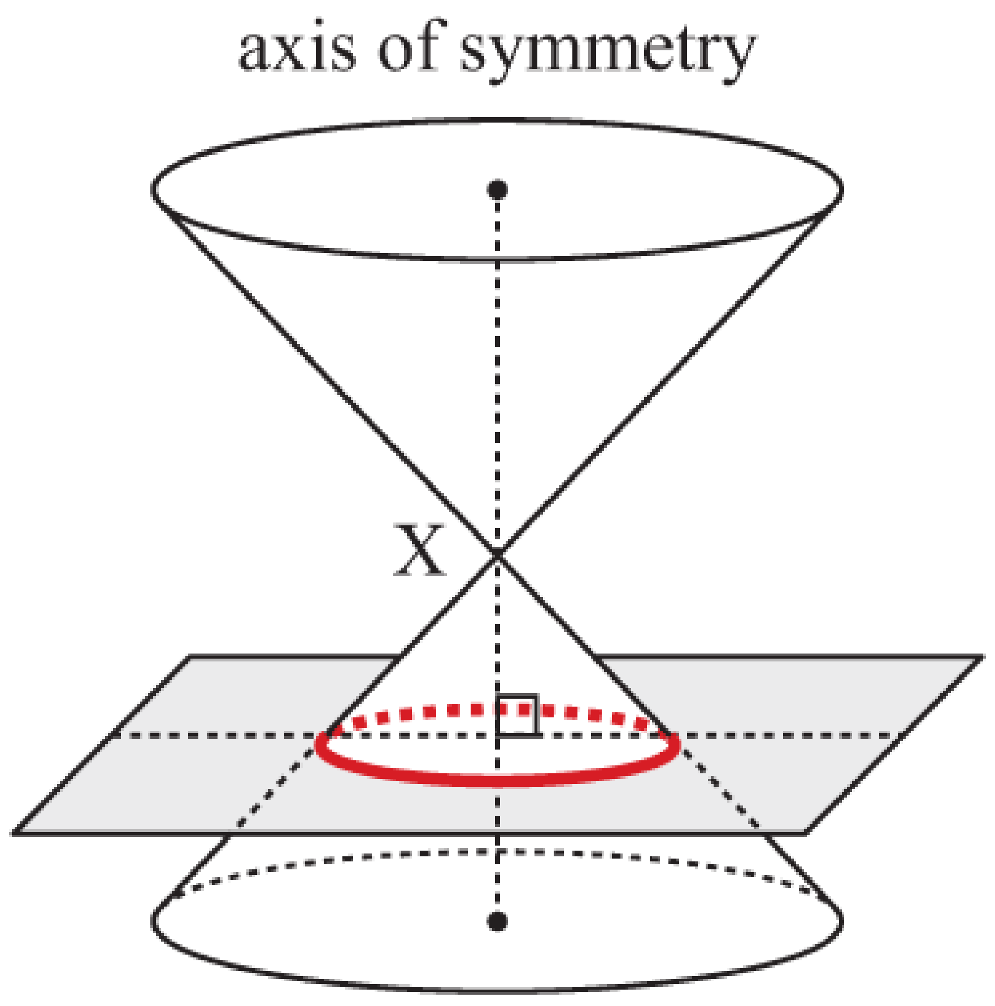
2 a **line** when the plane is tangential to the double-cone, for example (AB)



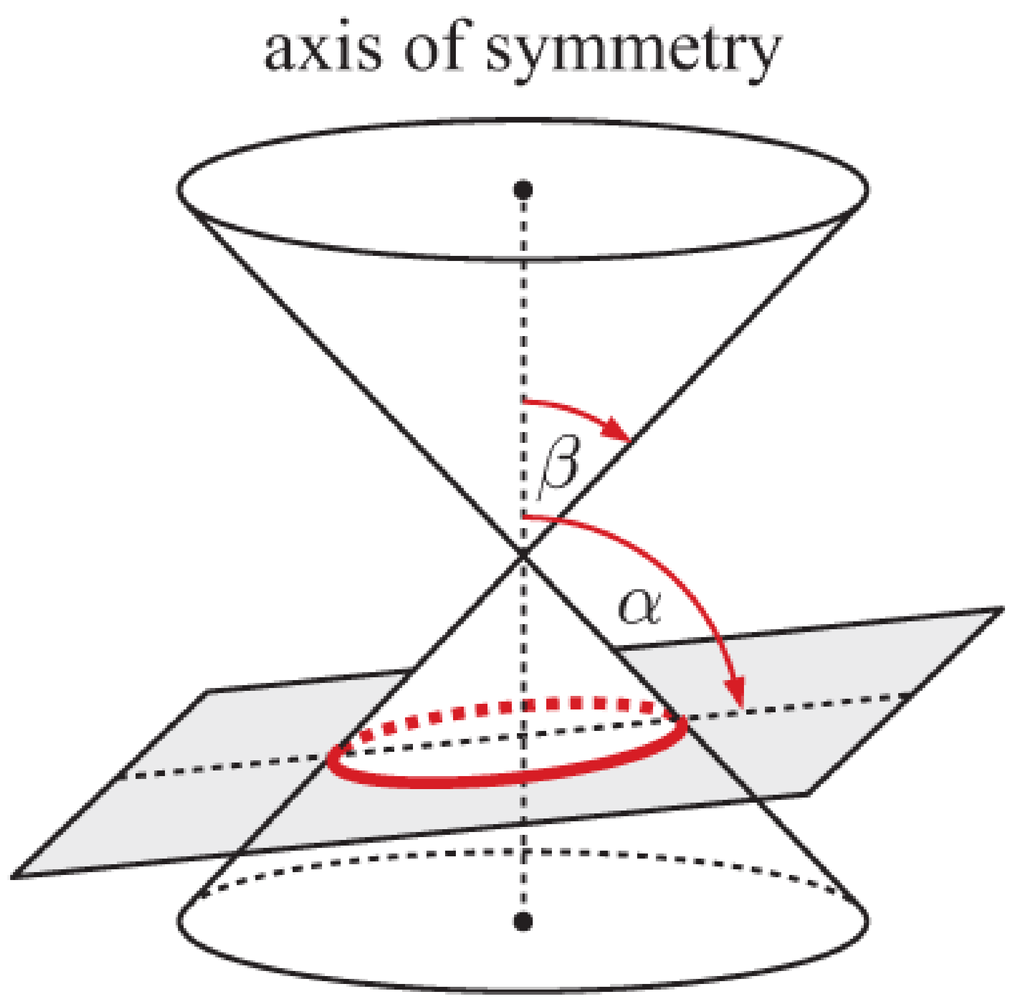
3 a **line-pair** when the plane contains an axis of symmetry of the double-cone



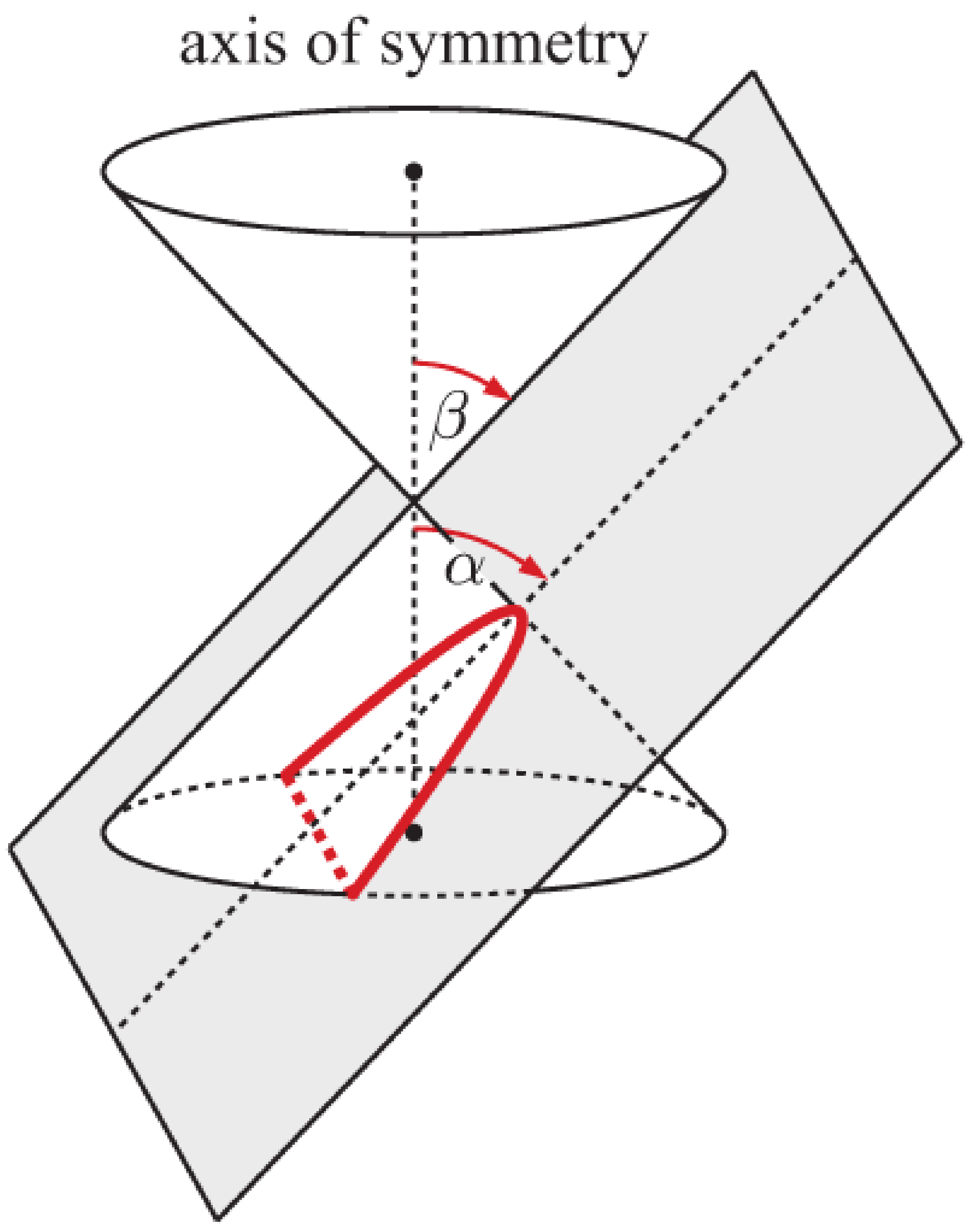
4 a **circle** when the plane is perpendicular to the axis of symmetry, and not through X



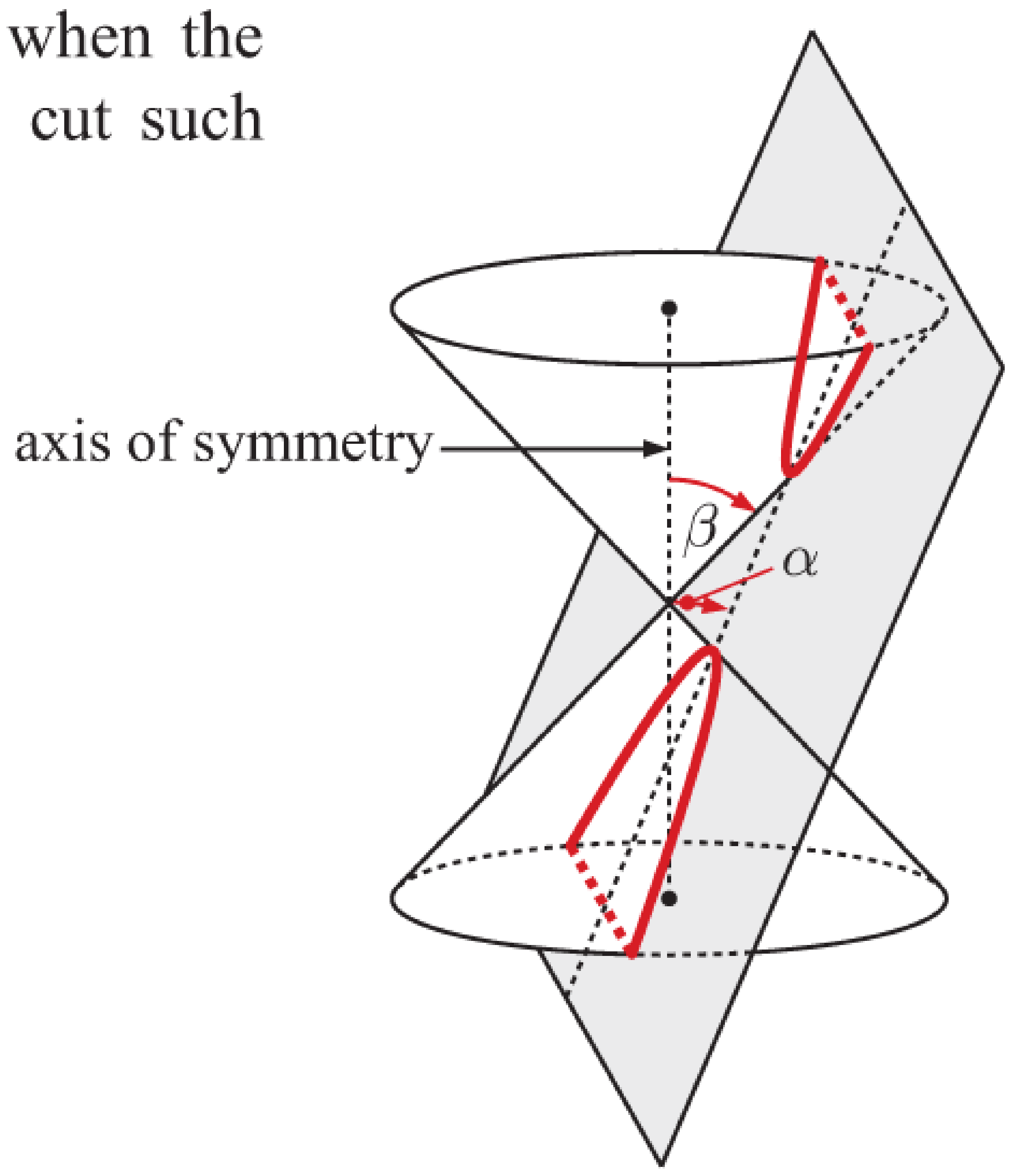
5 an **ellipse** when the double-cone is cut such that $\alpha > \beta$



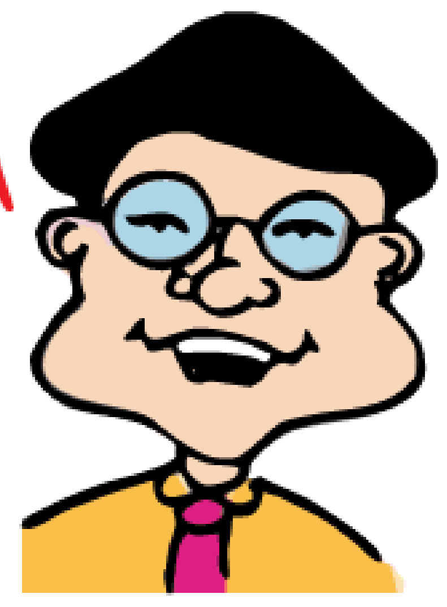
6 a **parabola** when the double-cone is cut such that $\alpha = \beta$



7 an **hyperbola** when the double-cone is cut such that $\alpha < \beta$.



1 , **2**, and **3** are called **degenerate conics** and **4** to **7** are called the **non-degenerate conics**.



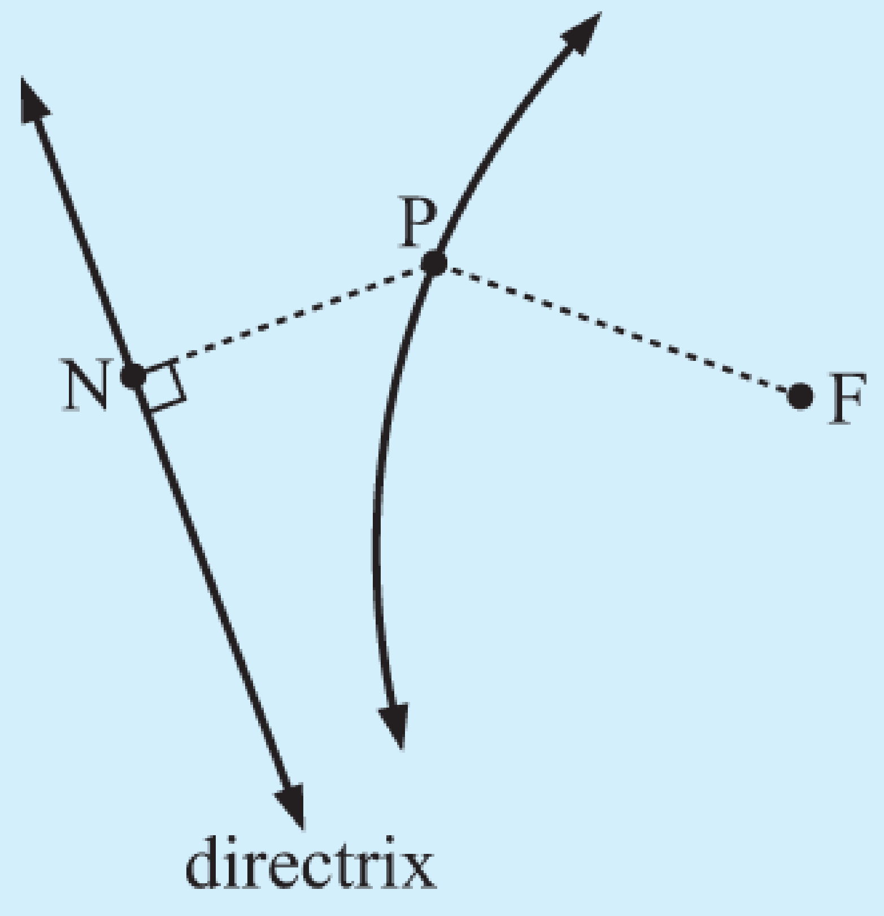
DEMO

FOCUS-DIRECTRIX DEFINITION OF AN ELLIPSE, HYPERBOLA, AND PARABOLA

Suppose $P(x, y)$ moves in the plane such that its distance from a fixed point F (called the **focus**) is a constant ratio e of its distance to a fixed line (called the **directrix**). The locus of P is a **conic** which is

- an **ellipse** if $0 < e < 1$
- a **parabola** if $e = 1$
- an **hyperbola** if $e > 1$.

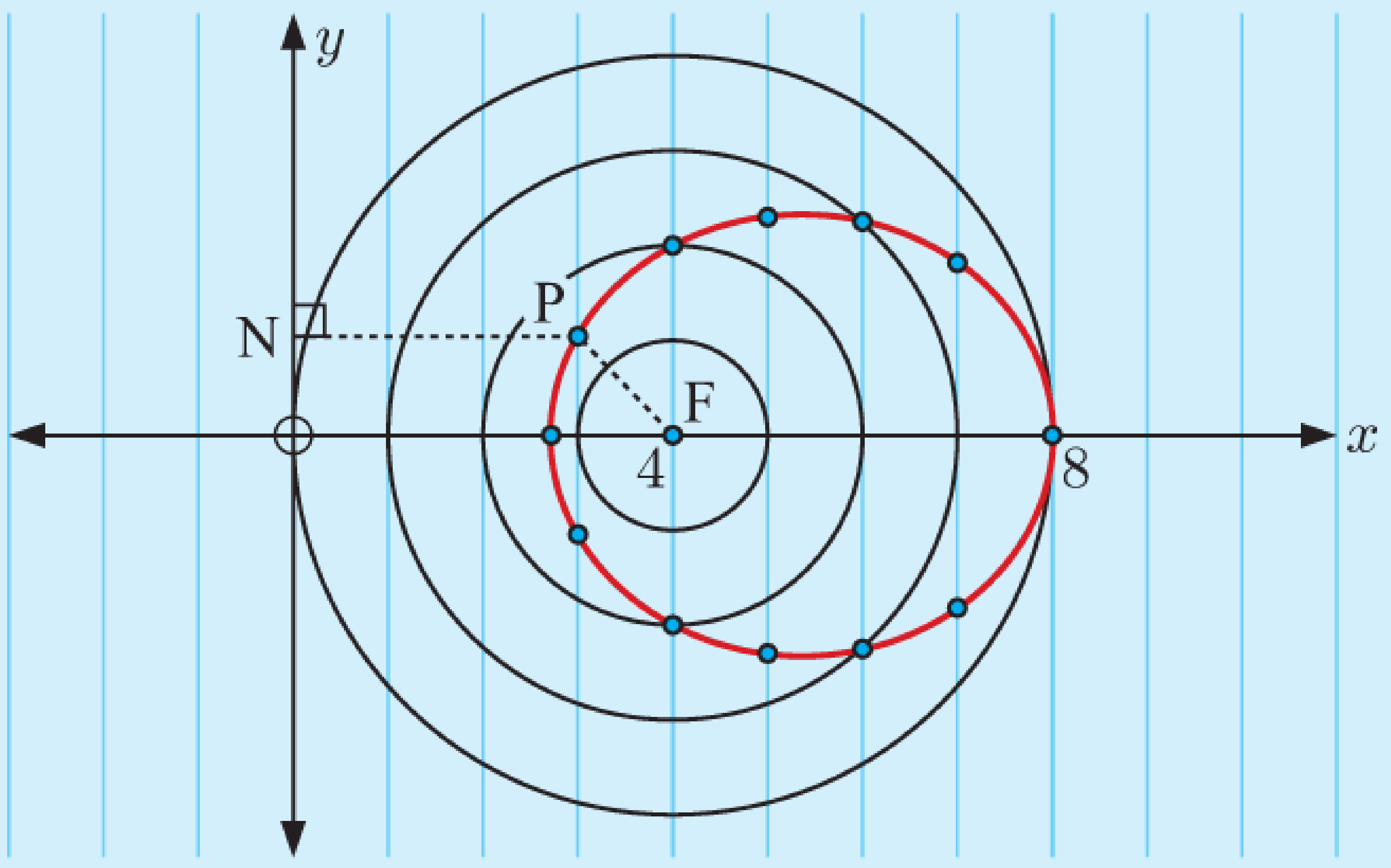
If N is the foot of the perpendicular from P to the directrix then $\dfrac{\text{PF}}{\text{PN}} = e$, and e is called the **eccentricity**.



Circular-linear graph paper is useful for graphing the non-degenerate conics.

Example 31

$P(x, y)$ moves so that $\dfrac{\text{PF}}{\text{PN}} = \frac{1}{2}$ where F is $(4, 0)$ and N is the foot of the perpendicular from P to the y-axis. Sketch the locus of P.

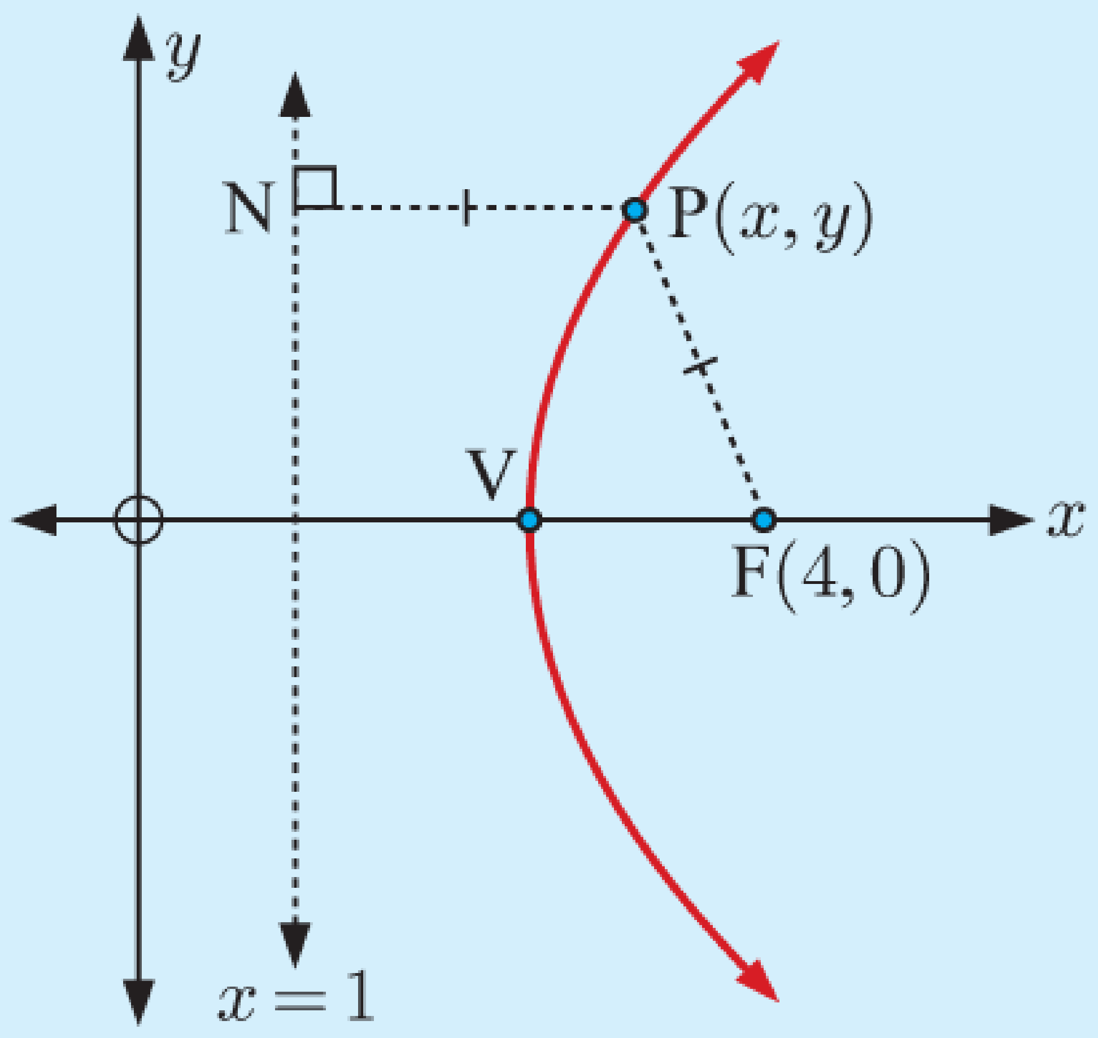


Example 32

Consider a parabola with focus $(4, 0)$ and directrix $x = 1$.

- **a** Sketch the parabola on a set of axes.
- **b** Find the location of the vertex.
- **c** Find the Cartesian equation of the parabola.

a



b The vertex is midway between the focus and the directrix.

$\therefore$ V is $(\frac{5}{2}, 0)$.

c For any point $P(x, y)$ on the parabola, $\frac{PF}{PN} = 1$ where N is $(1, y)$

$$\therefore \sqrt{(x-4)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (y-y)^2}$$

$$\therefore x^2 - 8x + 16 + y^2 = x^2 - 2x + 1$$

$$\therefore y^2 = 6x - 15$$

EXERCISE 2N.1

1. Click on the icon to obtain and print circular-linear graph paper.

   $P(x, y)$ moves in the plane so that $\frac{PF}{PN} = e$ where F is $(0, 3)$ and N is the foot of the perpendicular from P to the x-axis. Sketch the locus of P if:

   a $e = 1$ **b** $e = \frac{1}{2}$ **c** $e = 2$ (2 parts)

CIRCULAR-LINEAR GRAPH PAPER

2. Consider a parabola with vertex $(1, 1)$ and focus $(3, 3)$.
   - **a** Sketch the parabola on a set of axes.
   - **b** Find the equation of the directrix.
   - **c** Find the Cartesian equation of the parabola.

3. Find the Cartesian equation of a parabola with directrix $x + y = 4$ and focus $(1, 1)$.

THE PARABOLA

If F is a fixed point (called a **focus**) and $P(x, y)$ moves so that $PF = PN$ where N is the foot of the perpendicular from P to a fixed line (called the **directrix**), then the locus of P is a **parabola**.

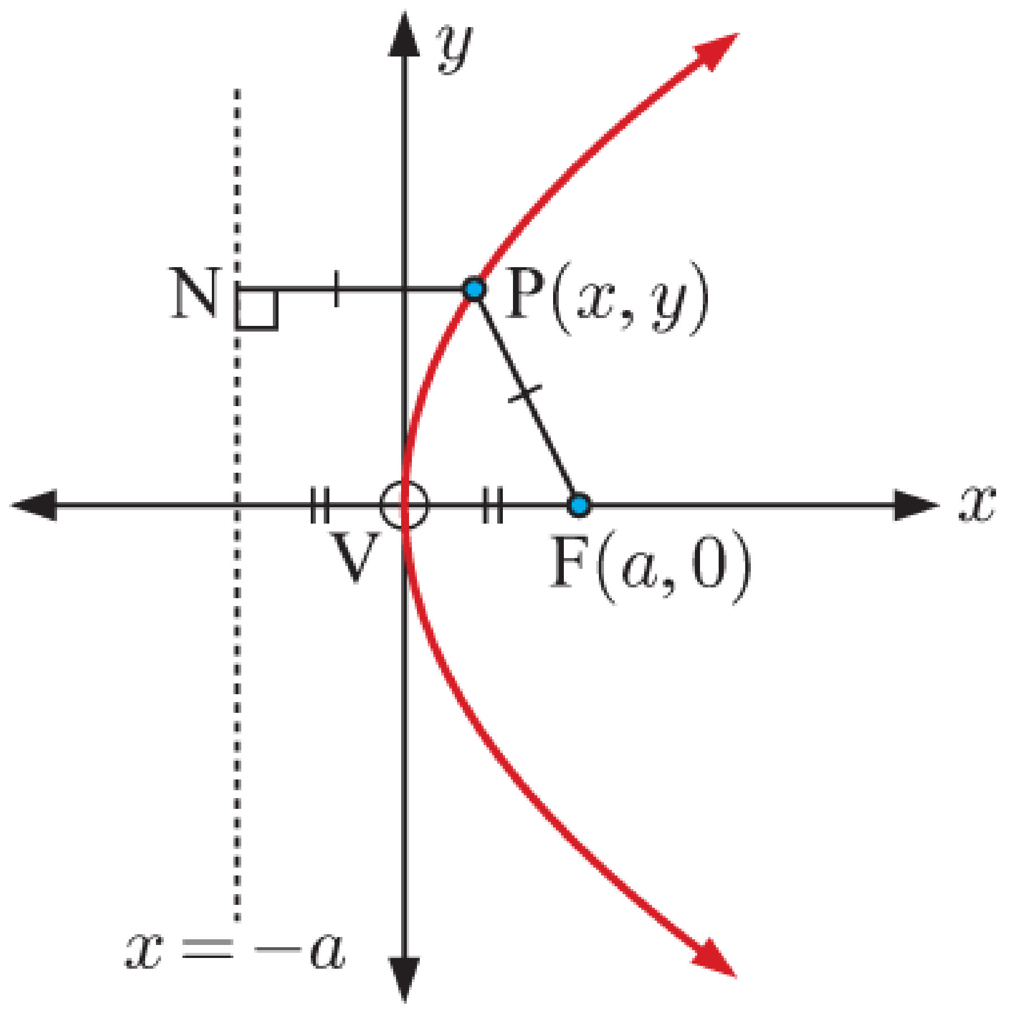
The **axis of symmetry** of the parabola is the line through the focus which is normal to the directrix.

Since $e = 1$ for a parabola, $PF = PN$.

For the simplest parabola we choose the focus to be $F(a, 0)$ on the x-axis, $a > 0$, and the directrix to be the vertical line $x = -a$.

The x-axis is the axis of symmetry of the parabola, and the vertex of the parabola is the origin O.

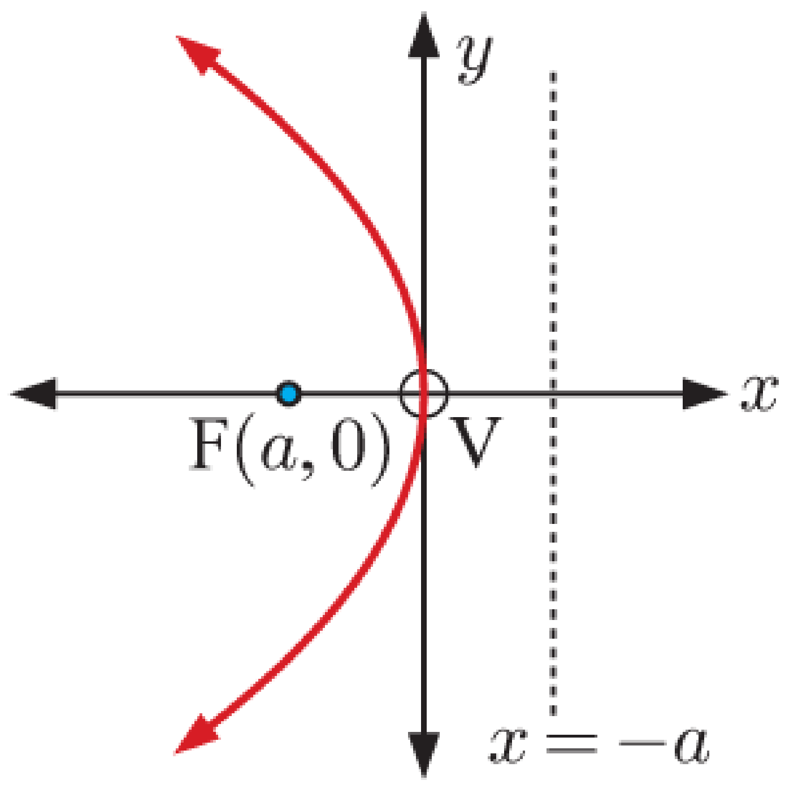
For a given point $P(x, y)$, the coordinates of N are $(-a, y)$.



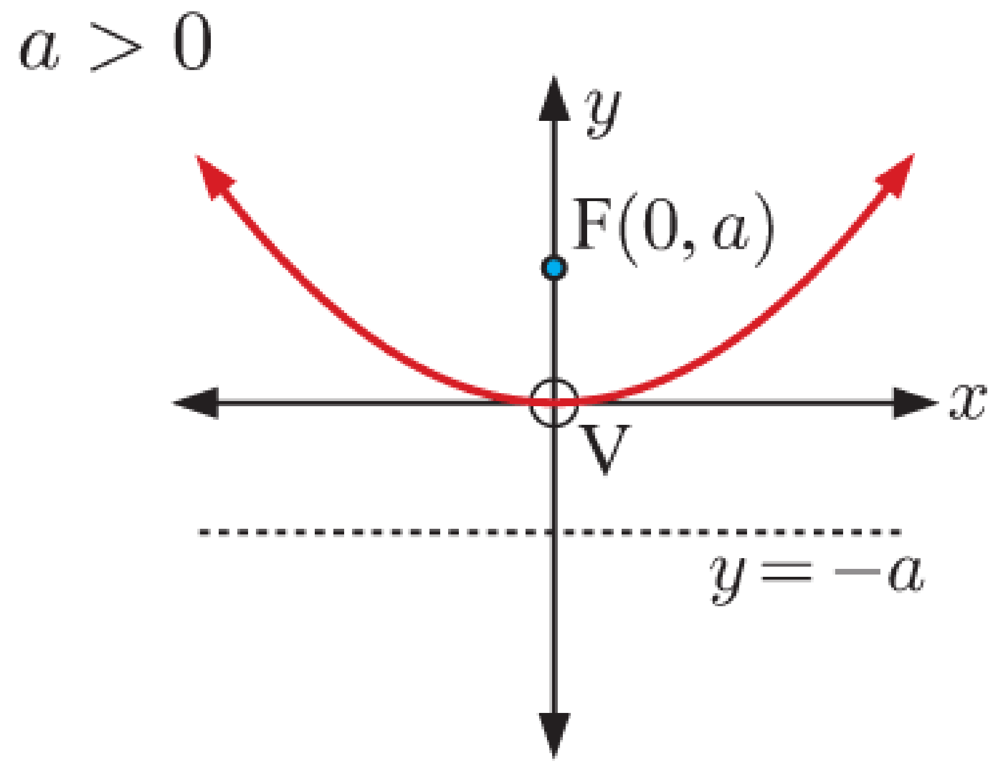
$$\begin{aligned} \text{Since} \qquad \text{PF} &= \text{PN}, \\ \sqrt{(x-a)^2+y^2} &= x-(-a) \\ \therefore \quad \cancel{x^2} - 2ax + \cancel{a^2} + y^2 &= \cancel{x^2} + 2ax + \cancel{a^2} \\ \therefore \quad y^2 &= 4ax \end{aligned}$$

For the case $a < 0$, the graph opens in the negative direction.

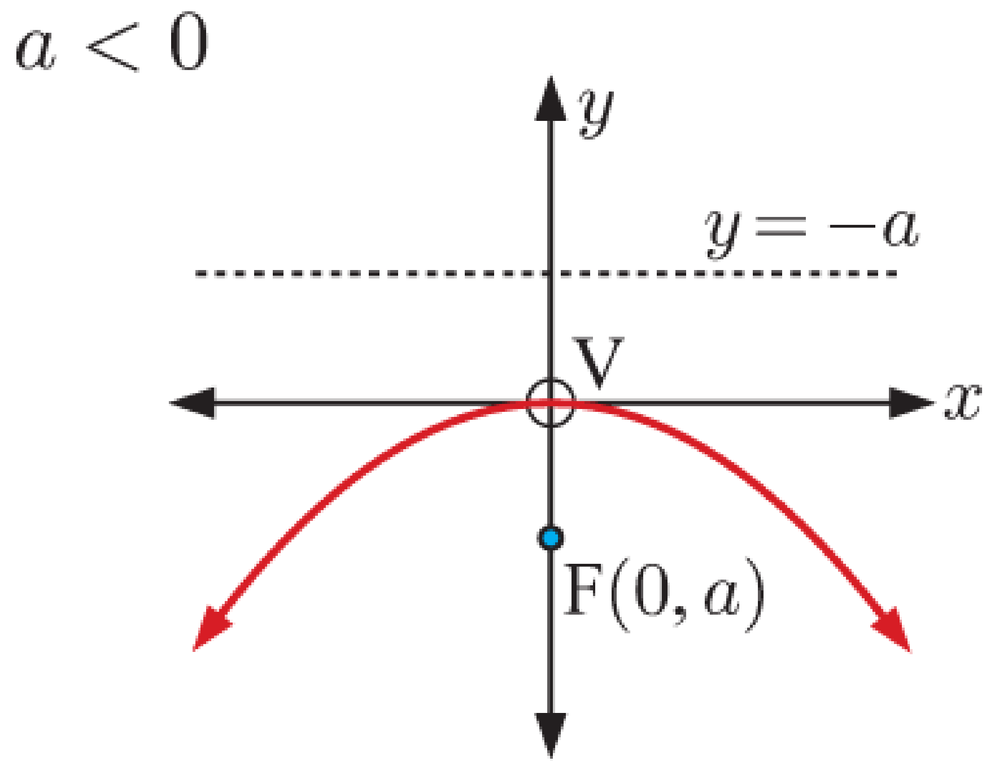
Using the same reasoning we also obtain $y^2 = 4ax$ in this case.



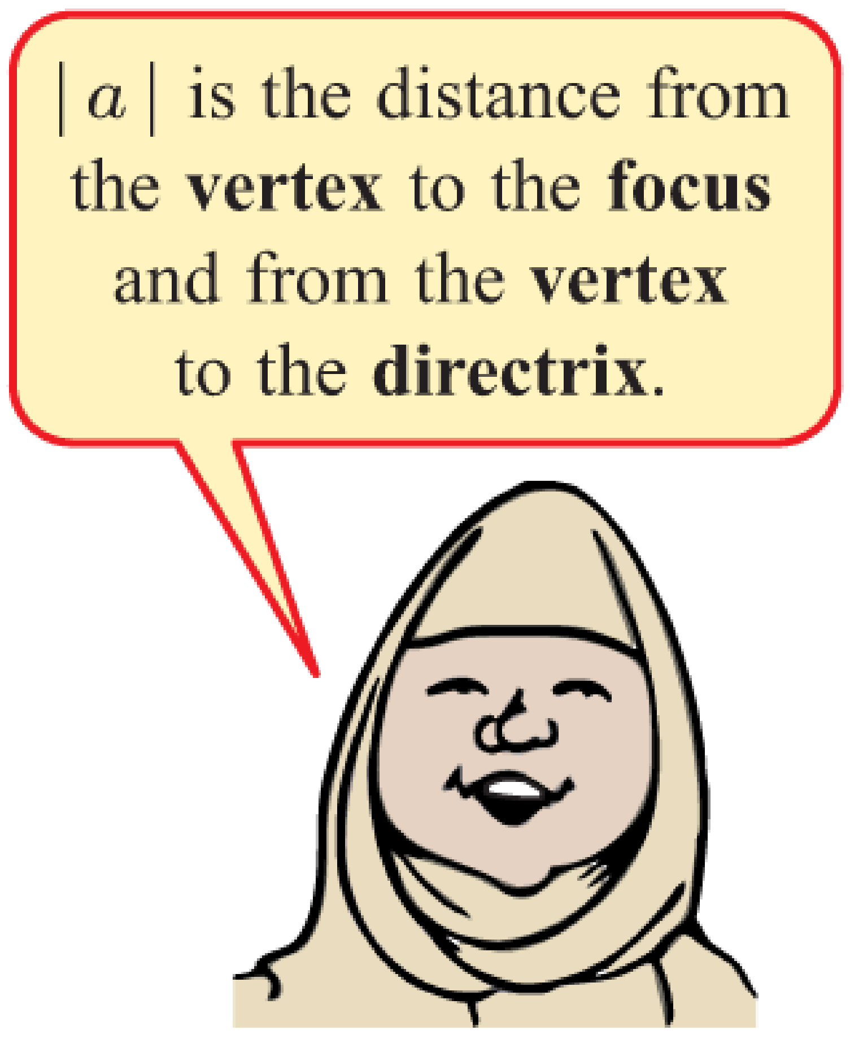
If we rotate the parabolas anti-clockwise through $\frac{\pi}{2}$ we obtain the familiar quadratics $x^2 = 4ay$.



$a > 0$
y
F$(0, a)$
x
V
$y = -a$



$a < 0$
y
$y = -a$
V
x
F$(0, a)$



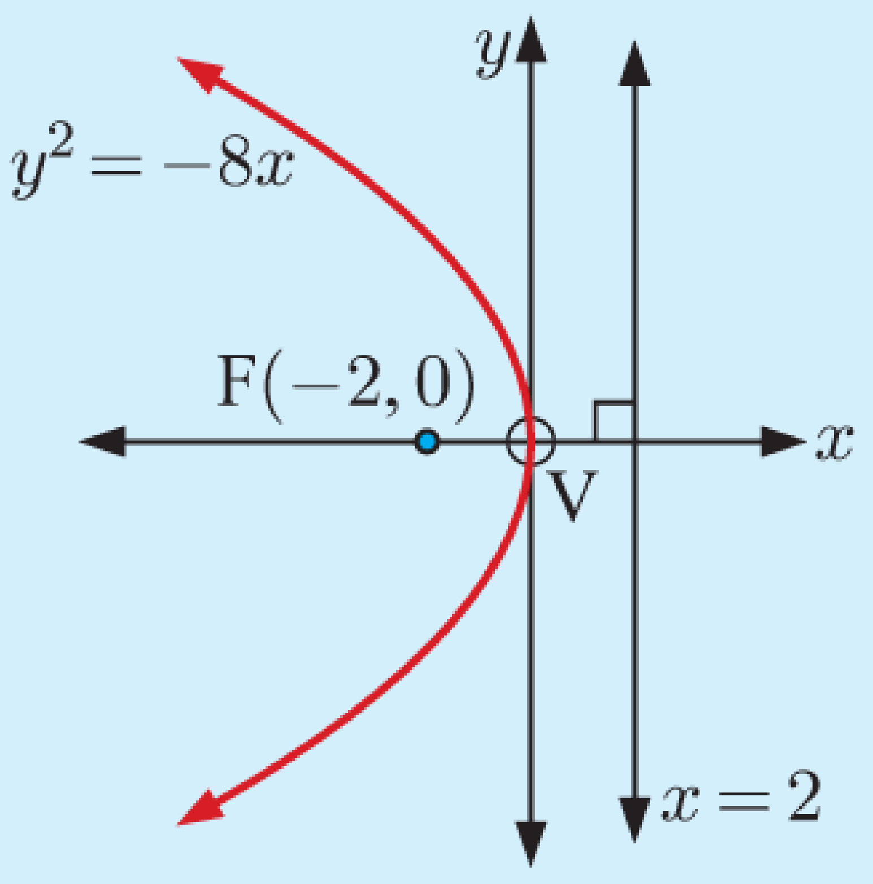
$|\,a\,|$ is the distance from the **vertex** to the **focus** and from the **vertex** to the **directrix**.

Example 33

Sketch $y^2 = -8x$. State the coordinates of the focus and the equation of the directrix.

$$\begin{aligned} y^2 &= -8x \\ \therefore \quad 4a &= -8 \qquad \{y^2 = 4ax\} \\ \therefore \quad a &= -2 \end{aligned}$$

$\therefore$ the focus is F$(-2, 0)$ and the directrix is $x = 2$.



y
$y^2 = -8x$
F$(-2, 0)$
x
V
$x = 2$

EXERCISE 2N.2

1 Sketch each parabola, stating the coordinates of its focus and the equation of its directrix:

a $y^2 = 8x$ **b** $y^2 + 10x = 0$ **c** $x^2 = 12y$ **d** $2x^2 + 5y = 0$

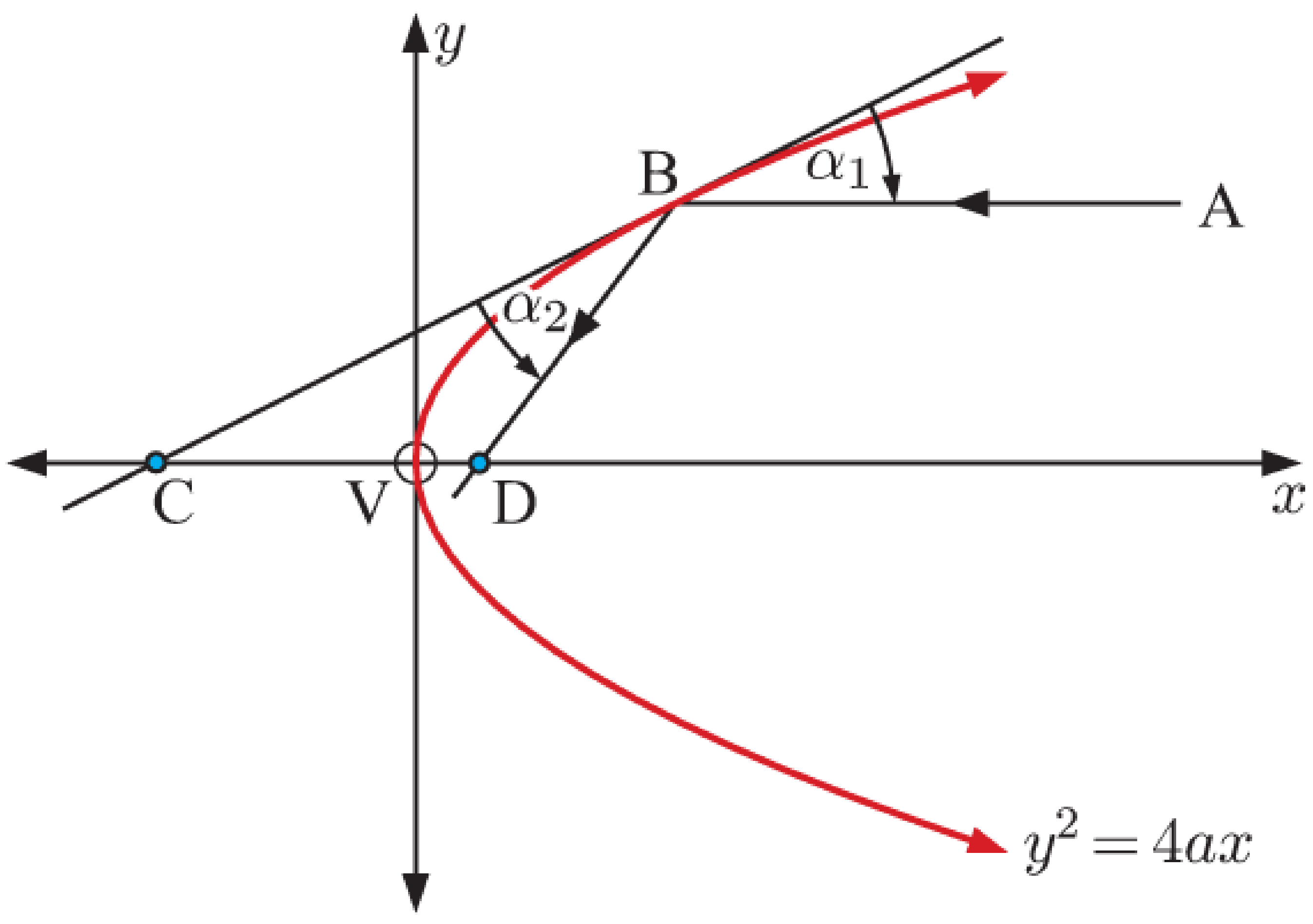
2 Find the equation of the parabola with:

- **a** $V(0,\ 0)$ and $F(3,\ 0)$
- **b** $V(0,\ 0)$ and $F(0,\ 3)$
- **c** $V(0,\ 0)$ and directrix $x = -5$
- **d** $F(-2,\ 0)$ and directrix $x = 2$
- **e** $F(0,\ -5)$ and directrix $y = 5$
- **f** $F(0,\ -2)$ and directrix $y = 2$.

3 Consider the parabola $y^2 = 4ax$.

- **a** Show that the tangent at $(x_1,\ y_1)$ is $2ax - y_1y = -2ax_1$.
- **b** Show that the normal at $(x_1,\ y_1)$ is $y_1x + 2ay = x_1y_1 + 2ay_1$.
- **c** For $a > 0$, show that the x-intercept of the normal is $> 2a$.

4



The inner surface of this parabola is a mirror. [AB] is a ray of light which is parallel to the axis of symmetry.

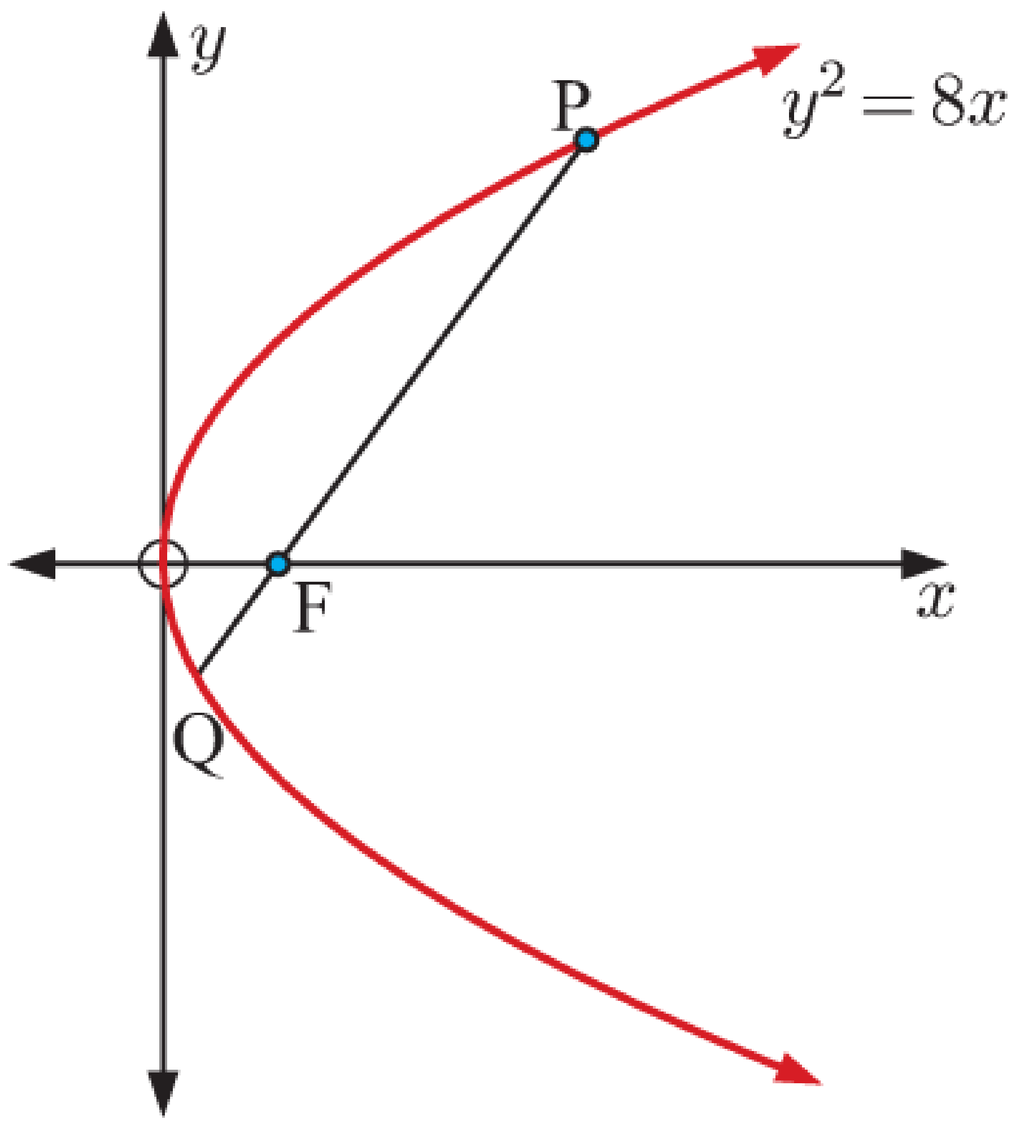
The ray is reflected at B so that $\alpha_1 = \alpha_2$, and it cuts the axis of symmetry at D.

[BC] is a tangent to $y^2 = 4ax$.

Prove that:

- **a** $\triangle$BCD is isosceles
- **b** D is the focus of the parabola.

5



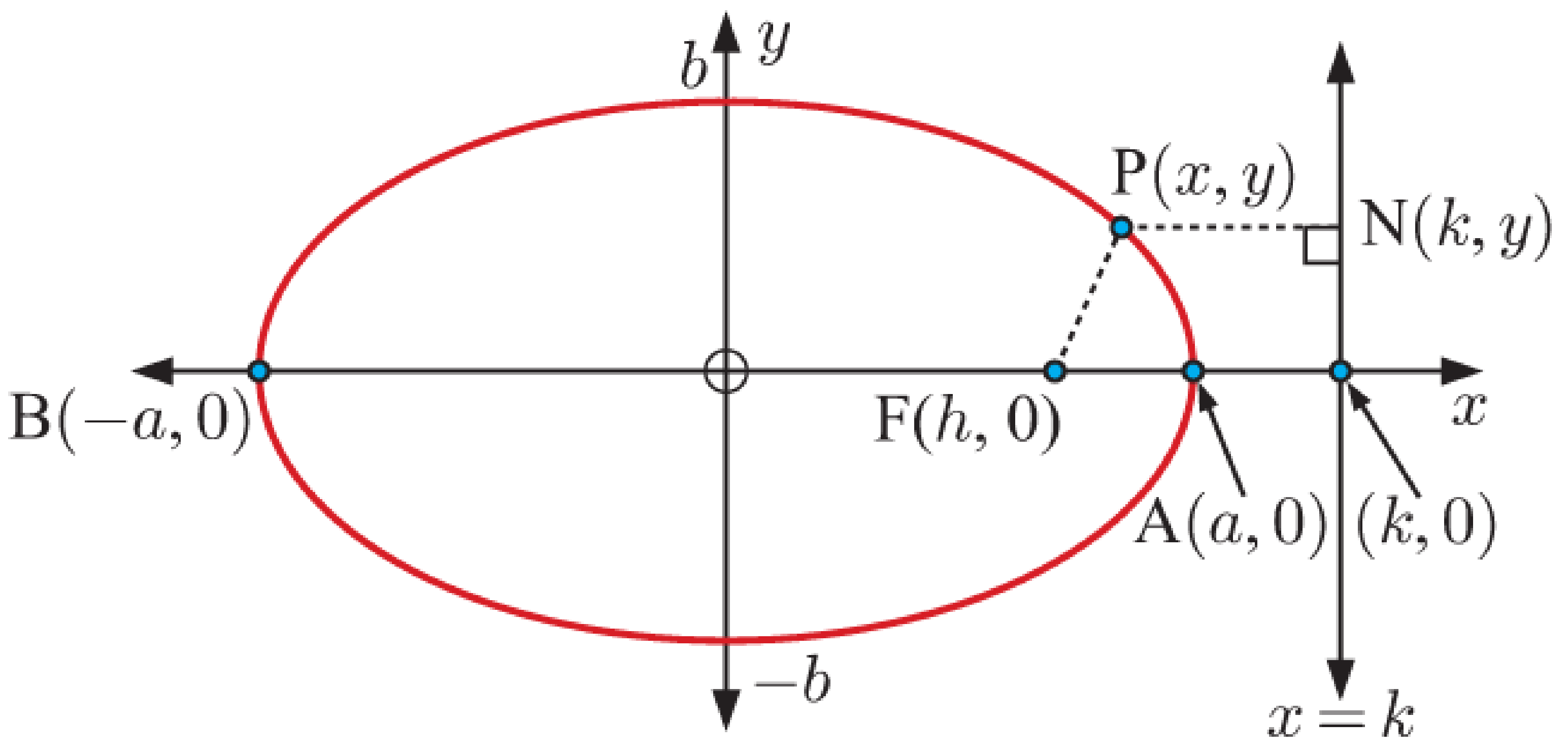
F is the focus of parabola $y^2 = 8x$ and [PQ] is a **focal chord** which passes through the focus.

- **a** Suppose P has x-coordinate 4.
  - **i** Find the coordinates of P and Q.
  - **ii** Find the equations of the tangents at P and Q.
  - **iii** Show that the tangents at P and Q meet on the directrix and are at right angles to each other.
- **b** Show that the property in **a iii** is true for any focal chord [PQ].

THE ELLIPSE

If F is a fixed point (called a **focus**) and $P(x,\ y)$ moves so that $\dfrac{PF}{PN} = e$ where $0 < e < 1$ and N is the foot of the perpendicular from P to a fixed line (called the **directrix**), then the locus of P is an **ellipse**, and e is called the **eccentricity**.

Consider $P(x,\ y)$ moving on an ellipse with centre O, focus $(h,\ 0)$, directrix $x = k$, x-intercepts $\pm a$, and y-intercepts $\pm b$.



As P moves around the ellipse, N moves along the directrix.

When P is at A, N is at $(k, 0)$ and $\text{AF} = e\text{AN}$ $\Rightarrow$ $a - h = e(k - a)$ .... (1)

When P is at B, N is at $(k, 0)$ and $\text{BF} = e\text{BN}$ $\Rightarrow$ $h + a = e(k + a)$ .... (2)

Adding (1) and (2) gives $2a = 2ek$ $\Rightarrow$ $k = \dfrac{a}{e}$

Subtracting (1) and (2) gives $-2h = -2ae$ $\Rightarrow$ $h = ae$

Hence, the focus is $\text{F}(ae, 0)$ and the directrix has equation $x = \dfrac{a}{e}$.

$$\text{Since} \quad \text{PF} = e\text{PN},$$

$$\sqrt{(x - ae)^2 + y^2} = e\sqrt{\left(x - \frac{a}{e}\right)^2}$$

$$\therefore \quad x^2 - 2aex + a^2e^2 + y^2 = e^2\left(x^2 - \frac{2ax}{e} + \frac{a^2}{e^2}\right)$$

$$\therefore \quad x^2 - \cancel{2aex} + a^2e^2 + y^2 = e^2x^2 - \cancel{2aex} + a^2$$

$$\therefore \quad (1 - e^2)x^2 + y^2 = a^2(1 - e^2)$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

The ellipse has y-intercepts $\pm a\sqrt{1 - e^2} = \pm b$

$\therefore$ the ellipse has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $b^2 = a^2(1 - e^2)$.

Since the equation remains unaltered by replacing x by $-x$, the ellipse has a second focus at $(-ae, 0)$ and a second directrix $x = -\dfrac{a}{e}$.

TERMINOLOGY

The **centre** of an ellipse is the point of intersection of its axes of symmetry.

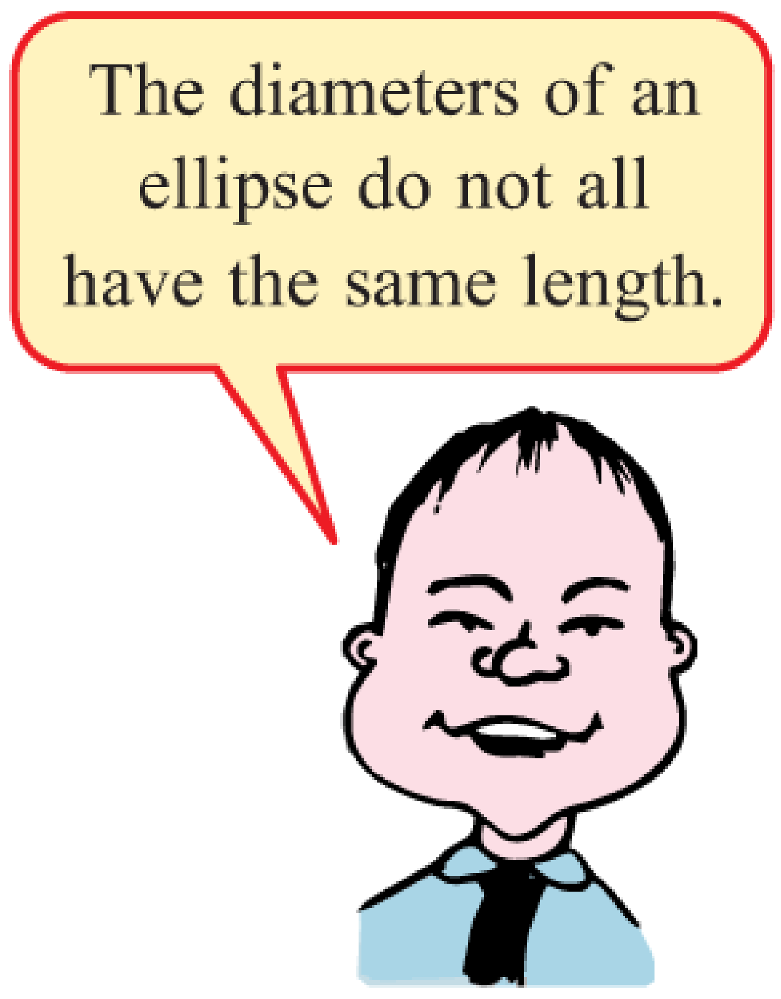
A **chord** of an ellipse is any line segment joining two points on the ellipse.

A **diameter** of an ellipse is any chord which passes through its centre.

The **major axis** of an ellipse is the diameter through its foci.

The **minor axis** is the diameter perpendicular to the major axis.

The **latus rectum** is the chord through a focus which is perpendicular to the axis of symmetry.


The diameters of an ellipse do not all have the same length.

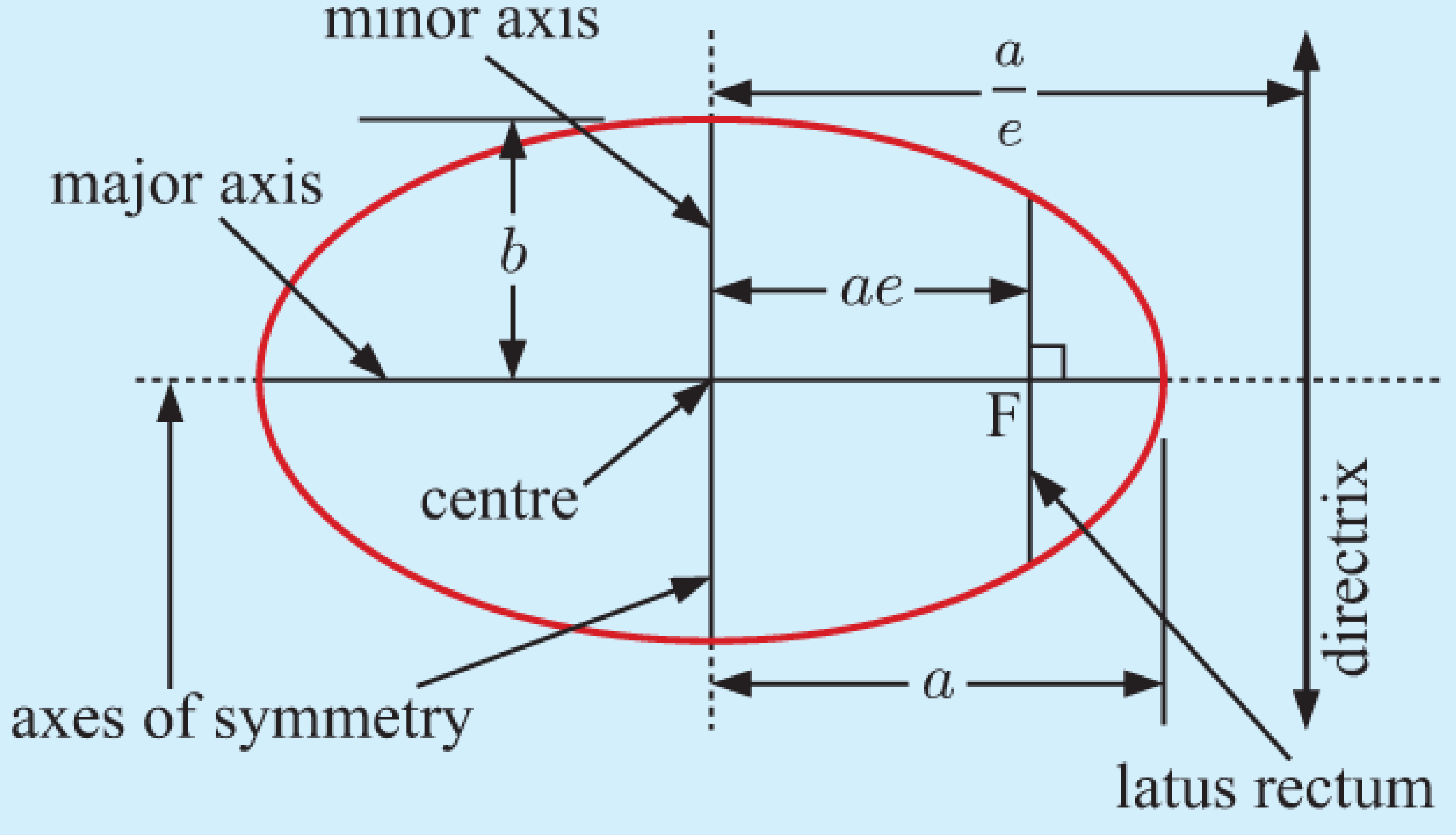
GEOMETRICAL PROPERTIES FOR $a > 0,\ b > 0,\ 0 < e < 1$

$a =$ half the length of the major axis.

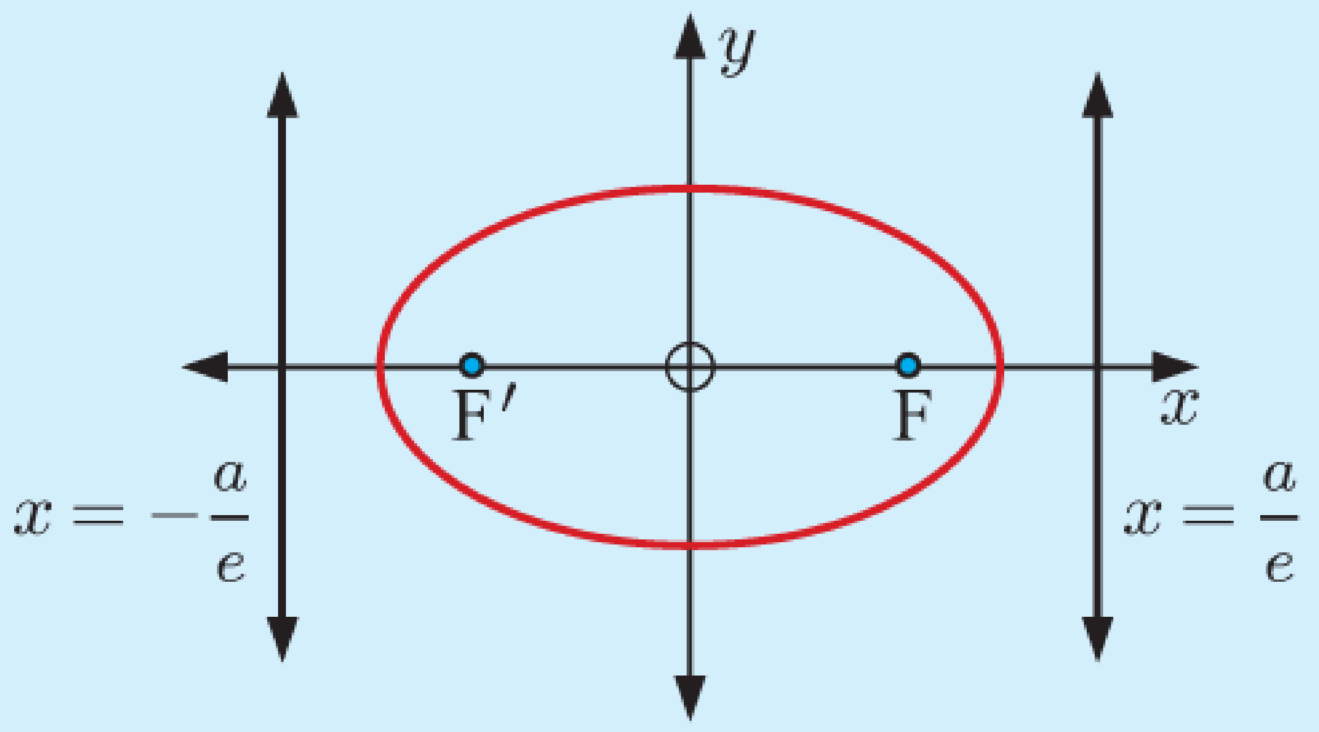
$ae =$ distance from centre to focus.

$\frac{a}{e} =$ distance from centre to directrix.

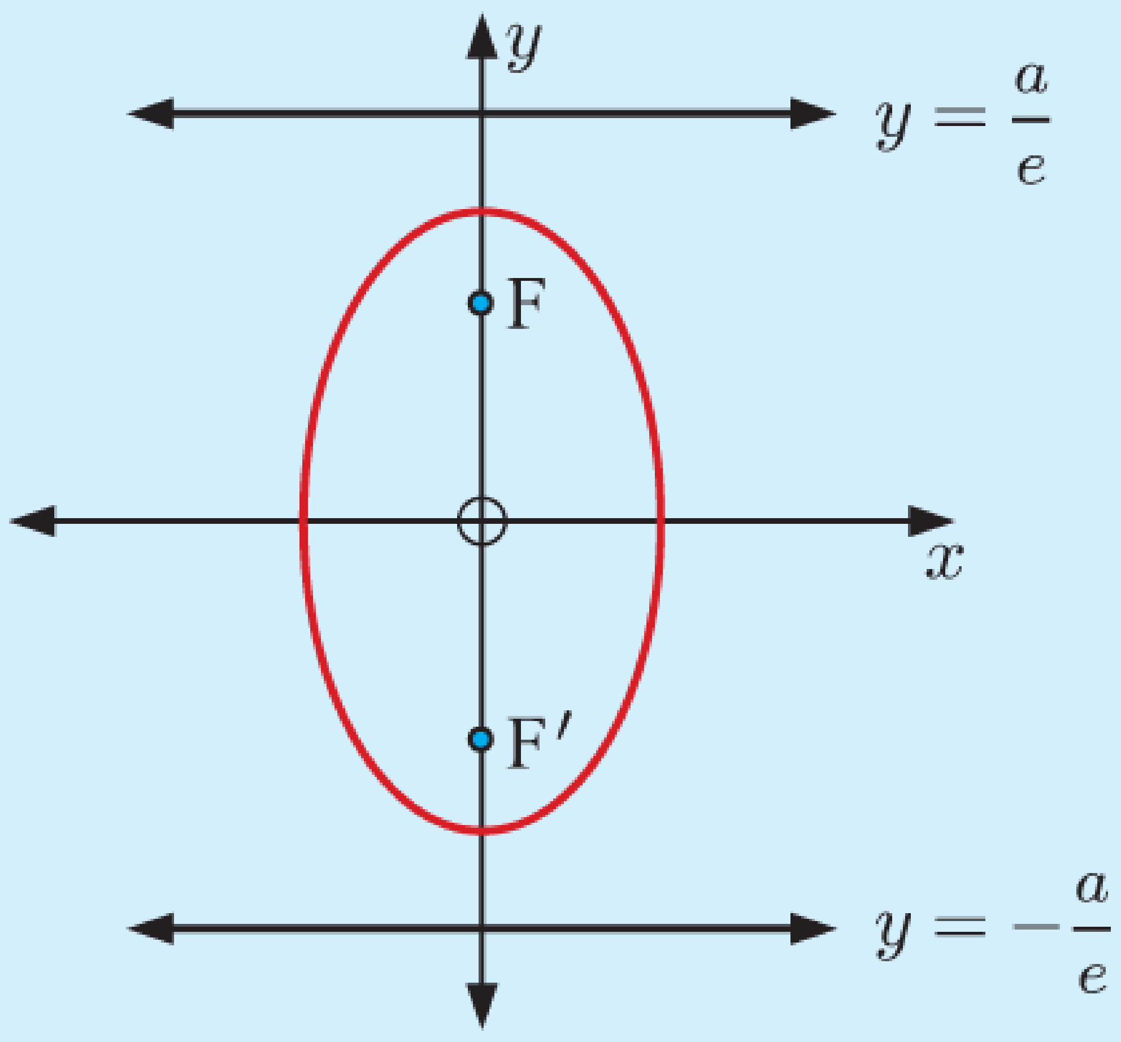
$b =$ half the length of the minor axis.


minor axis
major axis
$\frac{a}{e}$
b
ae
F
centre
directrix
axes of symmetry
a
latus rectum

SIMPLE ELLIPSES


y
x
F′
F
$x = -\frac{a}{e}$
$x = \frac{a}{e}$

centre:	$(0,\ 0)$
foci:	$(\pm ae,\ 0)$
directrices:	$x = \pm\frac{a}{e}$
inequality:	$a > b,\ \ 0 < e < 1$
identity:	$b^2 = a^2(1 - e^2)$
equation:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

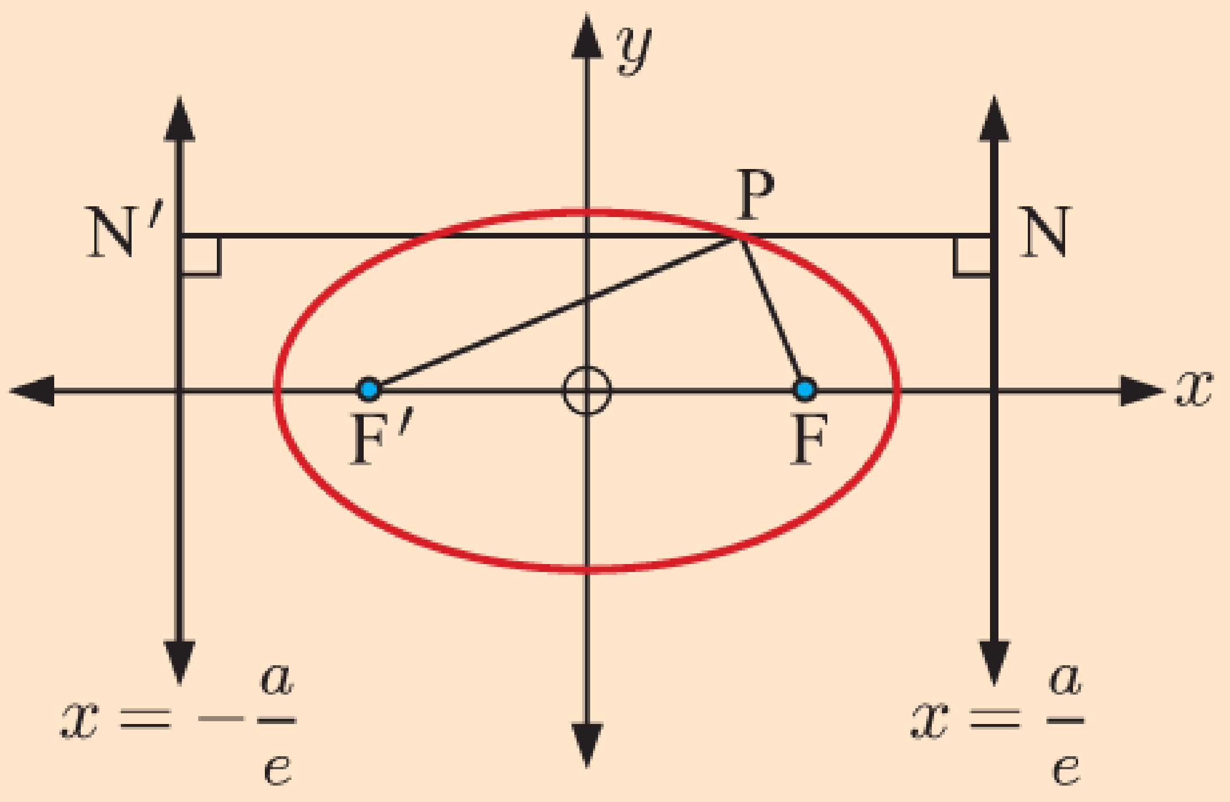

y
$y = \frac{a}{e}$
F
x
F′
$y = -\frac{a}{e}$

centre:	$(0,\ 0)$
foci:	$(0,\ \pm ae)$
directrices:	$y = \pm\frac{a}{e}$
inequality:	$a > b,\ \ 0 < e < 1$
identity:	$b^2 = a^2(1 - e^2)$
equation:	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

THE FOCAL-DISTANCE PROPERTY

As $\mathrm{P}(x,\ y)$ moves around an ellipse, the sum of the distances from P to the foci is $\mathrm{PF} + \mathrm{PF}' = 2a$.

Proof:

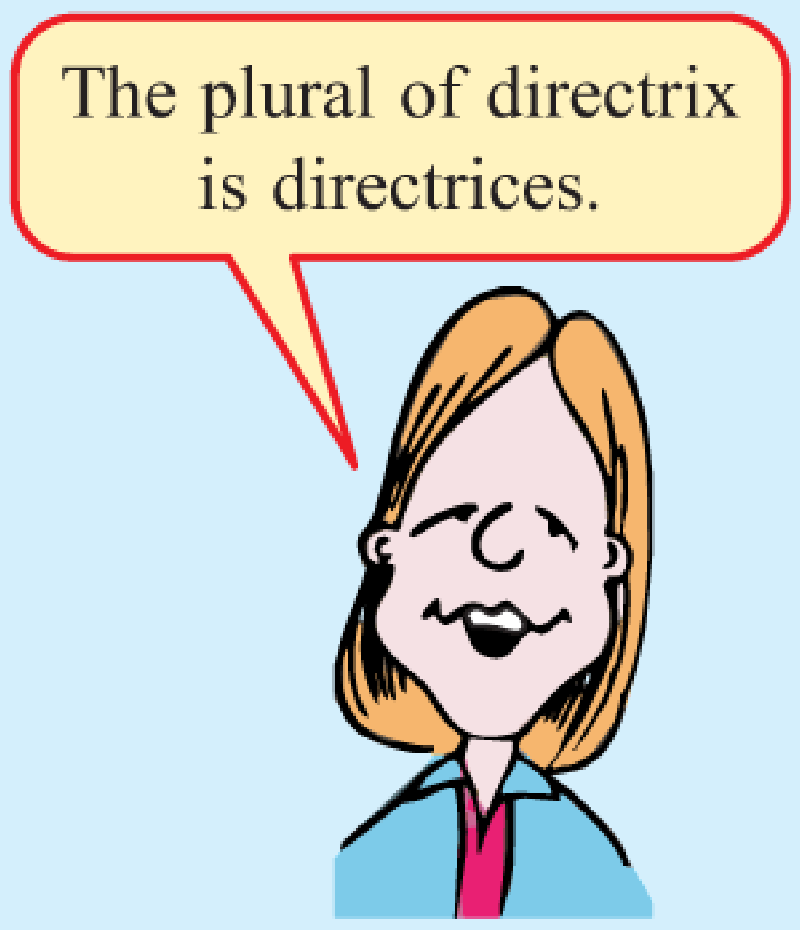

y
N′
P
N
x
F′
F
$x = -\frac{a}{e}$
$x = \frac{a}{e}$

$$\begin{aligned} &\mathrm{PF} + \mathrm{PF}' \\ &= e\mathrm{PN} + e\mathrm{PN}' \\ &= e(\mathrm{PN} + \mathrm{PN}') \\ &= e\left(\frac{a}{e} - \left(-\frac{a}{e}\right)\right) \\ &= 2a \end{aligned}$$

Example 34

Consider the ellipse with equation $2x^2 + 4y^2 = 16$.

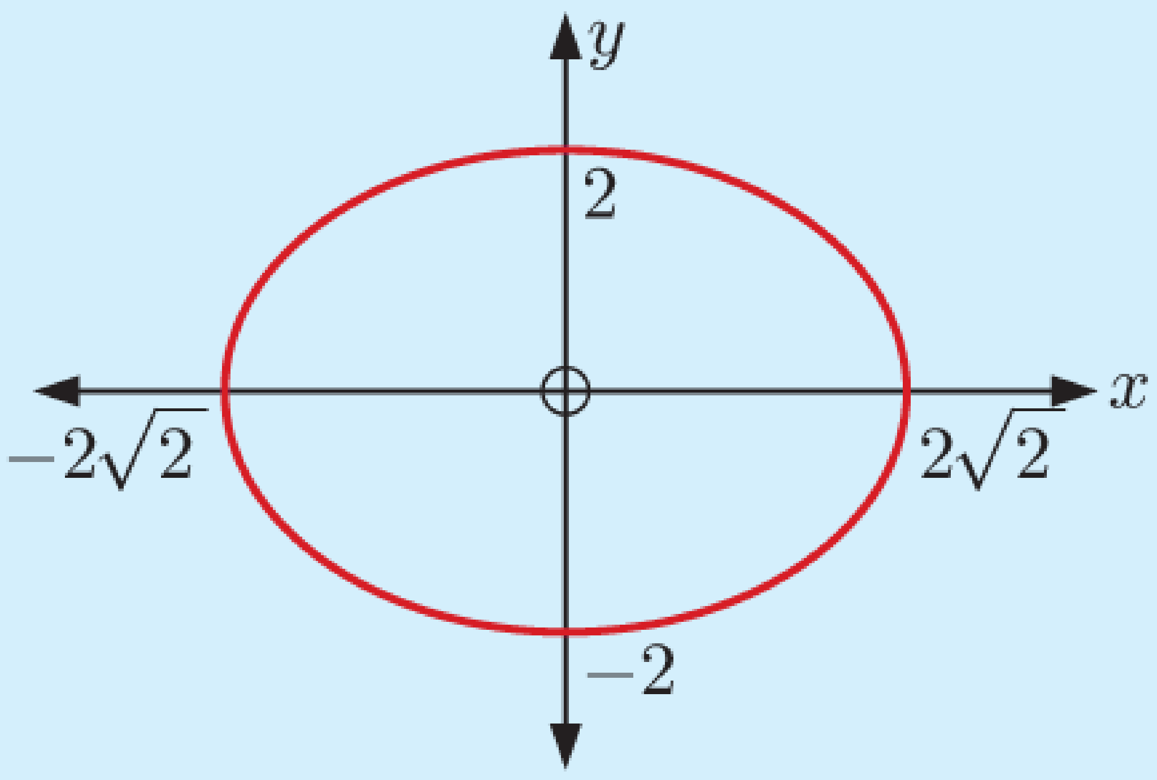
- **a** Find the axes intercepts.
- **b** Sketch the ellipse.
- **c** Find the eccentricity of the ellipse.
- **d** Find the coordinates of each focus and the equations of the corresponding directrices.


The plural of directrix is directrices.

a
$$2x^2 + 4y^2 = 16$$
$$\therefore \frac{x^2}{8} + \frac{y^2}{4} = 1$$
$$\therefore a^2 = 8 \text{ and } b^2 = 4$$

$\therefore$ the x-intercepts are $\pm 2\sqrt{2}$
y-intercepts are ± 2.

b



c
$$b^2 = a^2(1-e^2)$$
$$\therefore 4 = 8(1-e^2)$$
$$\therefore 1 - e^2 = \tfrac{1}{2}$$
$$\therefore e^2 = \tfrac{1}{2}$$
$$\therefore e = \tfrac{1}{\sqrt{2}} \qquad \{\text{as } e > 0\}$$

d $ae = 2\sqrt{2} \times \frac{1}{\sqrt{2}} = 2$ and $\frac{a}{e} = \frac{2\sqrt{2}}{\frac{1}{\sqrt{2}}} = 4$

The focus $(2, 0)$ has corresponding directrix $x = 4$.

The focus $(-2, 0)$ has corresponding directrix $x = -4$.

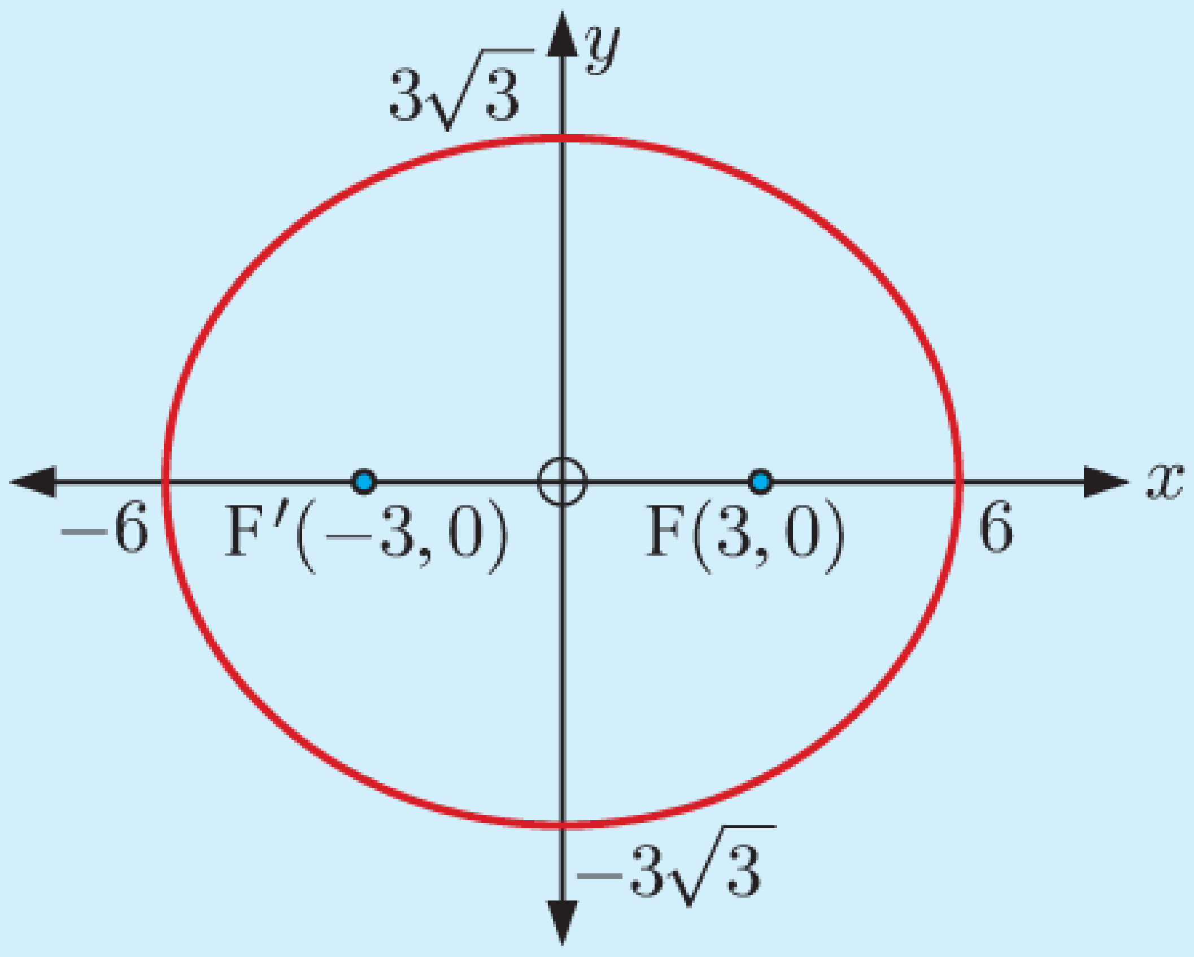
Example 35

Find the equation of the ellipse with foci $(\pm 3, 0)$ and eccentricity $\frac{1}{2}$.

As the foci are $(\pm 3, 0)$, $ae = 3$
$$\therefore a = 6.$$

Now $b^2 = a^2(1-e^2)$
$$\therefore b^2 = 36(1 - \tfrac{1}{4}) = 27$$

$\therefore$ the ellipse has equation $\frac{x^2}{36} + \frac{y^2}{27} = 1$.



EXERCISE 2N.3

1 For each ellipse:

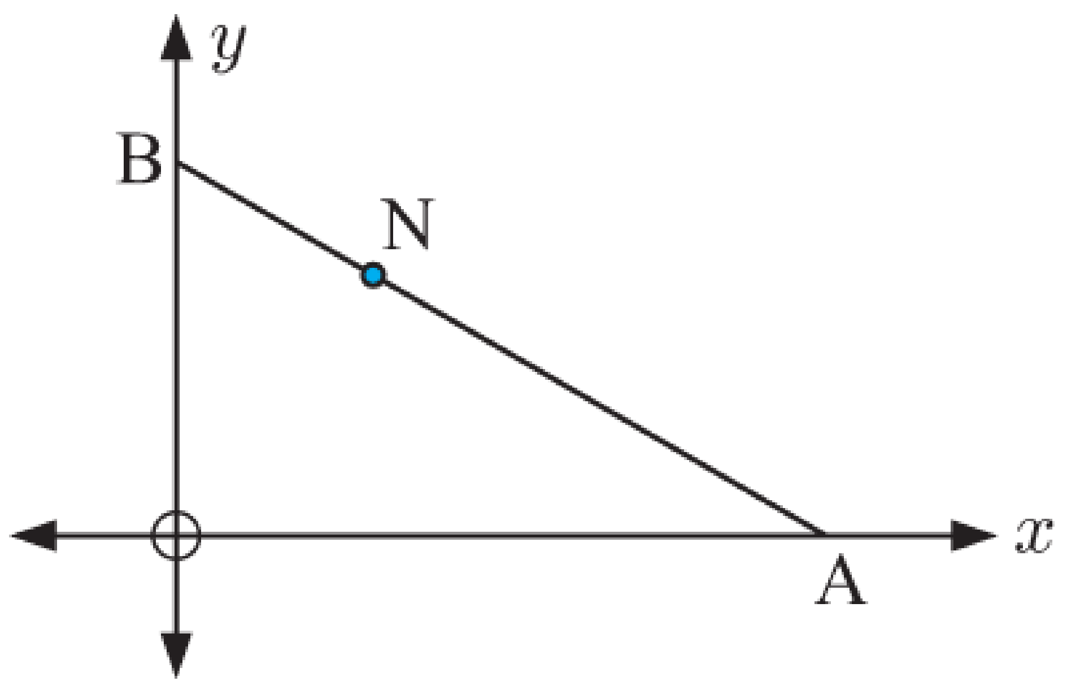
- **i** Find the axes intercepts.
- **ii** Sketch the ellipse.
- **iii** Find the eccentricity.
- **iv** Find the coordinates of each focus and the equations of the corresponding directrices.

a $4x^2 + 9y^2 = 36$ **b** $4x^2 + 3y^2 = 12$

2 Find the equation of the ellipse with the properties:

- **a** centre $(0, 0)$, focus $(3, 0)$, major axis of length 8 units
- **b** foci $(0, \pm 4)$, $e = \frac{1}{2}$
- **c** major axis on the x-axis, x-intercepts ± 3, $e = \frac{2}{3}$
- **d** foci $(0, \pm 4)$, directrices $y = \pm 5$
- **e** $\text{PF} + \text{PF}' = 10$, foci $(\pm 4, 0)$
- **f** centre $(0, 0)$, foci $(0, \pm 3)$, minor axis of length 8 units
- **g** extremities of minor axis $(0, \pm 3)$, $e = \frac{1}{2}$
- **h** foci $(\pm 3, 0)$, directrix $x = 5$.

3 Find the length of the latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci $(\pm ae, 0)$.

4 

[AB] is a line segment of fixed length k, where A is free to move on the x-axis and B is free to move on the y-axis. N lies on [AB] such that AN : NB = 2 : 1. Determine the nature of the locus of N.

5 Explain how to draw an ellipse using two pegs and a piece of inelastic string.

Example 36

Find the equation of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at the point $\left(2, -\frac{3\sqrt{3}}{2}\right)$.

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$\therefore \quad \frac{2x}{16} + \frac{2y}{9}\frac{dy}{dx} = 0 \qquad \{\text{implicit differentiation}\}$$

$$\therefore \quad \frac{y}{9}\frac{dy}{dx} = -\frac{x}{16}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{9x}{16y}$$

At $\left(2, -\frac{3\sqrt{3}}{2}\right)$, $\frac{dy}{dx} = \frac{-9(2)}{16\left(-\frac{3\sqrt{3}}{2}\right)} = \frac{\sqrt{3}}{4}$

$\therefore$ the equation of the tangent is $\sqrt{3}x - 4y = \sqrt{3}(2) - 4\left(-\frac{3\sqrt{3}}{2}\right)$

which is $\sqrt{3}x - 4y = 8\sqrt{3}$

6 Consider the ellipse $\frac{x^2}{3} + \frac{y^2}{12} = 1$. Find the equation of:

- **a** the tangents to the ellipse at the point where $x = \sqrt{2}$
- **b** the normals to the ellipse at the point where $y = 2$.

7 **a** Show that the equation of the tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$.

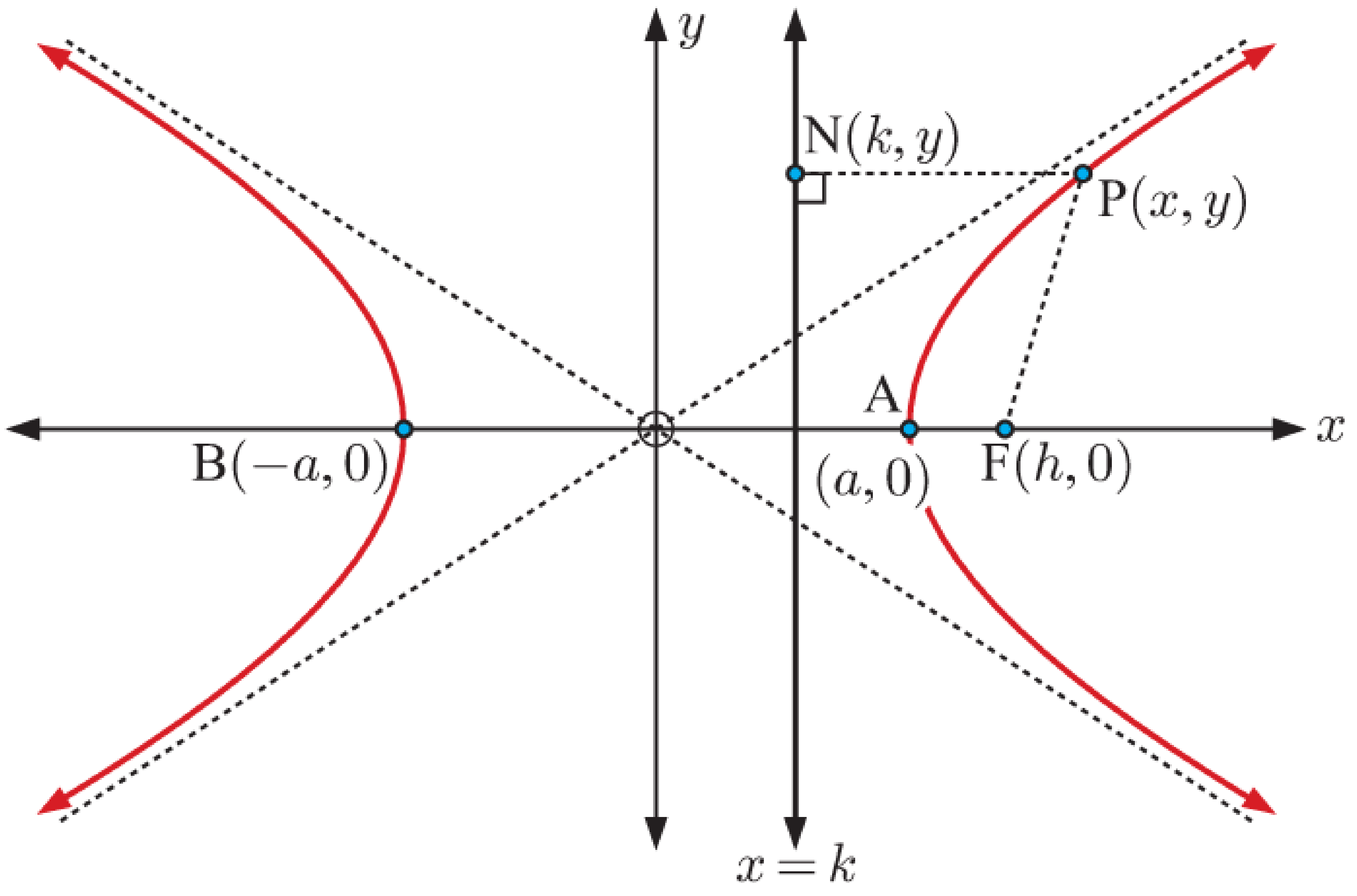
b Hence, show that the equation of the tangent at the end of the latus rectum in the first quadrant is $ex+y=a$.

c Find the equation of the normal at the point (x_1, y_1).

THE HYPERBOLA

If F is a fixed point (called a **focus**) and $P(x, y)$ moves so that $\frac{\text{PF}}{\text{PN}}=e$ where $e>1$ and N is the foot of the perpendicular from P to a fixed line (called the **directrix**), then the locus of P is an **hyperbola**.

Consider $P(x, y)$ moving on an hyperbola with centre O, focus $(h, 0)$, directrix $x=k$, and x-intercepts $\pm a$.



As P moves on the hyperbola, N moves along the directrix.

When P is at A, N is at $(k, 0)$ and $\text{AF}=e\text{AN}$ $\Rightarrow$ $h-a=e(a-k)$ .... (1)

When P is at B, N is at $(k, 0)$ and $\text{BF}=e\text{BN}$ $\Rightarrow$ $h+a=e(k+a)$ .... (2)

(1) and (2) are the same equations as for the ellipse

$\therefore$ $h=ae$ and $k=\frac{a}{e}$

Hence, the focus is $F(ae, 0)$ and the directrix has equation $x=\frac{a}{e}$.

Using $\text{PF}=e\text{PN}$, we also get the same equation of the hyperbola as we obtained for the ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1.$$

However, as $e>1$, $1-e^2<0$ and therefore $a^2(1-e^2)<0$.

Let $a^2(1-e^2)=-b^2$ for some constant b.

$\therefore$ the equation becomes $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

Once again the equation is unaltered by replacing x by $-x$, so there is a second focus at $(-ae, 0)$ and a second directrix $x=-\frac{a}{e}$.

ASYMPTOTES

From the graph of the basic hyperbola, we observe there are no y-intercepts.

$\therefore$ clearly b has a different geometrical interpretation from an ellipse.

However, we notice from the graph that the basic hyperbola approaches oblique asymptotes for large values of x and y.

Oblique means at an angle. An oblique asymptote is neither horizontal nor vertical.

For large values of x and y, $\dfrac{y^2}{b^2} = \dfrac{x^2}{a^2} - 1$ becomes $\dfrac{y^2}{b^2} \approx \dfrac{x^2}{a^2}$.

This means that y approaches, but never equals, $\pm\dfrac{b}{a}x$.

$\therefore$ the asymptotes of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are $y = \pm\dfrac{b}{a}x$

TERMINOLOGY

The **centre** of an hyperbola is the point of intersection of its axes of symmetry.

A **diameter** of an hyperbola is any chord which passes through its centre.

The **transverse axis** is the diameter on the axis containing the foci.

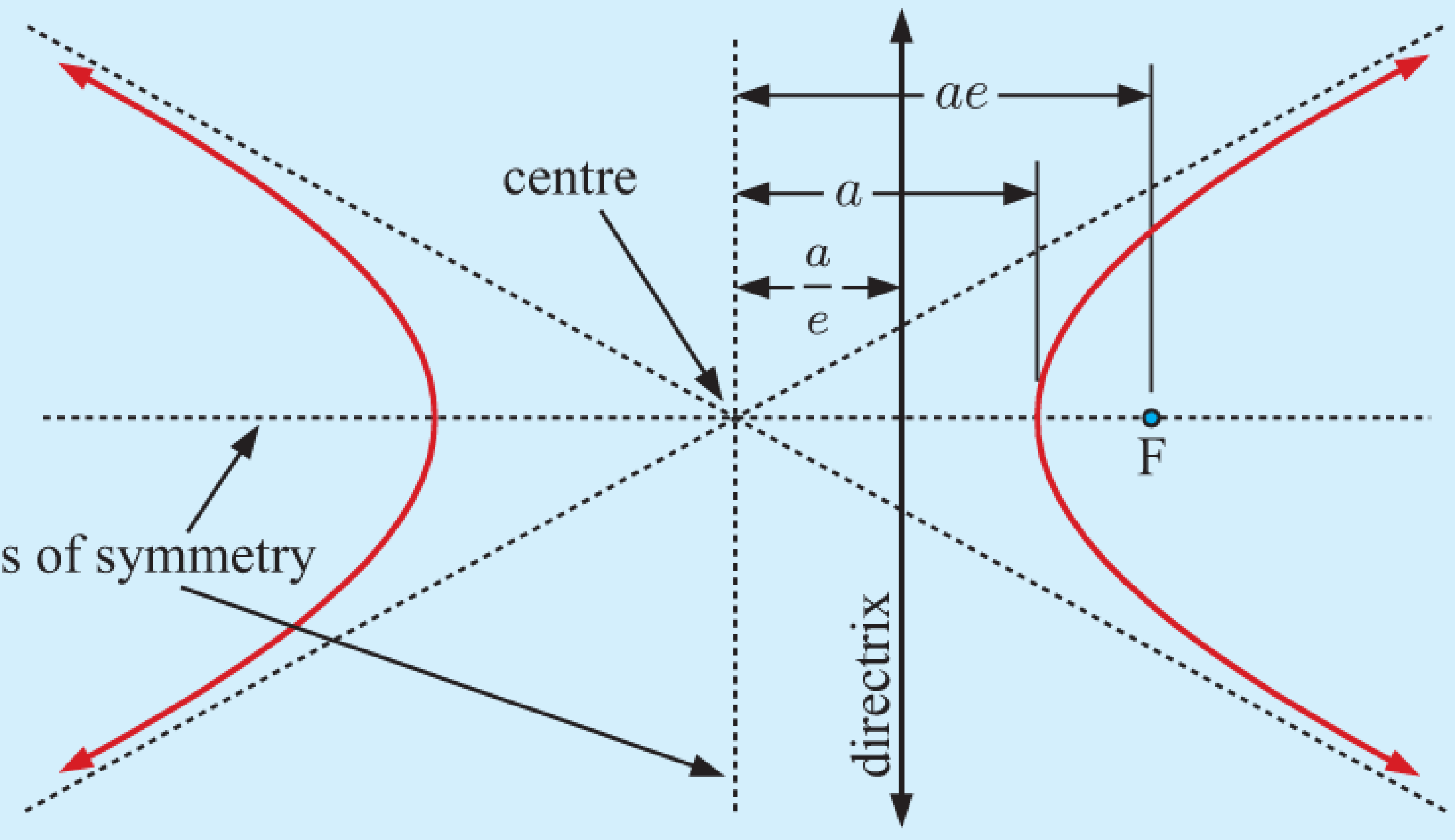
The **latus rectum** is the chord through a focus which is perpendicular to the transverse axis.

GEOMETRICAL PROPERTIES FOR $a > 0,\ b > 0,\ e > 1$

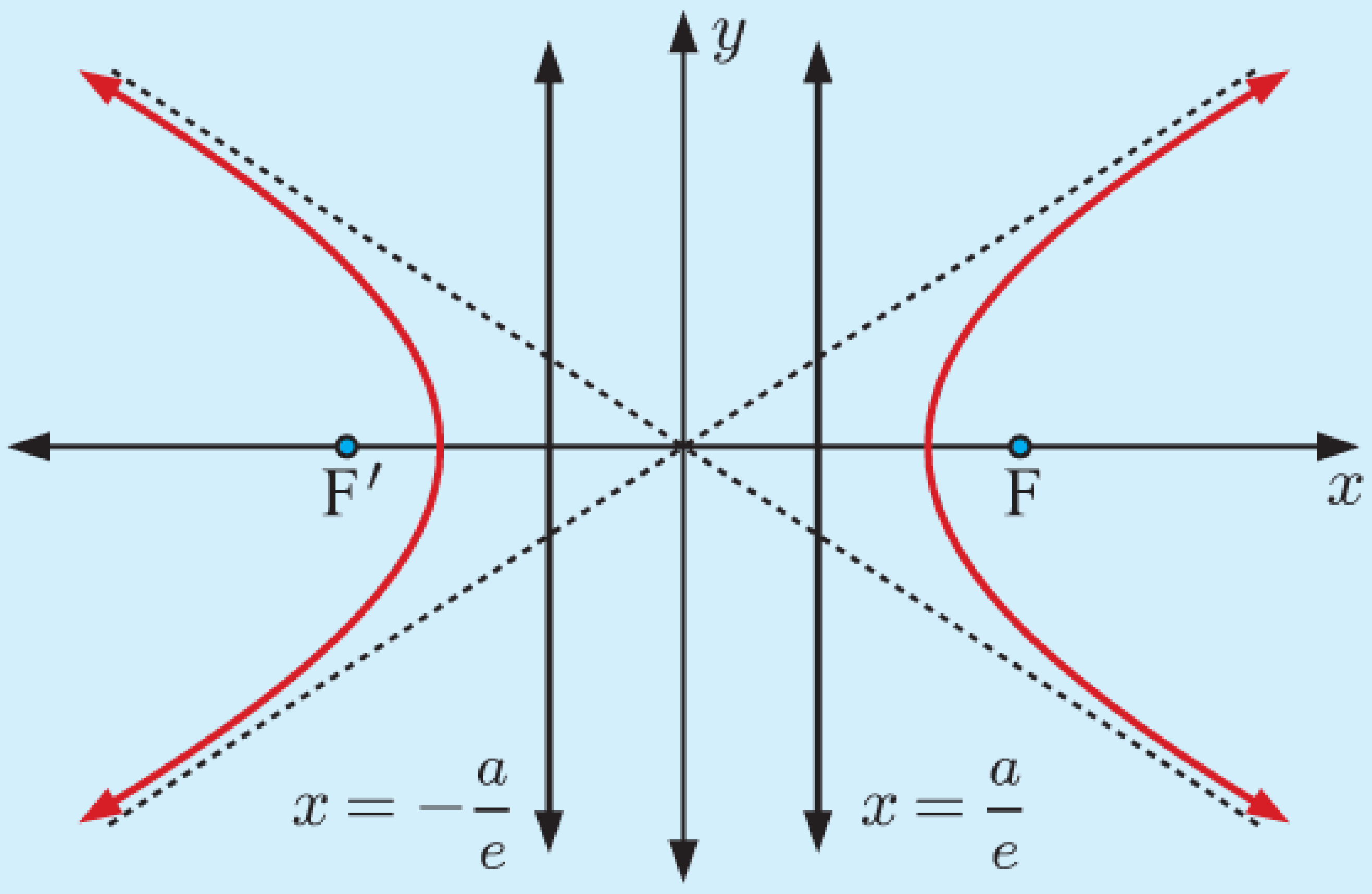
$a =$ half the length of the transverse axis.

$ae =$ distance from centre to focus.

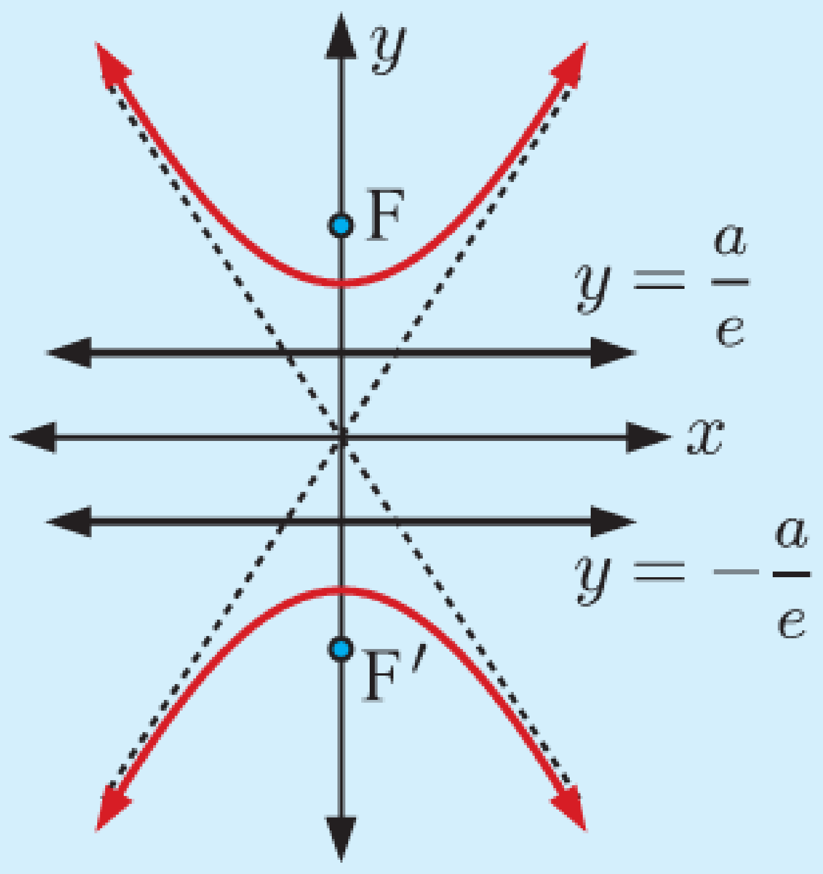
$\dfrac{a}{e} =$ distance from centre to directrix.



SIMPLE HYPERBOLAE



foci:	$(\pm ae,\ 0)$
vertices:	$(\pm a,\ 0)$
directrices:	$x = \pm\dfrac{a}{e}$
asymptotes:	$y = \pm\dfrac{b}{a}x$
identity:	$b^2 = a^2(e^2 - 1)$
equation:	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



foci: $(0,\ \pm ae)$
vertices: $(0,\ \pm a)$
directrices: $y=\pm\frac{a}{e}$
asymptotes: $y=\pm\frac{a}{b}x$
identity: $b^2=a^2(e^2-1)$
equation: $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

THE FOCAL-DISTANCE PROPERTY

As $\mathrm{P}(x,\ y)$ moves along the hyperbola, $|\mathrm{PF}-\mathrm{PF}'|=2a$.

RECTANGULAR HYPERBOLAE

An hyperbola is **rectangular** if its asymptotes are perpendicular.

Since the asymptotes are $y=\pm\frac{b}{a}x$, their gradients are $\pm\frac{b}{a}$.

$\therefore$ for the asymptotes to be perpendicular, $\frac{b}{a}\times-\frac{b}{a}=-1$

$$\therefore\ a^2=b^2$$

$$\therefore\ a=b \qquad \{\text{since both are } >0\}$$

$\therefore$ the equation of a rectangular hyperbola can be $x^2-y^2=a^2$ or $y^2-x^2=a^2$.

Since $b^2=a^2(e^2-1)$, $e^2-1=1$

$$\therefore\ e=\sqrt{2} \qquad \{\text{since } e>0\}$$

So, every rectangular hyperbola has eccentricity $\sqrt{2}$.

In the HL Core course, we saw rectangular hyperbolae with equations of the form $xy=k$ where k is a constant.

In fact, $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$ under a rotation of $\pm\frac{\pi}{4}$ becomes $xy=\pm\frac{a^2}{2}$ where a is a constant. These are the only rectangular hyperbolae for which y can be written as a function of x.

Example 37

Sketch $4x^2-9y^2=36$ by finding the axes intercepts and asymptotes.

$\frac{x^2}{9}-\frac{y^2}{4}=1$ cuts the x-axis when $y=0$.

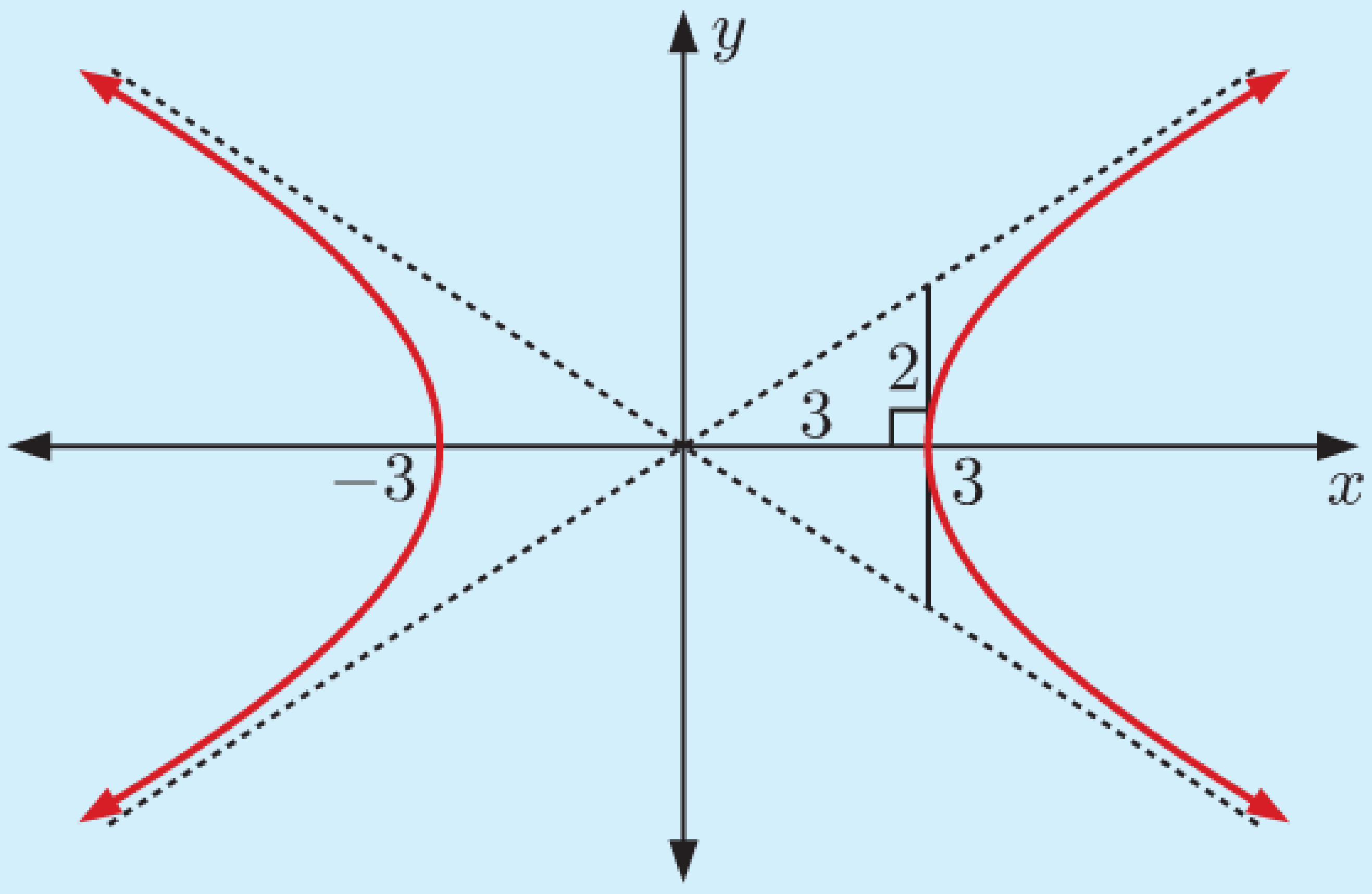
$$\therefore\ \frac{x^2}{9}=1$$

$$\therefore\ x^2=9$$

$$\therefore\ x=\pm 3$$

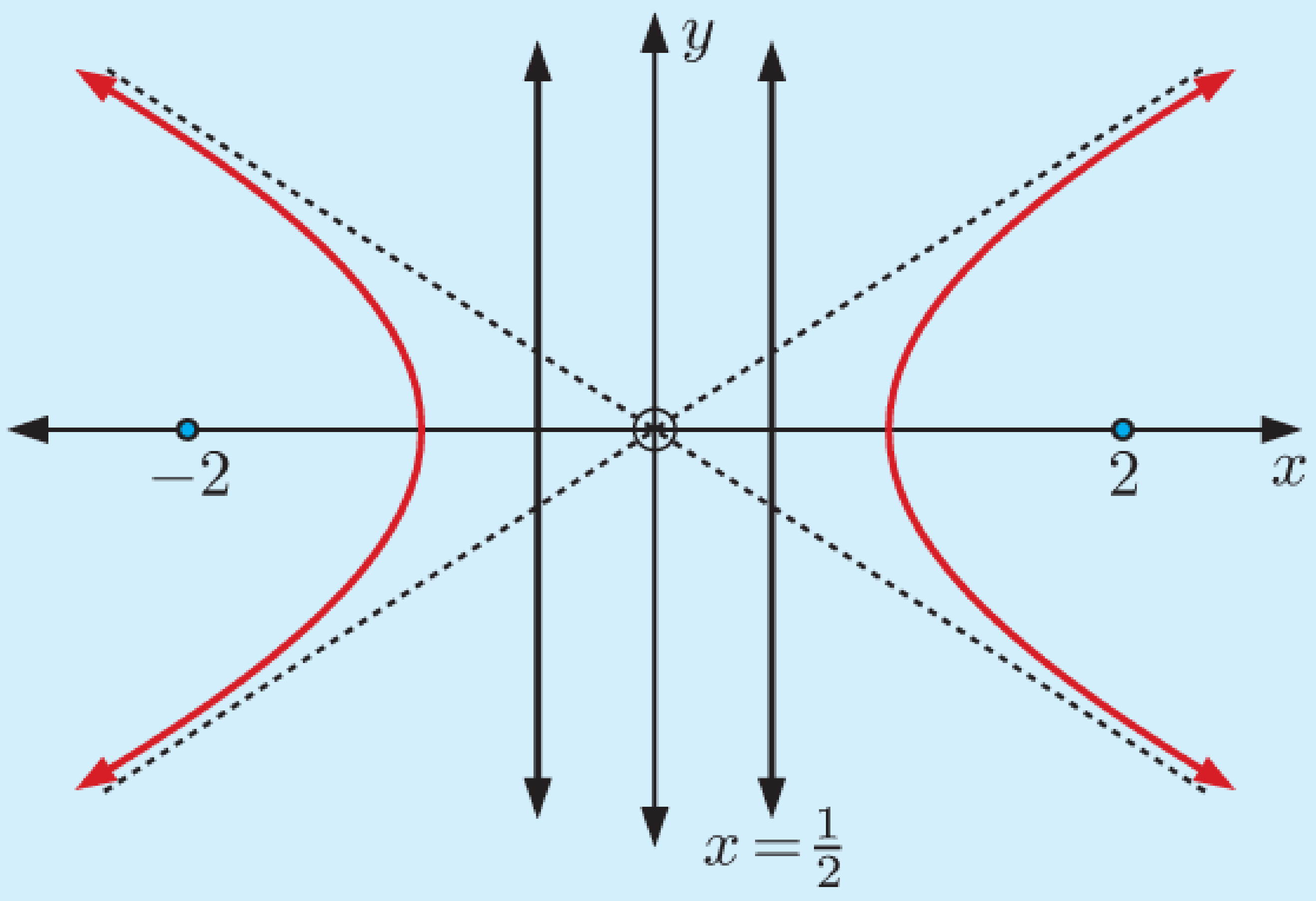
$\therefore$ the graph cuts the x-axis at $(3,\ 0)$ and $(-3,\ 0)$.

The asymptotes are $\frac{x^2}{9}=\frac{y^2}{4}$, which are $y=\pm\frac{2}{3}x$



Example 38

An hyperbola has foci $(\pm 2,\ 0)$ and directrices $x = \pm\frac{1}{2}$. At what points does it cut the axes?



The foci lie on the x-axis and the centre is $(0,\ 0)$.

Now $ae = 2$ and $\frac{a}{e} = \frac{1}{2}$

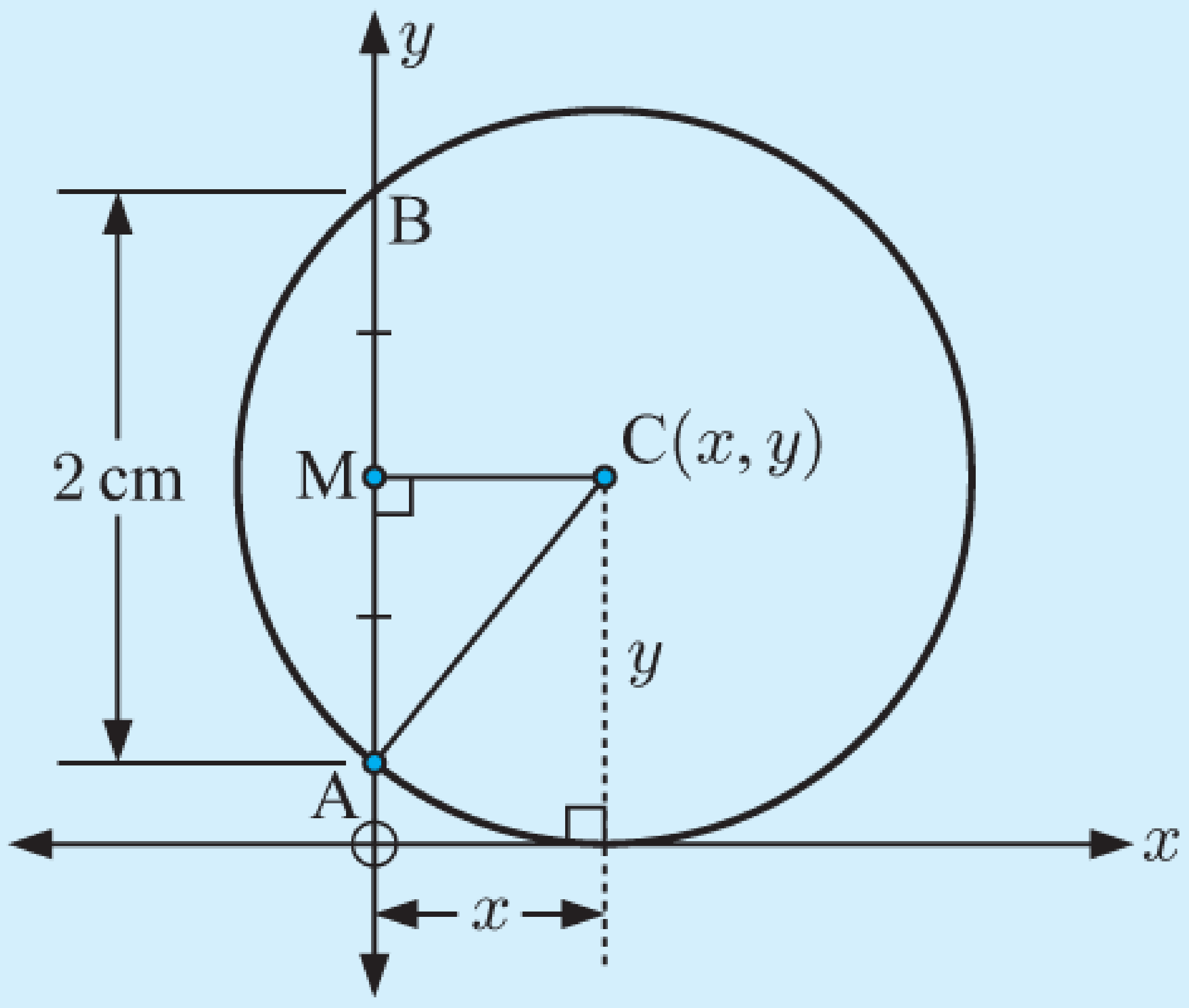
$\therefore\ a^2 = 2 \times \frac{1}{2} = 1$

$\therefore\ a = 1$ $\{$as $a > 0\}$

$\therefore$ the hyperbola cuts the x-axis at $(\pm 1,\ 0)$ but does not cut the y-axis.

Example 39

Each circle in a set touches the x-axis, and the y-axis cuts off a chord of length 2 cm from each circle. Find the nature of the locus of the centres of all such circles.



Let the centre of one of the circles be $C(x,\ y)$, and let M be the midpoint of [AB].

Now $CM = x$ and $CA = y$ $\{$radius of circle$\}$

$\therefore\ y^2 = x^2 + 1$ $\{$Pythagoras$\}$

$\therefore$ the locus of C is $y^2 - x^2 = 1$ which is the equation of a rectangular hyperbola

$\therefore$ all centres lie on a rectangular hyperbola.

EXERCISE 2N.4

1 For each hyperbola:

- **i** Find the axes intercepts.
- **ii** Find the foci and the equations of the corresponding directrices.
- **iii** Find the equations of the asymptotes.
- **iv** Sketch the graph.

a $25x^2 - 16y^2 = 400$

b $4y^2 - x^2 = 16$

c $x^2 - y^2 = 4$

d $y^2 - x^2 = 9$

2 Find the equation of the hyperbola with the following properties:

a vertices $(\pm 4,\ 0)$, $e = 1\frac{1}{2}$

b centre O, y-intercept -2, directrix $y = \frac{8}{5}$

c foci $(\pm 12,\ 0)$, directrices $x = \pm\frac{3}{4}$

d vertices $(\pm\frac{4}{\sqrt{3}},\ 0)$, directrices $x = \pm\frac{2}{\sqrt{3}}$

e $|\mathrm{PF} - \mathrm{PF}'| = 2$, foci $(\pm 3,\ 0)$ **f** foci $(0,\ \pm\frac{5}{2})$, directrices $y = \pm\frac{8}{5}$

g asymptotes $y = \pm 2x$, vertices $(\pm 4,\ 0)$

h transverse axis 4 units long on y-axis, $e = \frac{5}{2}$.

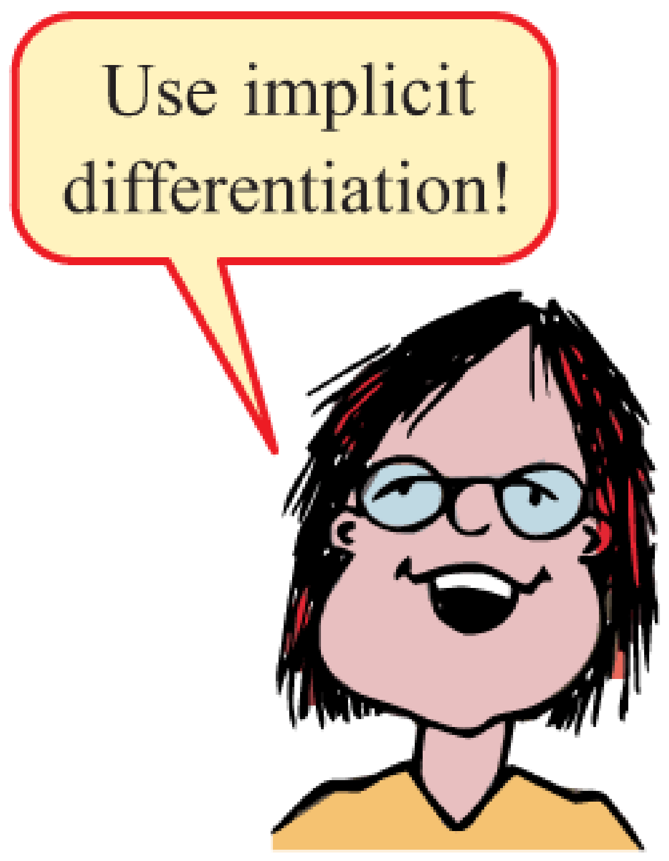
3 For any hyperbola:

a Prove that $|\mathrm{PF} - \mathrm{PF}'| = 2a$.

b Show that b can be interpreted as the shortest distance from a focus to an asymptote.

4 Find the equations of the tangent and normal to $4x^2 - 9y^2 = 36$ at the point:

a $(3\ \ 0)$ **b** $(3\sqrt{2}\ \ -2)$


Use implicit differentiation!

5 Find the equation of the tangent and normal to $x^2 - y^2 = 9$ at the points on the curve where $x = 5$.

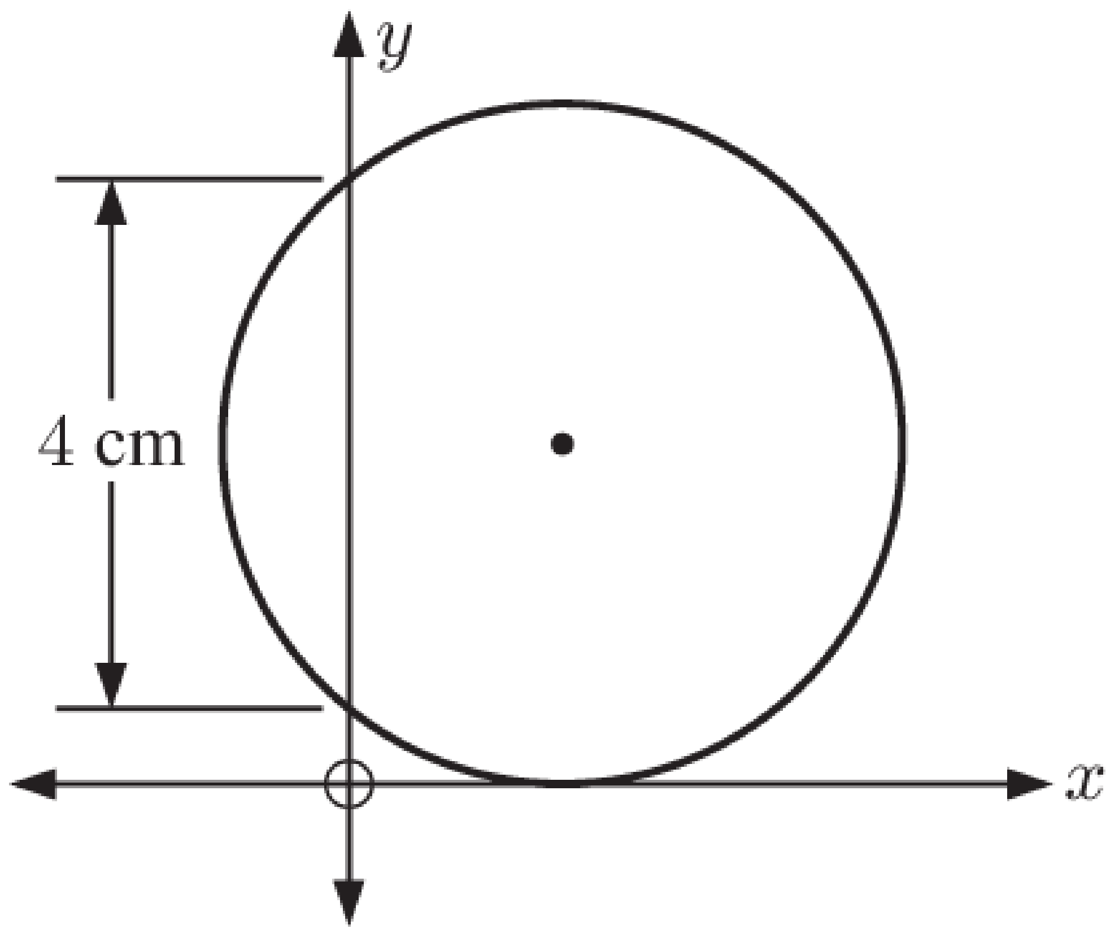
6 Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Suppose $\mathrm{P}(x_1,\ y_1)$ lies on the hyperbola.

a Show that the equation of the normal to the curve at P is $a^2 y_1 x + b^2 x_1 y = (a^2 + b^2) x_1 y_1$.

b Find the equation of the tangent to the curve at P.

c Suppose [PT] is a tangent to the curve and T lies on the asymptote with positive gradient. Show that T has coordinates $\left(\frac{bx_1 + ay_1}{b},\ \frac{bx_1 + ay_1}{a}\right)$.

7



Each circle in a set touches the x-axis, and the y-axis cuts off a chord of length 4 cm from each circle. Find the nature of the locus of the centres of all such circles.

TRANSLATING CONICS

In general, if a conic is translated $\binom{h}{k}$, we replace x by $x - h$, and y by $y - k$ in its equation.

Under the translation $\binom{h}{k}$

- the parabola $y^2 = 4ax$ becomes $(y - k)^2 = 4a(x - h)$
- the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
- the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ becomes $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
- the rectangular hyperbola $xy = c^2$ becomes $(x - h)(y - k) = c^2$.

If a conic has an equation which can be put into one of these forms, we can then sketch it and find details of any foci and directrices.

Example 40

Sketch the ellipse $\dfrac{(x+2)^2}{9}+\dfrac{(y-1)^2}{4}=1$ and give details of its foci and directrices.

$\dfrac{(x+2)^2}{9}+\dfrac{(y-1)^2}{4}=1$ comes from $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ under the translation $\begin{pmatrix}-2\\1\end{pmatrix}$.

Now $a^2=9$ and $b^2=4$.

Since $b^2=a^2(1-e^2)$,

$1-e^2=\frac{4}{9}$

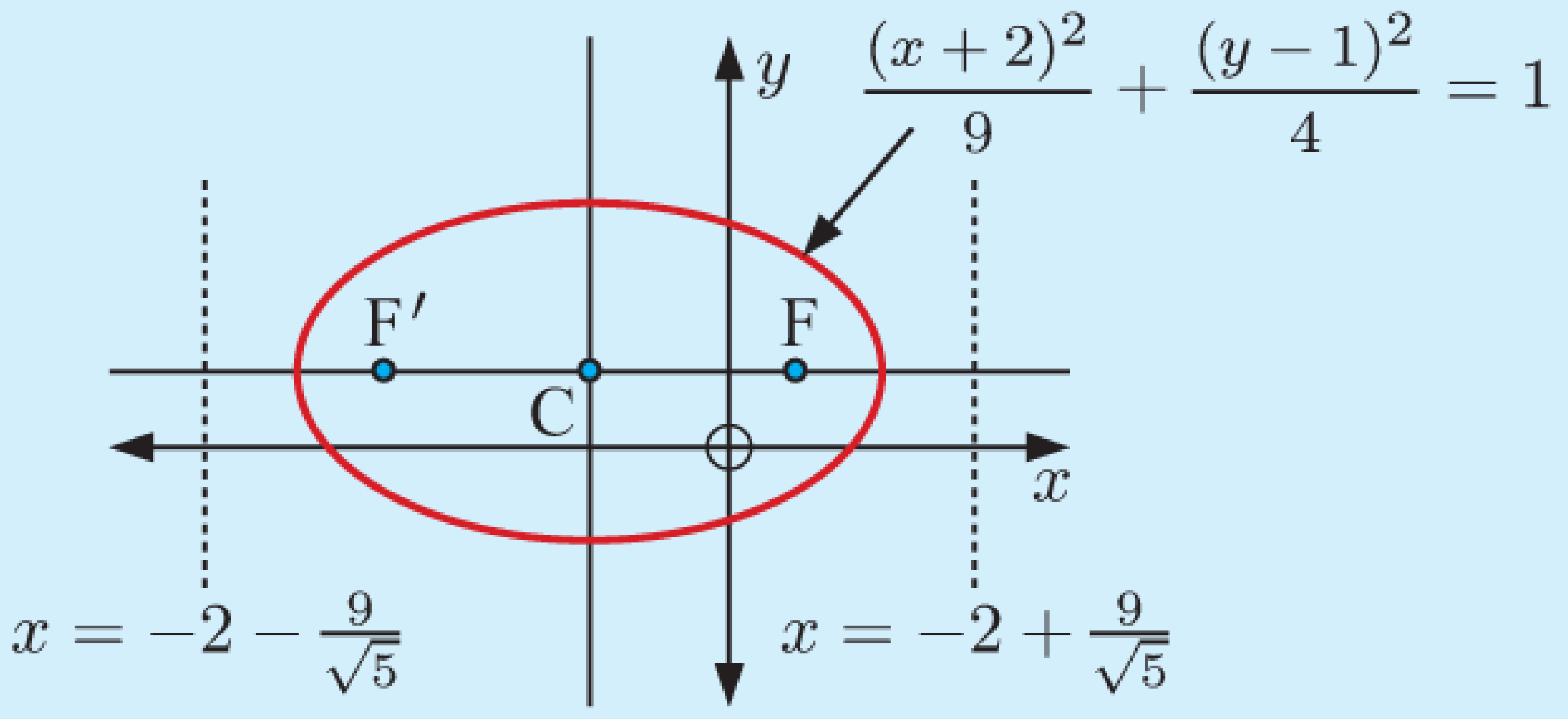
$\therefore\ e^2=\frac{5}{9}$

$\therefore\ e=\frac{\sqrt{5}}{3}$ $\{e>0\}$

Now $ae=\sqrt{5}$ and $\dfrac{a}{e}=\dfrac{9}{\sqrt{5}}$

$\therefore$ $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ has foci $(\pm\sqrt{5},\ 0)$

and directrices $x=\pm\frac{3}{\sqrt{5}}$.


y
$\dfrac{(x+2)^2}{9}+\dfrac{(y-1)^2}{4}=1$
F′
F
C
x
$x=-2-\frac{9}{\sqrt{5}}$
$x=-2+\frac{9}{\sqrt{5}}$

Hence $\dfrac{(x+2)^2}{9}+\dfrac{(y-1)^2}{4}=1$ has foci $(-2\pm\sqrt{5},\ 1)$ and directrices $x=-2\pm\frac{9}{\sqrt{5}}$.

EXERCISE 2N.5

1 Sketch each conic, giving details of any foci and directrices:

a $\dfrac{(x-1)^2}{16}+\dfrac{(y+3)^2}{9}=1$ **b** $(y+4)^2=-8(x+2)$ **c** $(x+2)^2-\dfrac{(y-1)^2}{4}=1$

2 Find the Cartesian equations of the ellipse with:

- **a** foci $(-5,\ 2)$ and $(1,\ 2)$, and eccentricity $\frac{2}{3}$
- **b** focus $(-3,\ 4)$ with corresponding axis extremity $(-5,\ 4)$, and eccentricity $\frac{1}{3}$
- **c** foci $(-1,\ -3)$ and $(5,\ -3)$, and one directrix $x=13$.

3 Find the equation of the hyperbola with:

- **a** centre $(2,\ -1)$, focus $(1,\ -1)$, and eccentricity 2
- **b** focus $(2,\ 3)$ with corresponding directrix $x=-1$, and eccentricity 2
- **c** foci $(2,\ -2)$ and $(6,\ -2)$, where $|\mathrm{PF}-\mathrm{PF}'|=2$.

4 Consider the curve with equation $xy-2x+3y-10=0$.

- **a** Write the equation in the form $(x-h)(y-k)=c^2$.
- **b** Identify the curve and sketch its graph.
- **c** Check the position of the graph by finding the axes intercepts from the original equation.

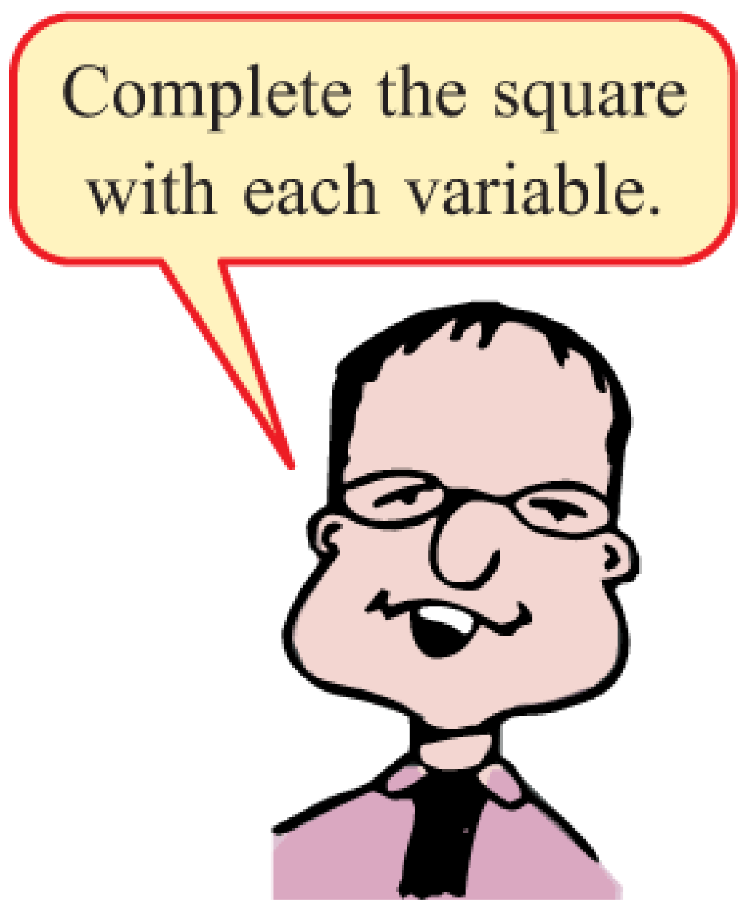
5 Consider the curve with equation $y^2 - 8x + 6y + 22 = 0$.

- **a** Write the equation in the form $(y-k)^2 = 4a(x-h)$.
- **b** Identify the curve and sketch its graph.
- **c** Find the axes intercepts from the original equation.
- **d** Find the coordinates of any foci and the equations of any directrices.

6 For each conic:

- **i** Write the equation in a suitable form so that the curve can be identified and graphed.
- **ii** Sketch the graph of the curve.
- **iii** Find the coordinates of any foci and the equations of any directrices.

- **a** $x^2 + 4y^2 - 6x + 32y + 69 = 0$
- **b** $4x^2 - 9y^2 + 16x + 18y = 9$


Complete the square with each variable.

7 Explain why $3x^2 + y^2 - 6x - 4y + 40 = 0$ does not have a graph.

O PARAMETRIC EQUATIONS

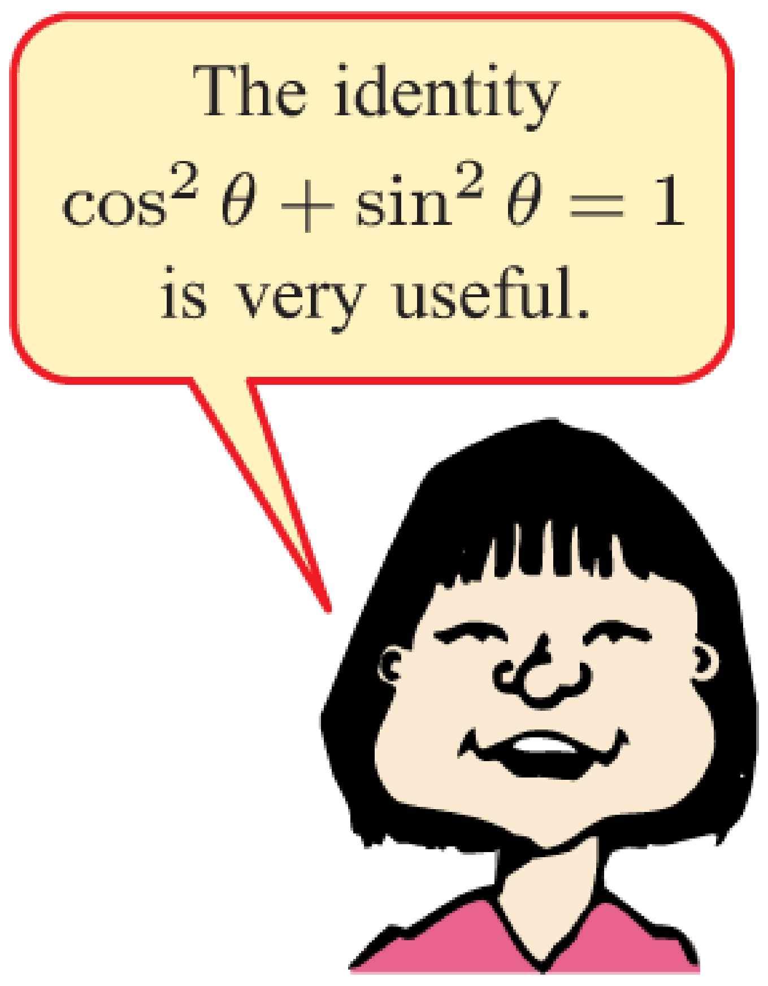
Parametric equations are equations where both x and y are expressed in terms of another variable called the **parameter**. The parameter takes all real values unless otherwise specified, and often represents an angle θ, or a time t.

For example, $x = 2\cos\theta$, $y = 2\sin\theta$ is a parametric representation for the circle $x^2 + y^2 = 4$. The parameter is θ where $\theta \in \mathbb{R}$ is the angle measured anticlockwise from the positive x-axis to the point P(x, y).

However, notice that $x = 2\sin\theta$, $y = 2\cos\theta$ would also be a suitable parametric representation for this locus. In this case θ would have a different meaning.

There may be infinitely many parametric representations for the one Cartesian equation.

For example, for the line with equation $x + y = 8$ we could use $x = t$, $y = 8 - t$ or $y = t$, $x = 8 - t$ or $x = 1 - t$, $y = 7 + t$, and so on.



Example 41

Find the Cartesian equation of the curve with parametric equations:

a $x = \sin\theta$, $y = 2\cos\theta$

b $x = \sin\theta - \cos\theta$, $y = \sin 2\theta$

c $x = 1 + \frac{2}{t}$, $y = 2 - 3t$ where $t \neq 0$.

a $\sin^2\theta + \cos^2\theta = 1$

$$\therefore\ x^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\therefore\ x^2 + \frac{y^2}{4} = 1$$

b We use the identities $\cos^2\theta + \sin^2\theta = 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\begin{aligned}\text{Now}\quad x^2 &= (\sin\theta - \cos\theta)^2\\ &= \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta\\ &= 1 - \sin 2\theta\\ &= 1 - y\end{aligned}$$

$$\therefore\ y = 1 - x^2$$

c Since $x - 1 = \frac{2}{t}$ and $y - 2 = -3t$, we can eliminate t by multiplying.

$$(x-1)(y-2) = \left(\frac{2}{t}\right)(-3t) = -6$$

$$\therefore\ xy - 2x - y + 2 + 6 = 0$$

$$\therefore\ xy - 2x - y + 8 = 0$$

or alternatively, $y - 2 = \frac{-6}{x-1}$

$$\therefore\ y = 2 - \frac{6}{x-1}$$

Example 42

Find a suitable parametric representation for:

a $xy = 5$ **b** $y^2 = 10x$ **c** $\frac{x^2}{4} - \frac{y^2}{9} = 1$

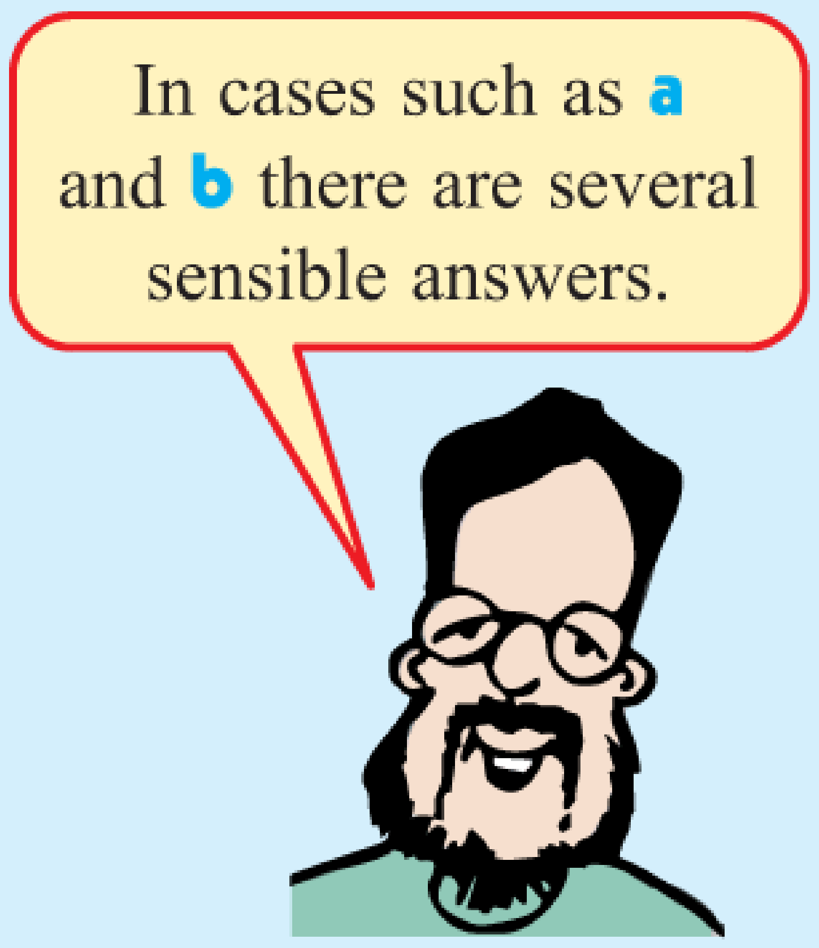
a If $x = t$ then $y = \frac{5}{t}$, $t \neq 0$. **b** If $y = t$ then $x = \frac{t^2}{10}$

c
$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$
$$\therefore \left(\frac{x}{2}\right)^2 - \left(\frac{y}{3}\right)^2 = 1$$
$$\therefore \left(\frac{x}{2}\right)^2 = \left(\frac{y}{3}\right)^2 + 1$$

But $\sec^2\theta = \tan^2\theta + 1$ for all θ.

$\therefore$ we let $\frac{x}{2} = \sec\theta$ and $\frac{y}{3} = \tan\theta$

$\therefore$ $x = 2\sec\theta$, $y = 3\tan\theta$ are the parametric equations.


In cases such as **a** and **b** there are several sensible answers.

PARAMETRIC DIFFERENTIATION

Consider a curve with parametric equations $x = g(t)$, $y = h(t)$.

Using the Chain Rule, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

$$\therefore h'(t) = \frac{dy}{dx}g'(t)$$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ or $\frac{h'(t)}{g'(t)}$ is the gradient of the tangent at any point with parameter t on the curve.

Example 43

A curve has parametric equations $x = t^2 - t$, $y = 2t - 3$.

Find the equation of the tangent to the curve which has gradient $\frac{2}{5}$.

$\frac{dx}{dt} = 2t - 1$ and $\frac{dy}{dt} = 2$

$$\therefore \frac{dy}{dx} = \frac{2}{2t-1}$$

Since the gradient of the tangent is $\frac{2}{5}$, $\frac{2}{2t-1} = \frac{2}{5}$

$$\therefore t = 3$$

When $t = 3$, $x = 6$ and $y = 3$

$\therefore$ $(6, 3)$ is the point of contact.

Thus, the equation of the tangent is $2x - 5y = 2(6) - 5(3)$

which is $2x - 5y = -3$.

EXERCISE 20

1 Find the Cartesian equation of the curve with parametric equations:

- **a** $x = t,\ y = \frac{9}{t}$
- **b** $x = t,\ y = 1 - 5t$
- **c** $x = 1 + 2t,\ y = 3 - t$
- **d** $x = t,\ y = t^2 - 1$
- **e** $x = t^2,\ y = t^3$
- **f** $x = t^2,\ y = 4t$

2 Find the Cartesian equation of the curve with parametric equations:

- **a** $x = 2\cos\theta,\ y = 3\sin\theta$
- **b** $x = 2 + \cos\theta,\ y = \sin\theta$
- **c** $x = \cos\theta,\ y = \cos 2\theta$
- **d** $x = \sin\theta,\ y = \cos 2\theta$
- **e** $x = \tan\theta,\ y = 2\sec\theta$
- **f** $x = \cos\theta,\ y = \sin 2\theta$

3 Find a suitable parametric representation for:

- **a** $x + 4y = 5$
- **b** $xy = -8$
- **c** $y^2 = 9x$
- **d** $x^2 + y^2 = 9$
- **e** $4x^2 + y^2 = 16$
- **f** $x^2 = -4y$
- **g** $3x^2 + 5y^2 = 15$
- **h** $\frac{x^2}{4} = 1 + \frac{y^2}{9}$
- **i** $\frac{x^2}{16} - \frac{y^2}{9} = 1$

4 Consider the curve represented by $x = 2t^2,\ y = t$.

- **a** Where does the line $x + y = 3$ meet the curve?
- **b** Check your answer by first converting the parametric equations into Cartesian form.

5 Find the equation of the tangent to:

- **a** $x = 3t,\ y = t^2 - 3t$ at $t = 2$
- **b** $x = 2\cos\theta,\ y = 5\sin\theta$ at $\theta = \frac{\pi}{4}$
- **c** $x = \sec\theta,\ y = \tan\theta$ at $\theta = \frac{\pi}{3}$.

6 Find the equation of the tangent to:

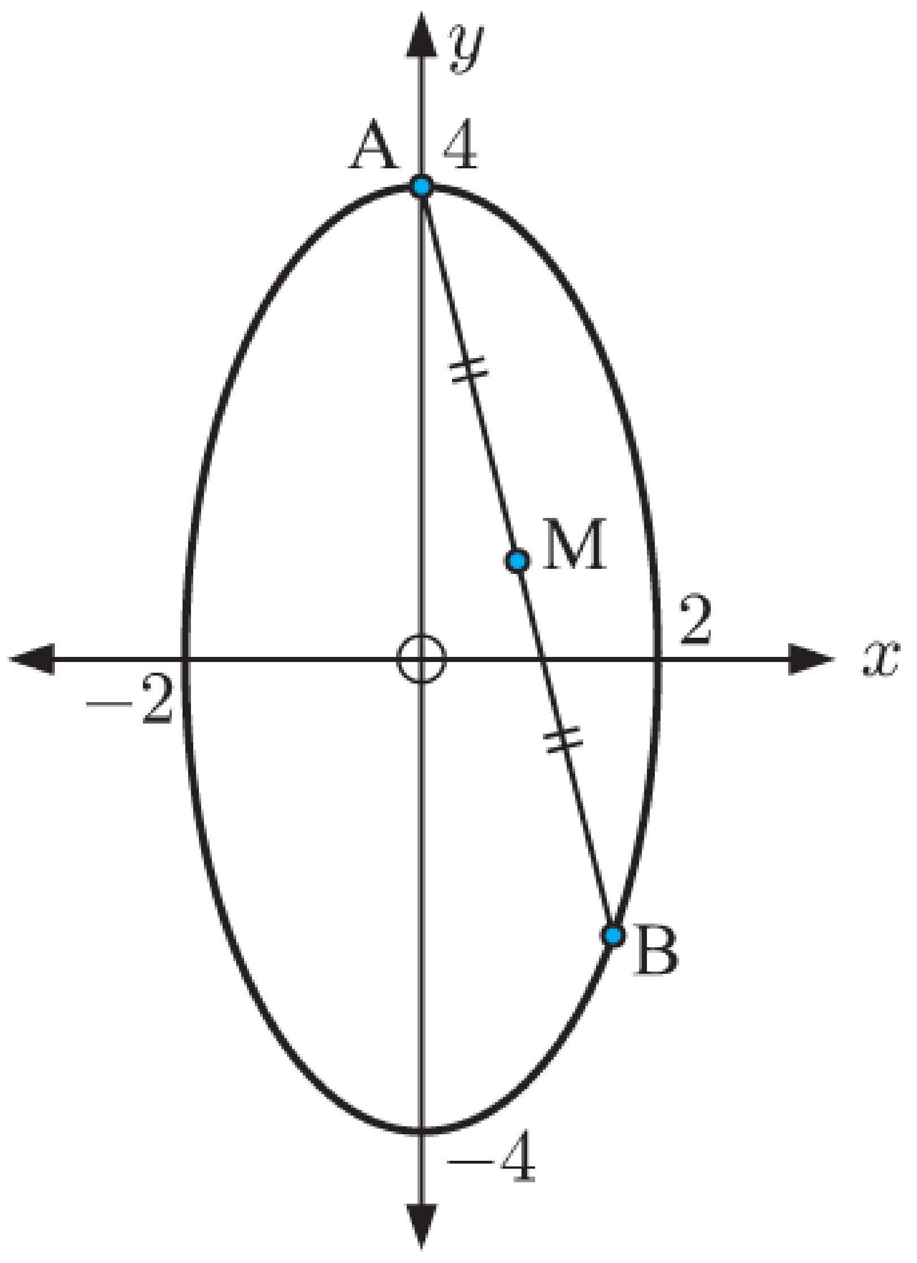
- **a** $x = 1 - t^2,\ y = 4t$ with gradient 4
- **b** $x = 1 - t,\ y = t^3$ passing through $(1, 0)$.

7 Find the coordinates of the points where the line $x + 2y = 3$ meets the curve with parametric equations $x = 1 + \sin\theta,\ y = 1 - \cos\theta$.

8 A curve has parametric equations $x = t + \frac{1}{t},\ y = t - \frac{1}{t},\ t \neq 0$. Find:

- **a** the Cartesian equation of the curve
- **b** the equation of the normal to the curve at the point where $t = 2$.

9



The illustrated ellipse has equation $\frac{x^2}{4} + \frac{y^2}{16} = 1$. A is $(0, 4)$.

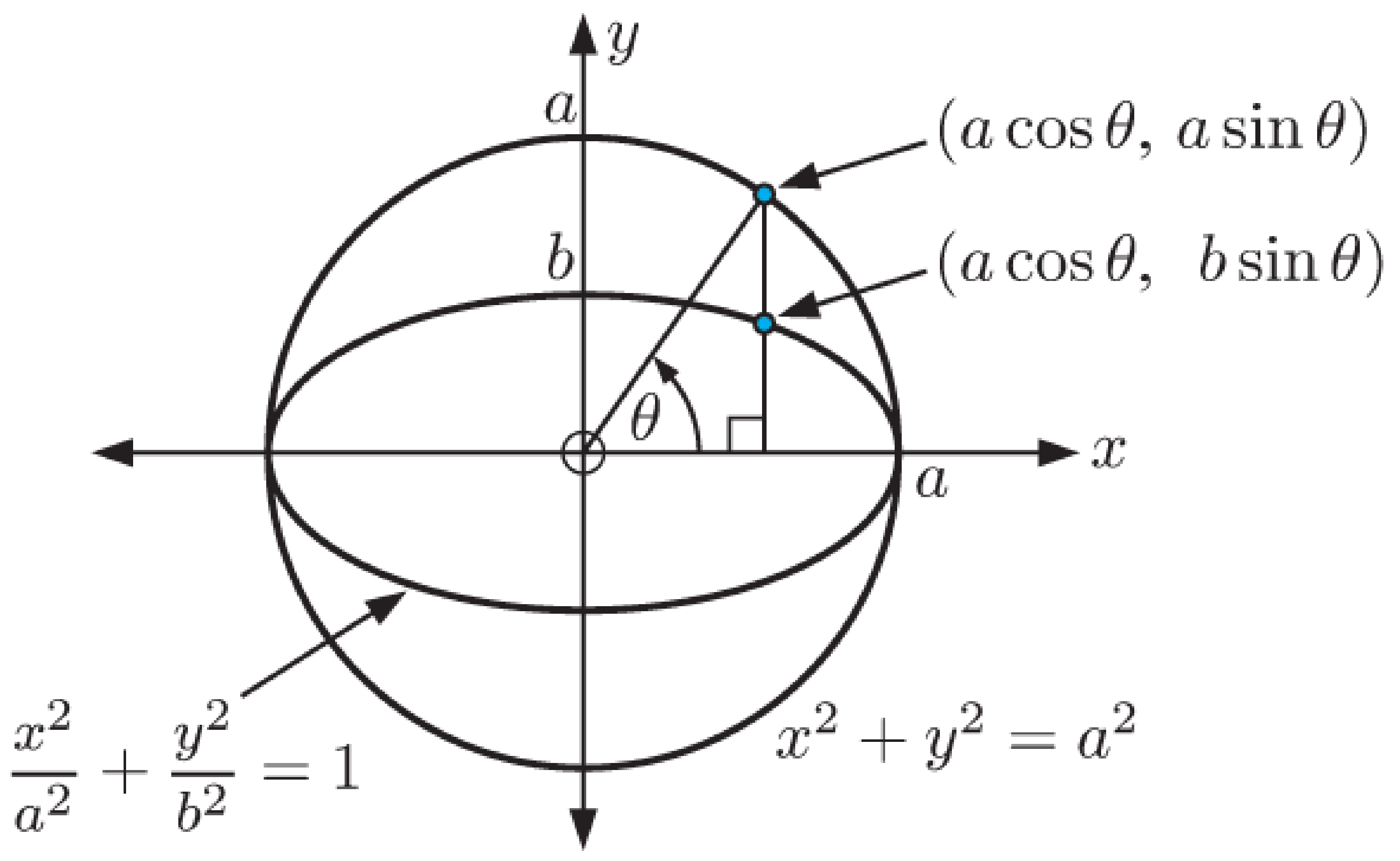
Find the nature of the locus of the midpoints of all chords from A to the ellipse.

Hint: Write the coordinates of B in terms of parameter θ.
Then write the coordinates of M in terms of θ.

P PARAMETRIC EQUATIONS FOR CONICS

The **standard parametric equations** for the non-degenerate conics are:

- $\boldsymbol{x = a\cos\theta,\ \ y = a\sin\theta}$ for the **circle** $x^2 + y^2 = a^2$
- $\boldsymbol{x = at^2,\ \ y = 2at}$ for the **parabola** $y^2 = 4ax$
- $\boldsymbol{x = a\cos\theta,\ \ y = b\sin\theta}$ for the **ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
- $\boldsymbol{x = ct,\ \ y = \dfrac{c}{t}},\ \ t \neq 0$ for the **rectangular hyperbola** $xy = c^2$
- $\boldsymbol{x = a\sec\theta,\ \ y = b\tan\theta}$ for the **hyperbola** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$


y
a
$(a\cos\theta,\ a\sin\theta)$
b
$(a\cos\theta,\ \ b\sin\theta)$
θ
a
x
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
$x^2 + y^2 = a^2$

The parameter θ is called the **eccentric angle**.

$x^2 + y^2 = a^2$ is called the **auxiliary circle**.

SUMMARY OF TANGENTS AND NORMALS TO CONICS

In the following **Exercise** we will prove these equations of tangents and normals for the non-degenerate conics:

	Tangent	Normal
Circle $x^2 + y^2 = a^2$ at $(a\cos\theta,\ a\sin\theta)$	$(\cos\theta)x + (\sin\theta)y = a$	$y = (\tan\theta)x$
Parabola $y^2 = 4ax$ at $(at^2,\ 2at)$	$x - ty = -at^2$	$tx + y = at^3 + 2at$
Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at $(a\cos\theta,\ \ b\sin\theta)$	$(b\cos\theta)x + (a\sin\theta)y = ab$	$(a\sin\theta)x - (b\cos\theta)y = (a^2 - b^2)\sin\theta\cos\theta$
Rectangular hyperbola $xy = c^2$ at $\left(ct,\ \dfrac{c}{t}\right)$	$x + t^2y = 2ct$	$t^2x - y = ct^3 - \dfrac{c}{t}$
Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at $(a\sec\theta,\ b\tan\theta)$	$bx - (a\sin\theta)y = ab\cos\theta$	$(a\sin\theta)x + by = (a^2 + b^2)\tan\theta$

Example 44

Find the equation of the normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\theta,\ b\tan\theta)$.

Let $x = a\sec\theta$ and $y = b\tan\theta$

$\therefore\ \frac{dx}{d\theta} = a\sec\theta\tan\theta$ and $\frac{dy}{d\theta} = b\sec^2\theta$

$$\begin{aligned}\therefore\ \frac{dy}{dx} &= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}\\ &= \frac{b\sec\theta}{a\tan\theta}\\ &= \frac{b}{a}\left(\frac{1}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)\\ &= \frac{b}{a\sin\theta}\end{aligned}$$

$\therefore$ the gradient of the normal is $-\frac{a\sin\theta}{b}$

$\therefore$ the equation of the normal is
$$\begin{aligned}(a\sin\theta)x + by &= (a\sin\theta)(a\sec\theta) + b(b\tan\theta)\\ &= a^2\sin\theta\left(\frac{1}{\cos\theta}\right) + b^2\tan\theta\\ &= a^2\tan\theta + b^2\tan\theta\\ &= \tan\theta(a^2+b^2)\end{aligned}$$

Thus, $(a\sin\theta)x + by = (a^2+b^2)\tan\theta$.

EXERCISE 2P

1 Suppose $P(a\cos\theta,\ a\sin\theta)$ is any point on the circle $x^2+y^2=a^2$. Show that:

- **a** the equation of the tangent at P is $(\cos\theta)x + (\sin\theta)y = a$
- **b** the equation of the normal at P is $y = (\tan\theta)x$.

2 Find the equations of the tangent and normal to the circle $x^2+y^2=9$ at the point where the eccentric angle is $\frac{\pi}{3}$.

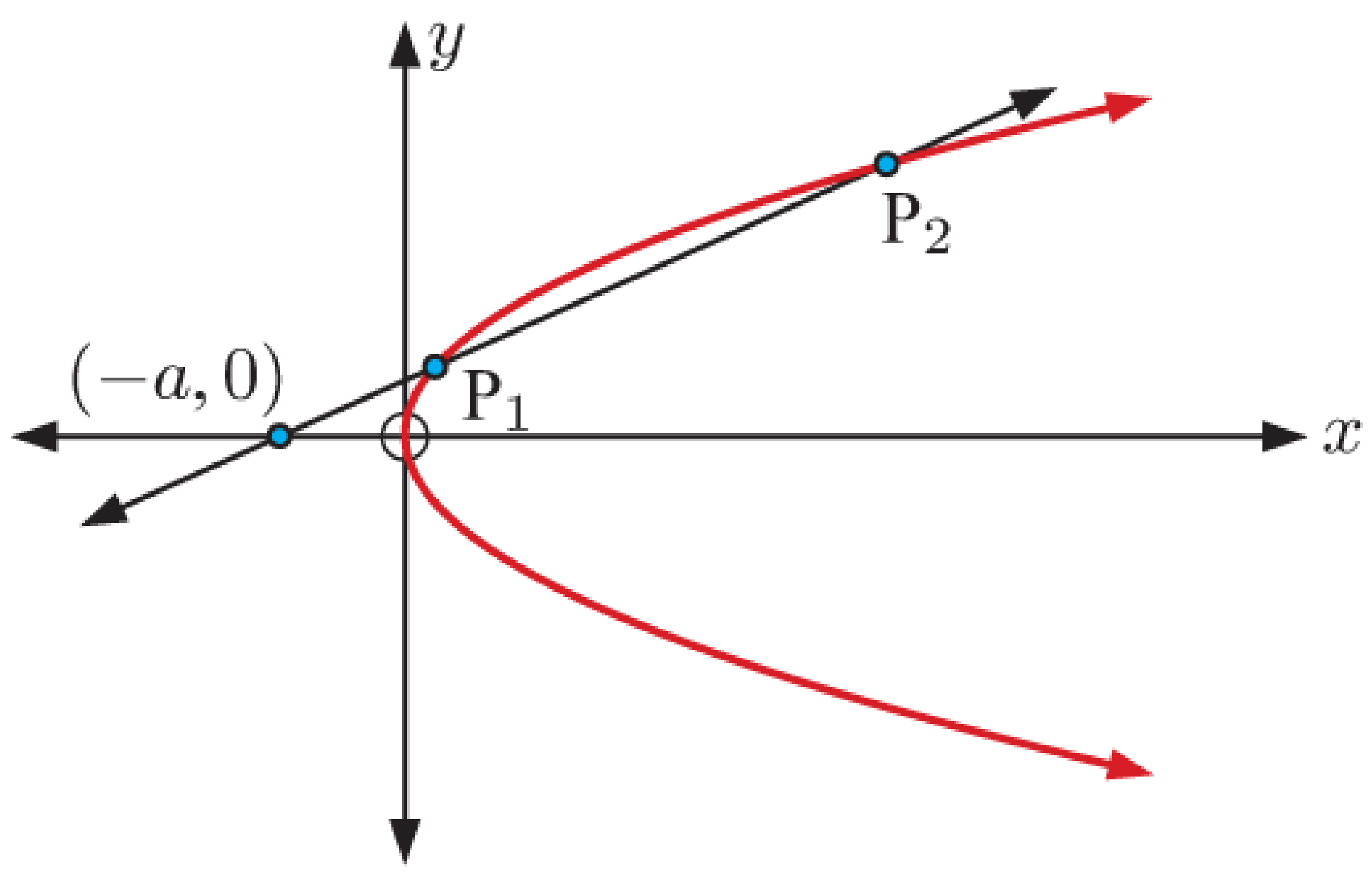
3 Suppose $P(at^2,\ 2at)$ is any point on the parabola $y^2=4ax$. Show that:

- **a** the equation of the tangent at P is $x - ty = -at^2$
- **b** the equation of the normal at P is $tx + y = at^3 + 2at$.

4 $A(at_1^2,\ 2at_1)$ and $B(at_2^2,\ 2at_2)$ lie on the parabola $y^2=4ax$.

- **a** Show that the chord [AB] has equation $2x - (t_1+t_2)y = -2at_1t_2$.
- **b** If [AB] is a focal chord, prove that $t_1t_2 = -1$.
- **c** Hence prove that the tangents at the extremities of the focal chord always intersect at right angles on the directrix.
- **d** If the tangents at the ends of a focal chord meet the y-axis at C and D, prove that [CD] subtends a right angle at the focus.
- **e** Determine the nature of the locus of the midpoints of the focal chords of $y^2=4ax$.

5

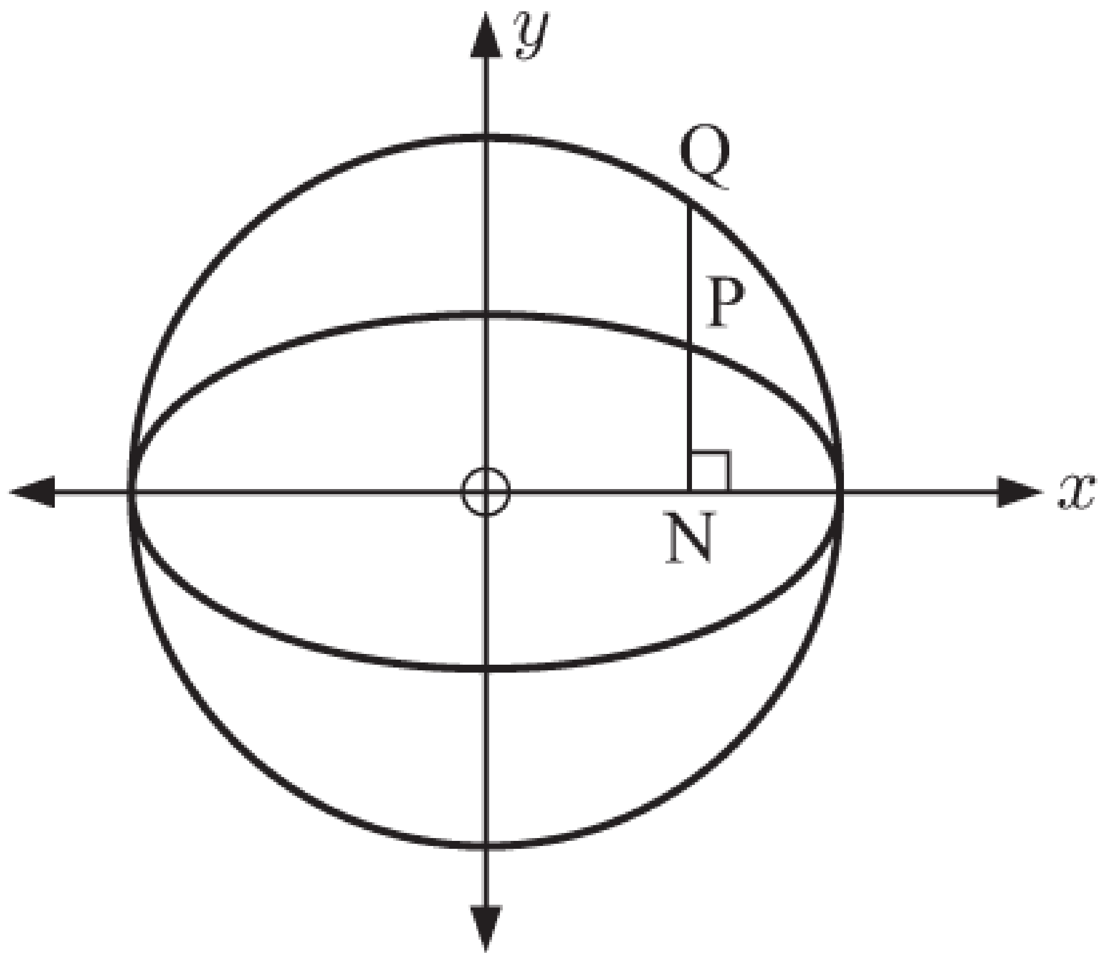


A chord of $y^2 = 4ax$ from $(-a, 0)$ cuts the parabola at P_1 and P_2.
Find the locus of the points of intersection of the tangents at P_1 and P_2.

6 Suppose $P(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that:

- **a** the equation of the tangent at P is $(b\cos\theta)x + (a\sin\theta)y = ab$
- **b** the equation of the normal at P is $(a\sin\theta)x - (b\cos\theta)y = (a^2 - b^2)\sin\theta\cos\theta$.

7



Suppose P is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
[NQ] is a vertical line which passes through P, where Q lies on the auxiliary circle $x^2 + y^2 = a^2$.
Prove that:

- **a** PN : QN $= b : a$ for all positions of N
- **b** the tangents at P and Q meet on the x-axis provided N is *not* at $(0, 0)$.

8 Suppose $P(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

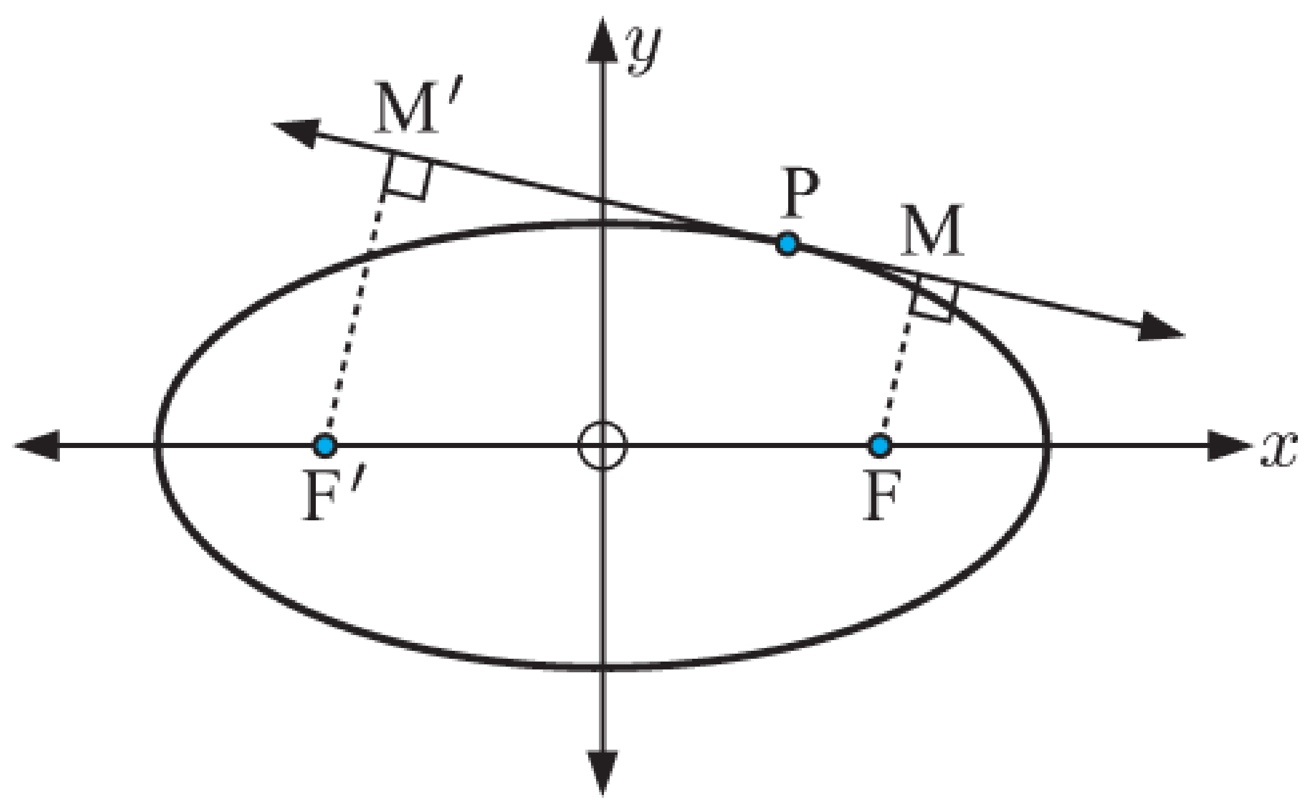
- **a** Show that the normal at P cuts the x-axis at $Q(ae^2\cos\theta, 0)$.
- **b** Prove that $PF = a(1 - e\cos\theta)$ and $PF' = a(1 + e\cos\theta)$ where F and F′ are the foci.
- **c** Hence, prove that the normal [PQ] bisects $F\widehat{P}F'$.
- **d** What is the significance of the result in **c**?

9 The tangent at $P(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the directrix at Q.
Prove that $P\widehat{F}Q$ is a right angle.

10 A tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axes at A and B. Find the equation of the locus of the midpoint of [AB].

11 Prove that the foot of the perpendicular from a focus to a variable tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lies on the auxiliary circle $x^2 + y^2 = a^2$.

12 Find the equation of the locus of the foot of the perpendicular from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent.

13 

A tangent is drawn at $(a\cos\theta,\, b\sin\theta)$ to the ellipse with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The ellipse has foci F and F′. M and M′ are the feet of the perpendiculars from F and F′ to the tangent, respectively. Prove that $\text{MF.MF}'=b^2$ for all tangents.

14 Prove that the equation of the tangent to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at the point $(a\sec\theta,\, b\tan\theta)$ has equation $bx-(a\sin\theta)y=ab\cos\theta$.

15 Suppose the normal to $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ cuts the axes at A and B, and that M is the midpoint of [AB]. Find the equation of the locus of M.

16 Suppose the tangent to $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at the point $\text{P}(a\sec\theta,\, b\tan\theta)$ meets the x-axis at Q.

a Find the coordinates of Q.

b If F and F′ are the foci of the hyperbola, prove that $\dfrac{\text{PF}'}{\text{PF}}=\dfrac{\text{QF}'}{\text{QF}}=\dfrac{e+\cos\theta}{e-\cos\theta}$.

c What can be deduced from **b**?

17 Suppose $\text{P}\left(ct,\, \dfrac{c}{t}\right)$ is any point on the rectangular hyperbola $xy=c^2$. Show that:

a the equation of the tangent at P is $x+t^2y=2ct$

b the equation of the normal at P is $t^2x-y=ct^3-\dfrac{c}{t}$.

18 A tangent to the rectangular hyperbola $xy=c^2$ meets the asymptotes at L and M. Find the coordinates of the midpoint of [LM]. Comment on your answer.

19 Consider the rectangular hyperbola $xy=c^2$. Find the coordinates of the foci and the equations of the corresponding directrices.

20 Find the equation of the locus of the foot of the perpendicular from O to a tangent of $xy=c^2$.

21 The normal to $xy=c^2$ cuts the x-axis at A and the y-axis at B. Find the equation of the locus of the midpoint of [AB].

22 Consider the part of the rectangular hyperbola $xy=c^2$ in the first quadrant.
Suppose P_1 and P_2 are any two distinct points on the curve which are extremities of a focal chord passing through focus F. Suppose Q is the point of intersection of the tangents at P_1 and P_2. Prove that the locus of Q is a straight line.

Q THE GENERAL CONIC EQUATION

The most general form for the equation of a conic section is $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$.

For the non-degenerate conics (circle, ellipse, parabola, hyperbola) we require that:

1. at least one of a, b, and c is non-zero
2. at least one of d, e, and f is non-zero.

For example:

- If $a = 1$, $b = 0$, $c = 1$, $d = 2$, $e = -4$, and $f = -5$ then we have the circle $x^2 + y^2 + 2x - 4y - 5 = 0$.
- If $a = c = 0$, $b = 1$, $d = 0$, $e = 0$, and $f = -6$ then we have the rectangular hyperbola $xy = 3$.
- If $a = 4$, $b = 0$, $c = 9$, $d = e = 0$, and $f = -36$ then we have the ellipse $4x^2 + 9y^2 = 36$ or $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

INVESTIGATION 4 GRAPHS OF CONIC SECTIONS

In this Investigation we explore the graphs of conic sections in the general form $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ where a, b, c, d, e, and f are constants.

PRINTABLE TABLE **GRAPHING PACKAGE**

What to do:

1. Use the **graphing package** to graph each conic. Hence copy and complete the table, identifying the type of each conic and whether axes of symmetry are parallel to the coordinate axes.

	Conic	*Type of conic*	*Axes of symmetry*
a	$x^2 - y^2 + 8x + 16 = 0$		
b	$x^2 - xy - 2y^2 + 5x - y + 6 = 0$		
c	$x^2 - xy - 2y^2 + 4x - 2y + 4 = 0$		
d	$xy = 4$		
e	$xy + 3x - 2y + 8 = 0$		
f	$xy + x + 2y + 6 = 0$		
g	$x^2 + y^2 - 6x + 10y + 18 = 0$		
h	$x^2 + y^2 - 2x - 6y + 6 = 0$		
i	$x^2 + y^2 - 6y + 9 = 0$		
j	$x^2 + y^2 + 4x - 2y + 6 = 0$		
k	$y^2 - 8x - 6y - 7 = 0$		
l	$3x^2 + 6xy + 3y^2 + 16y = 0$		
m	$9x^2 - 18xy + 9y^2 + 2x - 4y - 10 = 0$		

Conic	*Type of conic*	*Axes of symmetry*
n $4x^2 + y^2 - 8x + 6y + 4 = 0$		
o $x^2 - xy + y^2 - 4 = 0$		
p $7x^2 + 2xy + 5y^2 + 56x - 20y + 80 = 0$		
q $x^2 - 4y^2 - 3x + 5y + 4 = 0$		
r $4x^2 - 10xy + 3y^2 + 8x + 16y = 0$		
s $-8x^2 - 15xy + 6y^2 + 2x - 4y + 10 = 0$		

2 What feature(s) of the general conic equation determine:

- **a** what type of conic is produced
- **b** whether any axes of symmetry are parallel to the coordinate axes?

From the **Investigation** you should have noticed that when an xy-term is present in the equation of a conic, any axes of symmetry are not parallel to the coordinate axes. These conics can be considered as translations followed by rotations of standard conics.

THE QUADRATIC FORM

If $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, then

$$\begin{aligned} \mathbf{x}^T\mathbf{A}\mathbf{x} &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} ax+by & bx+cy \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= ax^2 + bxy + bxy + cy^2 \\ &= ax^2 + 2bxy + cy^2 \end{aligned}$$

If $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} d \\ e \end{pmatrix}$, then the general conic $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ becomes $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{v}^T\mathbf{x} + f = 0$.

We can use orthogonal diagonalisation to transform a general conic into standard form. This allows us to recognise the conic as an ellipse, hyperbola, or parabola.

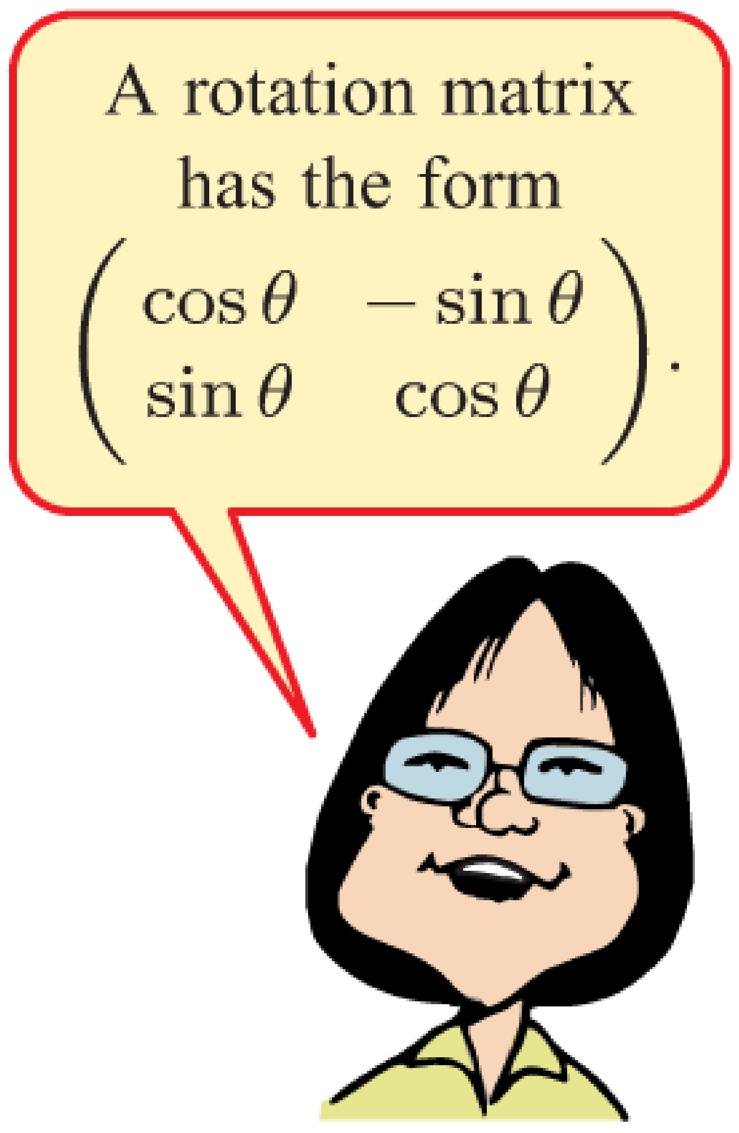
Step 1: Rewrite $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ in the form $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{v}^T\mathbf{x} = -f$ where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} d \\ e \end{pmatrix}$.

Step 2: Find the eigenvalues and corresponding eigenvectors of $\mathbf{A}$.

Step 3: Use the normalised eigenvectors as the columns of a rotation matrix $\mathbf{P}$ such that $\det \mathbf{P} = 1$.

Step 4: Let $\mathbf{x} = \mathbf{P}\mathbf{x}'$ where $\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ to obtain the standard conic equation.

Remember that $\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ since $\mathbf{P}$ is orthogonal.


A rotation matrix has the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Example 45

Using a suitable rotation, rewrite $5x^2+4xy+5y^2=21$ in standard form. Identify the conic and sketch its graph.

$5x^2+4xy+5y^2=21$ can be written as $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=21$

which has the form $\mathbf{x}^T\mathbf{A}\mathbf{x}=21$

Now $|\,\lambda\mathbf{I}-\mathbf{A}\,| = \begin{vmatrix} \lambda-5 & -2 \\ -2 & \lambda-5 \end{vmatrix} = \lambda^2-10\lambda+21=(\lambda-3)(\lambda-7)$

$\therefore\ \lambda=3,\ 7$ are the eigenvalues of $\mathbf{A}$.

When $\lambda=3$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$

$$\therefore\ \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ x+y=0$$

$$\therefore\ \mathbf{x}=\begin{pmatrix} -1 \\ 1 \end{pmatrix}t,\ \ t\in\mathbb{R}$$

When $\lambda=7$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$

$$\therefore\ \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore\ x-y=0$$

$$\therefore\ \mathbf{x}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}t,\ \ t\in\mathbb{R}$$

The normalised eigenvectors are $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ {length 1}

Let $\mathbf{P}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ where each column is a normalised eigenvector and

$\det\mathbf{P}=\left(\frac{1}{\sqrt{2}}\right)^2\times 2=1$ for a rotation. From the column order we have $\lambda_1=7$ and $\lambda_2=3$.

$\cos\theta=\frac{1}{\sqrt{2}}$ and $\sin\theta=\frac{1}{\sqrt{2}}$, so $\mathbf{P}$ corresponds to a rotation of $\frac{\pi}{4}$.

Let $\mathbf{x}=\mathbf{P}\mathbf{x}'$.

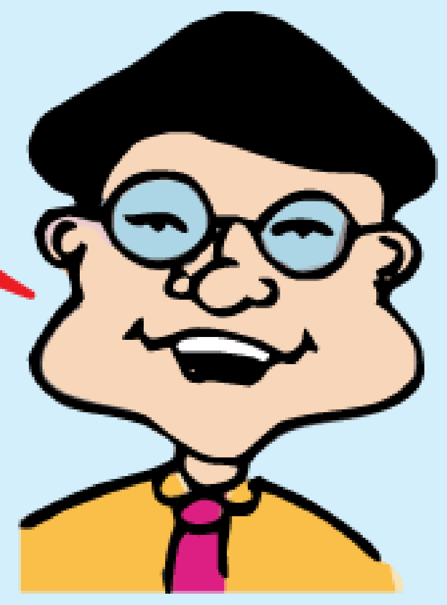
$$\therefore\ (\mathbf{P}\mathbf{x}')^T\mathbf{A}(\mathbf{P}\mathbf{x}')=21$$

$$\therefore\ \mathbf{x}'^{\,T}\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{x}'=21$$

$$\therefore\ \mathbf{x}'^{\,T}\begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{x}'=21$$

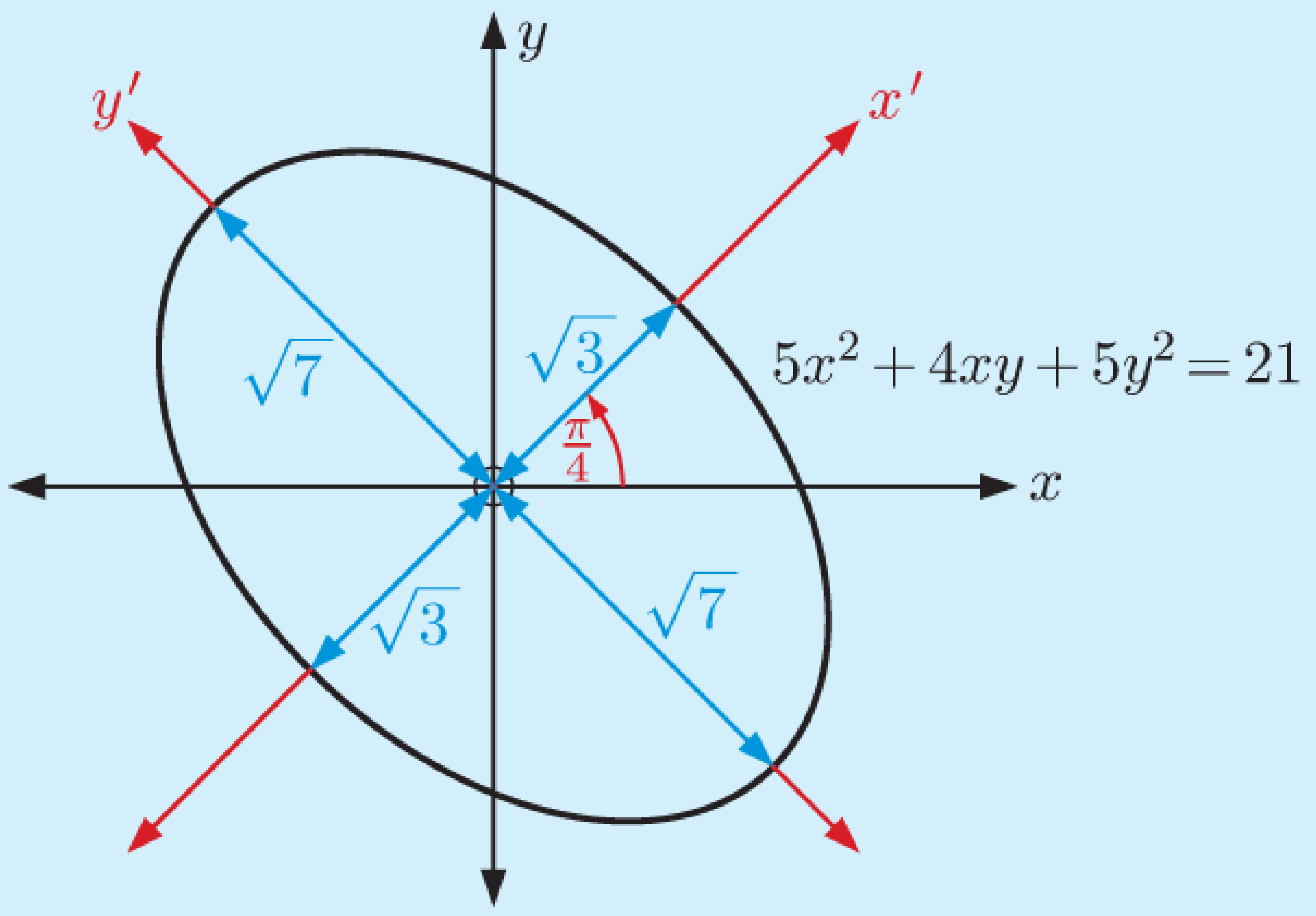
$$\therefore\ 7x'^{\,2}+3y'^{\,2}=21$$

$$\therefore\ \frac{x'^{\,2}}{3}+\frac{y'^{\,2}}{7}=1$$



Notice that $\mathbf{P}$ orthogonally diagonalises $\mathbf{A}$, $\mathbf{P}^T\mathbf{A}\mathbf{P}=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

$\therefore$ the conic is an ellipse with centre $(0,\ 0)$, rotated through $\frac{\pi}{4}$.



Example 46

Using a suitable rotation, rewrite $2x^2-4xy-y^2=-8$ in standard form. Identify the conic and graph it from its asymptotes and intercepts.

$2x^2-4xy-y^2=-8$ can be written as $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=-8$

which has the form $\mathbf{x}^T\mathbf{A}\mathbf{x}=-8$

Now $|\,\lambda\mathbf{I}-\mathbf{A}\,|=\begin{vmatrix} \lambda-2 & 2 \\ 2 & \lambda+1 \end{vmatrix}=\lambda^2-\lambda-6=(\lambda-3)(\lambda+2)$

$\therefore$ $\lambda=3,\ -2$ are the eigenvalues of $\mathbf{A}$.

When $\lambda=3$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$

$$\therefore \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore x+2y=0$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -2 \\ 1 \end{pmatrix}t,\quad t\in\mathbb{R}$$

When $\lambda=-2$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$

$$\therefore \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 2x-y=0$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}t,\quad t\in\mathbb{R}$$

The normalised eigenvectors are $\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ {length is 1}

Let $\mathbf{P}=\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ where each column is a normalised eigenvector and $\det\mathbf{P}=1$ for a rotation. $\lambda_1=-2$ and $\lambda_2=3$. $\cos\theta=\frac{1}{\sqrt{5}}$ and $\sin\theta=\frac{2}{\sqrt{5}}$, so $\tan\theta=2$.

Let $\mathbf{x}=\mathbf{P}\mathbf{x}'$.

$$\therefore (\mathbf{P}\mathbf{x}')^T\mathbf{A}(\mathbf{P}\mathbf{x}')=-8$$

$$\therefore \mathbf{x}'^T\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{x}'=-8$$

$$\therefore \mathbf{x}'^T\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{x}'=-8$$

$$\therefore -2x'^2+3y'^2=-8$$

$$\therefore 2x'^2-3y'^2=8$$

$$\therefore \frac{x'^2}{4}-\frac{y'^2}{\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)^2}=1$$ which is an hyperbola.

The graph cuts the x'-axis when $y'=0$

$\therefore$ $x'^2=4$, $x'=\pm2$

$\therefore$ the vertices for the rotated graph are

$\left(\frac{2}{\sqrt{5}},\frac{4}{\sqrt{5}}\right)$ and $\left(-\frac{2}{\sqrt{5}},-\frac{4}{\sqrt{5}}\right)$.

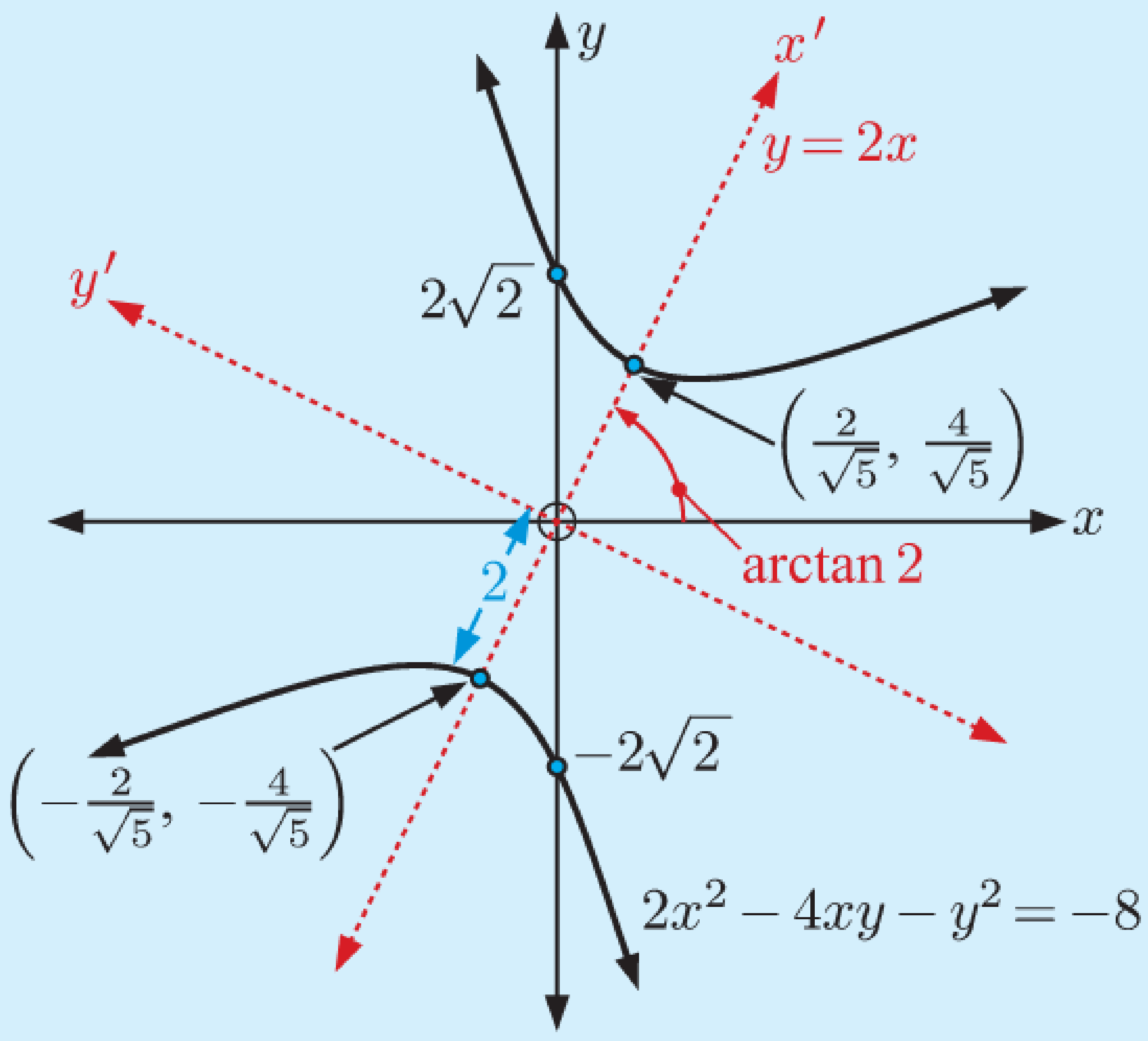
The graph cuts the x-axis when $y=0$

$\therefore$ $x^2=-4$ which is never.

The graph cuts the y-axis when $x=0$

$\therefore$ $y^2=8$

$\therefore$ $y=\pm2\sqrt{2}$



y
x'
$y=2x$
y'
$2\sqrt{2}$
$\left(\frac{2}{\sqrt{5}},\frac{4}{\sqrt{5}}\right)$
x
arctan 2
2
$-2\sqrt{2}$
$\left(-\frac{2}{\sqrt{5}},-\frac{4}{\sqrt{5}}\right)$
$2x^2-4xy-y^2=-8$

Example 47

Identify and sketch the conic $12x^2 - 7xy - 12y^2 - 30x + 40y = -50$.

To avoid working with fractions, consider $24x^2 - 14xy - 24y^2 - 60x + 80y = -100$.

This equation can be written in the form $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{v}^T\mathbf{x} = -100$ where
$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 24 & -7 \\ -7 & -24 \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} -60 \\ 80 \end{pmatrix}$.

Now $|\,\lambda\mathbf{I} - \mathbf{A}\,| = \begin{vmatrix} \lambda - 24 & 7 \\ 7 & \lambda + 24 \end{vmatrix} = \lambda^2 - 576 - 49 = \lambda^2 - 625 = (\lambda + 25)(\lambda - 25)$.

$\therefore$ $\lambda = 25, -25$ are the eigenvalues of $\mathbf{A}$.

When $\lambda = 25$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \begin{pmatrix} 1 & 7 \\ 7 & 49 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore x + 7y = 0$$
$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}t, \quad t \in \mathbb{R}$$

When $\lambda = -25$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \begin{pmatrix} -49 & 7 \\ 7 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore 7x - y = 0$$
$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}t, \quad t \in \mathbb{R}$$

The normalised eigenvectors are $\frac{1}{\sqrt{50}}\begin{pmatrix} -7 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{50}}\begin{pmatrix} 1 \\ 7 \end{pmatrix}$

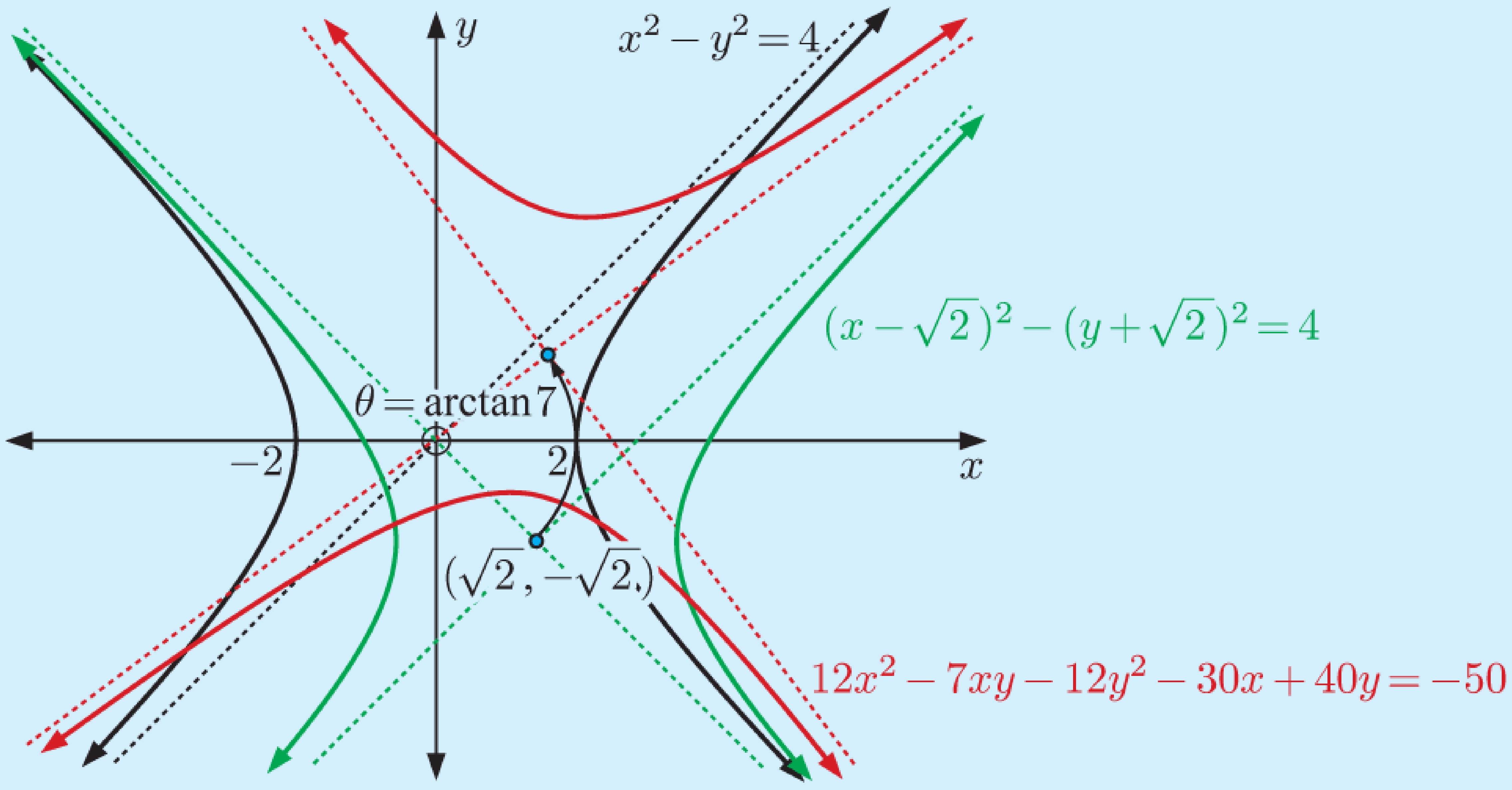
Let $\mathbf{P} = \frac{1}{\sqrt{50}}\begin{pmatrix} 1 & -7 \\ 7 & 1 \end{pmatrix}$ where $\det\mathbf{P} = 1$. $\lambda_1 = -25$ and $\lambda_2 = 25$. $\cos\theta = \frac{1}{\sqrt{50}}$ and $\sin\theta = \frac{7}{\sqrt{50}}$, so $\tan\theta = 7$.

Let $\mathbf{x} = \mathbf{P}\mathbf{x}'$.

$$\therefore (\mathbf{P}\mathbf{x}')^T\mathbf{A}(\mathbf{P}\mathbf{x}') + \mathbf{v}^T\mathbf{P}\mathbf{x}' = -100$$
$$\therefore \mathbf{x}'^T\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{x}' + \begin{pmatrix} -60 & 80 \end{pmatrix}\frac{1}{\sqrt{50}}\begin{pmatrix} 1 & -7 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = -100$$
$$\therefore \mathbf{x}'^T\begin{pmatrix} -25 & 0 \\ 0 & 25 \end{pmatrix}\mathbf{x}' + \frac{1}{5\sqrt{2}}\begin{pmatrix} 500 & 500 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = -100$$
$$\therefore -25x'^2 + 25y'^2 + \begin{pmatrix} 50\sqrt{2} & 50\sqrt{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = -100$$
$$\therefore -25x'^2 + 25y'^2 + 50\sqrt{2}x' + 50\sqrt{2}y' = -100$$
$$\therefore 25x'^2 - 25y'^2 - 50\sqrt{2}x' - 50\sqrt{2}y' = 100$$
$$\therefore x'^2 - y'^2 - 2\sqrt{2}x' - 2\sqrt{2}y' = 4$$
$$\therefore (x'^2 - 2\sqrt{2}x' + 2) - (y'^2 + 2\sqrt{2}y' + 2) = 4 + 2 - 2$$
$$\therefore (x' - \sqrt{2})^2 - (y' + \sqrt{2})^2 = 4$$

which is $X^2 - Y^2 = 4$ translated $\begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}$.

$\therefore$ the original conic is a rectangular hyperbola.



ANIMATION

EXERCISE 2Q

1 Using a suitable rotation, rewrite the conic:

a $6x^2-4xy+9y^2=80$ **b** $8x^2+28xy-13y^2+40=0$ in standard form.

Hence, identify the conic and sketch its graph.

2 Write each conic in the form $A(x'-h)^2+B(y'-k)^2+f=0$. Hence, identify and sketch the conic.

a $x^2-xy+y^2-2x+y-3=0$

b $x^2+4xy-2y^2+2\sqrt{5}x-\sqrt{5}y-5=0$

c $3x^2-6xy-5y^2+3x+9y=10$

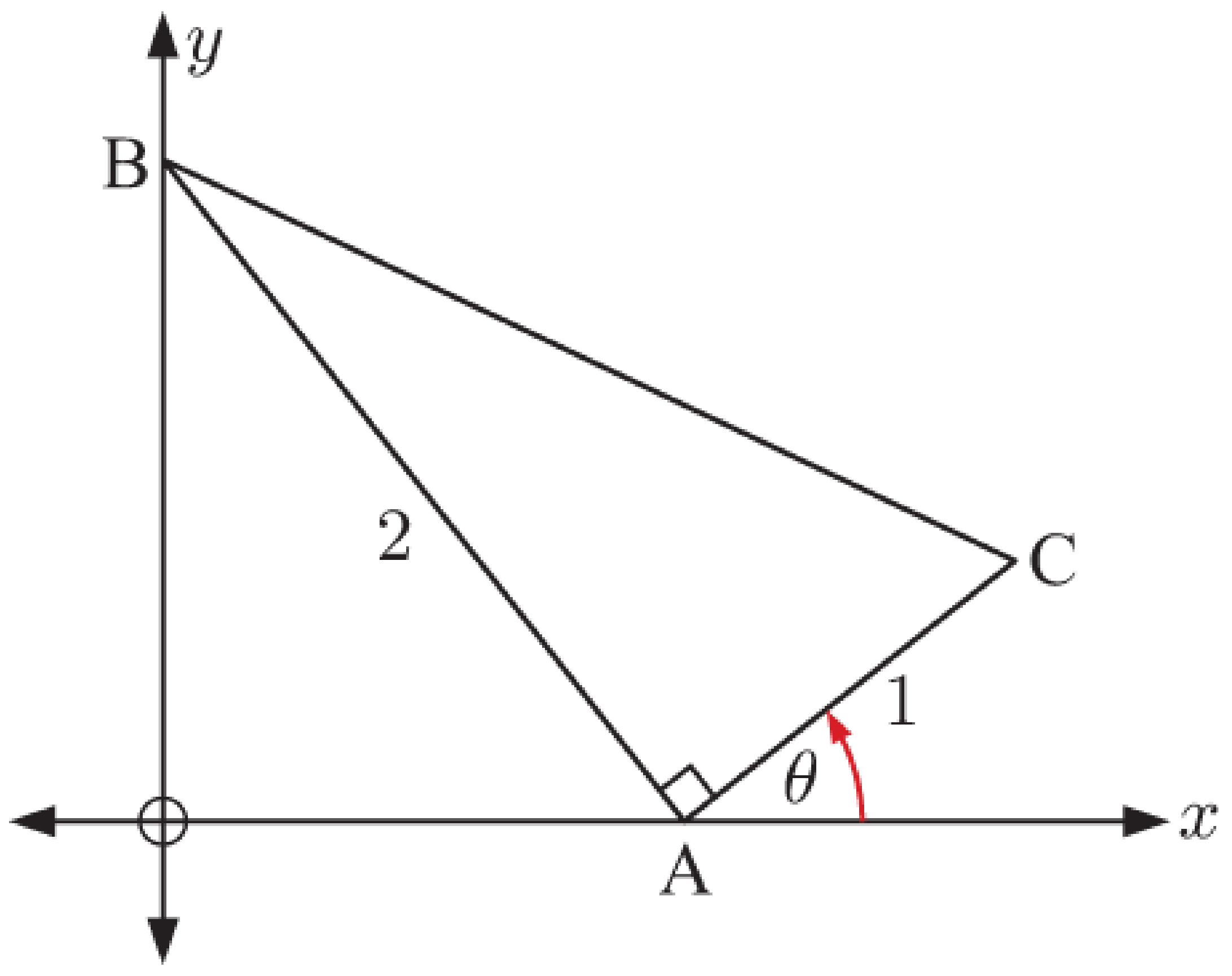
d $2x^2-4xy+5y^2+4x-2y=1$

3 **a** Use questions **1** and **2** to complete the following table:

Question	$a+c$	λ_1 *and* λ_2	$\lambda_1+\lambda_2$	*Sign of* $\lambda_1\lambda_2$	*Type of conic*
1 a	15	5, 10	15	>0	ellipse
1 b					
2 a					
2 b					
2 c					
2 d					

b Use the table to make conjectures dealing with the eigenvectors λ_1 and λ_2.

4



ABC is a set square which is right angled at A. [AB] and [AC] have lengths 2 units and 1 unit respectively.

The set square is free to move so that A always lies on the x-axis and B on the y-axis.

[AC] makes an angle θ with the x-axis as shown.

a Find the coordinates of C in terms of θ.

b Find the Cartesian equation of the locus of C.

c Find possible transformed equations for the locus of C.

d Use your observations from question **3** to identify the conic. Give reasons for your answer.

INVESTIGATION 5 THE DISCRIMINANT OF A CONIC

For the conic section with equation $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$, the value $b^2 - ac$ is called the **discriminant**. In this Investigation we will observe how the value of the discriminant allows us to quickly identify the conic type.

What to do:

1 We have seen that the conic section with equation $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ can be written in the form $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{v}^T\mathbf{x} + f = 0$ where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} d \\ e \end{pmatrix}$. Show that $\det \mathbf{A}$ is the negative of the discriminant of the conic.

2 Suppose the conic $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ is rotated through angle θ to produce $a'x'^{\,2} + 2b'x'y' + c'y'^{\,2} + d'x' + e'y' + f = 0$ such that the constant f is equal in both equations. Show that:

a the trace of matrix $\mathbf{A}$ is conserved, so $a' + c' = a + c$

b the determinant of matrix $\mathbf{A}$ is conserved, so $a'c' - b'^{\,2} = ac - b^2$

c the discriminant of the conics are the same, so $b'^{\,2} - a'c' = b^2 - ac$.

3 Consider the following proof that if $b^2 - ac > 0$ then the conic is either a hyperbola or a line-pair:

Suppose $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$ is rotated through angle θ to obtain $a'x'^{\,2} + 2b'x'y' + c'y'^{\,2} + d'x' + e'y' + f = 0$.

We choose θ so that $b' = 0$.

Using **2 c**, $b^2 - ac = b'^{\,2} - a'c' = -a'c'$

$\therefore \quad -a'c' > 0$

$\therefore \quad a'$ and c' are opposite in sign .... (1)

The rotated conic is $a'x'^{\,2} + c'y'^{\,2} + d'x' + e'y' + f = 0$

$$\therefore \quad \frac{x'^{\,2}}{c'} + \frac{d'}{a'c'}x' + \frac{y'^{\,2}}{a'} + \frac{e'}{a'c'}y' = -\frac{f}{a'c'}$$

$$\therefore \quad \frac{1}{c'}\left(x'^{\,2} + \frac{d'}{a'}x'\right) + \frac{1}{a'}\left(y'^{\,2} + \frac{e'}{c'}y'\right) = -\frac{f}{a'c'}$$

This equation is of the form $\frac{1}{c'}(x' - h)^2 + \frac{1}{a'}(y' - k)^2 = C$

which is a hyperbola if $C \neq 0$ {using (1)}
or a line-pair if $C = 0$.

Prove that:

a if $b^2 - ac < 0$ then the conic is an ellipse, a circle, a point, or else has no graph

b if $b^2 - ac = 0$ then the conic is a parabola, a line, a pair of parallel lines, or else has no graph.

Worked Solutions

EXERCISE 1A

1 **a** It contains a product of two variables, x_3x_4.

b It contains the squared term $-2x_3^{\,2}$.

c It contains the square root term $-\sqrt{x_2}$.

2 **a** Let $x = t$

$\therefore \; 8t - y = 3$

$\therefore \; y = 8t - 3$

$\therefore$ the solution set is $x = t,\; y = 8t - 3,\; t \in \mathbb{R}$.

b Let $x_2 = s$ and $x_3 = t$

$\therefore \; x_1 - 2s + t = 10$

$\therefore \; x_1 = 2s - t + 10$

$\therefore$ the solution set is $x_1 = 2s - t + 10,\; x_2 = s,\; x_3 = t$, where $s, t \in \mathbb{R}$.

c Let $x_2 = r,\; x_3 = s,$ and $x_4 = t$

$\therefore \; x_1 + r - 2s + t = -2$

$\therefore \; x_1 = -r + 2s - t - 2$

$\therefore$ the solution set is $x_1 = -r + 2s - t - 2,\; x_2 = r,\; x_3 = s,\; x_4 = t$, where $r, s, t \in \mathbb{R}$.

3 **a**

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_2 &= -4 \\ 2x_1 + x_2 &= -1 \end{aligned}$$

This system is overspecified as it has more equations (3) than unknowns (2).

b

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 &= 4 \\ 2x_1 + x_2 + 3x_3 - x_4 &= 3 \end{aligned}$$

This system is underspecified as it has more unknowns (4) than equations (2).

c

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 5 \\ x_3 + x_4 &= 6 \\ 2x_3 \qquad &= 8 \\ x_4 &= 2 \end{aligned}$$

This system is neither underspecified nor overspecified as it has the same number of equations and unknowns.

4 **a** The system has AM $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 1 & -1 & 1 & 8 \\ 2 & 1 & -3 & 0 \end{array}\right)$

b The system has AM $\left(\begin{array}{ccc|c} 1 & 1 & -2 & 7 \\ 3 & 0 & 1 & 2 \end{array}\right)$

c The system has AM $\left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 5 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 4 & -1 & 6 \end{array}\right)$

5 **a** No solutions exist when $a \neq 12,\; a \in \mathbb{R}$ (the lines are parallel).

b Infinitely many solutions exist when $a = 12$ (the lines are coincident).

c There is no value of $a \in \mathbb{R}$ for which exactly one solution exists.

6 The system is consistent if it has at least one solution

$\therefore \; k = 4$

7 Adding the first two equations gives $2x + y + 3z = p + q$ and the third equation is $2x + y + 3z = r$

$\therefore$ the system is consistent if $p + q = r$.

(If $p + q \neq r$, the two planes are parallel and no solutions would exist.)

8 **a** Solving the first two equations, $x = 3,\; y = 0$ but this does not satisfy the third equation

$\therefore$ the system is inconsistent.

b The system is overspecified as it has more equations than unknowns.

9 **a** The system is

$$\begin{aligned} x_1 + x_2 - x_3 &= 7 \\ x_1 - x_2 + 2x_3 &= 9 \end{aligned}$$

with $b_1 = 7$ and $b_2 = 9$

$\therefore$ the system is not homogeneous.

b The system is

$$\begin{aligned} x_1 + x_2 - x_3 &= a \\ 2x_1 - x_2 + x_3 &= b + 8 \end{aligned}$$

It is homogeneous if $a = 0$ and $b + 8 = 0$

$\therefore \; a = 0$ and $b = -8$.

EXERCISE 1B.1

1 **a** The system has AM $\left(\begin{array}{cc|c} 1 & -3 & 2 \\ 2 & 1 & -3 \end{array}\right)$

b $\sim \left(\begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 7 & -7 \end{array}\right)$

c Using row 2, $7y = -7$

$\therefore \; y = -1$

Substituting into row 1, $x - 3(-1) = 2$

$\therefore \; x = -1$

$\therefore$ the unique solution is $x = -1,\; y = -1$.

2 **a** The second equation is a multiple of the first.

$\therefore$ the lines are coincident and have infinitely many points of intersection.

$\therefore$ the system has infinitely many solutions.

b The lines are neither coincident nor parallel

$\therefore$ they intersect in exactly one point.

$\therefore$ the system has a unique solution.

c The lines are not coincident, but they have the same gradient

$\therefore$ they are parallel and never meet.

$\therefore$ the system has no solutions.

d If $a = 4$, the lines are coincident.

$\therefore$ the system has infinitely many solutions.

If $a \neq 4,\; a \in \mathbb{R}$, the lines are parallel.

$\therefore$ the system has no solutions.

3 **a** The system has AM

$$\left(\begin{array}{cc|c} 1 & -3 & -8 \\ 4 & 5 & 19 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -3 & -8 \\ 0 & 17 & 51 \end{array}\right) \quad R_2 - 4R_1 \to R_2$$

Using row 2, $17y = 51$

$\therefore \; y = 3$

Substituting into row 1, $x - 3(3) = -8$

$\therefore \; x = 1$

$\therefore$ the unique solution is $x = 1,\; y = 3$.

b The system has AM

$$\left(\begin{array}{cc|c} 1 & 7 & -17 \\ 2 & -1 & 11 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 7 & -17 \\ 0 & -15 & 45 \end{array}\right) \quad R_2 - 2R_1 \to R_2$$

Using row 2, $-15y = 45$

$\therefore \; y = -3$

Substituting into row 1, $x + 7(-3) = -17$

$\therefore \; x = 4$

$\therefore$ the unique solution is $x = 4,\; y = -3$.

c The system has AM

$$\left(\begin{array}{cc|c} 2 & 3 & -8 \\ 1 & 4 & -9 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 2 & 3 & -8 \\ 0 & 5 & -10 \end{array}\right) \quad 2R_2 - R_1 \to R_2$$

Using row 2, $5y = -10$

$\therefore\ y = -2$

Substituting into row 1, $2x + 3(-2) = -8$

$\therefore\ 2x = -2$

$\therefore\ x = -1$

$\therefore$ the unique solution is $x = -1,\ y = -2$.

d The system has AM

$$\left(\begin{array}{cc|c} 3 & -1 & 9 \\ 4 & 3 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 3 & -1 & 9 \\ 0 & 13 & -39 \end{array}\right) \quad 3R_2 - 4R_1 \to R_2$$

Using row 2, $13y = -39$

$\therefore\ y = -3$

Substituting into row 1, $3x - (-3) = 9$

$\therefore\ 3x = 6$

$\therefore\ x = 2$

$\therefore$ the unique solution is $x = 2,\ y = -3$.

4 **a** The equations represent coincident lines, which meet at infinitely many points.

$\therefore$ the system has infinitely many solutions.

b The system has AM

$$\left(\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 6 & 8 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 0 & 0 \end{array}\right) \quad R_2 - 2R_1 \to R_2$$

The second equation is a multiple of the first, so we obtain a row of zeros when we try to use row operations. The second equation adds nothing to aid the solution of the system.

c Let $y = t,\ t \in \mathbb{R}$.

Using row 1, $x + 3t = 4$

$\therefore\ x = 4 - 3t$

$\therefore$ the solution set is $x = 4 - 3t,\ y = t,\ t \in \mathbb{R}$.

d Let $x = s,\ s \in \mathbb{R}$.

Using row 1, $s + 3y = 4$

$\therefore\ 3y = 4 - s$

$\therefore\ y = \dfrac{4-s}{3}$

$\therefore$ the solution set is $x = s,\ y = \dfrac{4-s}{3},\ s \in \mathbb{R}$.

e Let $\left(s',\ \dfrac{4-s'}{3}\right)$ be a point in the second solution set.

In the first solution set, when $y = \dfrac{4-s'}{3},\ t = \dfrac{4-s'}{3}$.

$\therefore\ x = 4 - 3\left(\dfrac{4-s'}{3}\right)$

$\therefore\ x = 4 - 4 + s'$

$\therefore\ x = s'$

$\therefore$ when the y-coordinates are the same, the x-coordinates are also the same.

$\therefore$ any point in the second solution set is also in the first solution set.

$\therefore$ the solution sets are equivalent.

5 **a** The system has AM

$$\left(\begin{array}{cc|c} 1 & -5 & 8 \\ 2 & -10 & a \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -5 & 8 \\ 0 & 0 & a-16 \end{array}\right) \quad R_2 - 2R_1 \to R_2$$

b When $a \neq 16$, the second row gives $0x + 0y \neq 0$ which is not possible.

$\therefore$ there are no solutions; the lines are parallel.

c When $a = 16$, the second row gives $0x + 0y = 0$ which is true for all $x, y \in \mathbb{R}$.

$\therefore$ there are infinitely many solutions to the system.

Let $y = t,\ t \in \mathbb{R}$.

Using row 1, $x - 5t = 8$

$\therefore\ x = 5t + 8$

$\therefore$ the solution set is $x = 5t + 8,\ y = t,\ t \in \mathbb{R}$.

6 The system has AM

$$\left(\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & a & b \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & a-6 & b-8 \end{array}\right) \quad R_2 - 2R_1 \to R_2$$

The second row gives $(a-6)y = b - 8$

If $a \neq 6$, $y = \dfrac{b-8}{a-6}$

$\therefore\ x + 3\left(\dfrac{b-8}{a-6}\right) = 4$

$\therefore\ x = \dfrac{4(a-6) - 3(b-8)}{a-6}$

$\therefore\ x = \dfrac{4a-3b}{a-6}$

$\therefore$ the unique solution is $x = \dfrac{4a-3b}{a-6},\ y = \dfrac{b-8}{a-6}$.

If $a = 6,\ b = 8$, $0x + 0y = 0$

$\therefore$ the lines are coincident and infinitely many solutions exist.

Let $y = t,\ t \in \mathbb{R}$.

$\therefore\ x + 3t = 4$

$\therefore\ x = 4 - 3t$

$\therefore$ the solution set is $x = 4 - 3t,\ y = t,\ t \in \mathbb{R}$.

If $a = 6,\ b \neq 8$, $0x + 0y \neq 0$ which is not possible.

$\therefore$ the lines are parallel and the system has no solutions.

EXERCISE 1B.2

1 **a** The system has AM

$$\left(\begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 1 & 6 & 17 & 9 \\ 1 & 4 & 8 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & -3 & -3 \end{array}\right) \quad \begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$$

Using row 3, $-3z = -3$

$\therefore\ z = 1$

Substituting into row 2, $2y + 6(1) = 2$

$\therefore\ 2y = -4$

$\therefore\ y = -2$

Substituting into row 1, $x + 4(-2) + 11(1) = 7$

$\therefore\ x = 4$

$\therefore$ the unique solution is $x = 4,\ y = -2,\ z = 1$.

b The system has AM

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 17\\ 2 & -2 & -5 & 4\\ 3 & 2 & 2 & 10 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & 3 & 17\\ 0 & -1 & -8 & -13\\ 0 & 7 & -5 & -31 \end{array}\right) \qquad \begin{array}{l} R_2 - R_1 \to R_2\\ 2R_3 - 3R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & 3 & 17\\ 0 & -1 & -8 & -13\\ 0 & 0 & -61 & -122 \end{array}\right) \qquad R_3 + 7R_2 \to R_3$$

Using row 3, $-61z = -122$

$\therefore\ z = 2$

Substituting into row 2, $-y - 8(2) = -13$

$\therefore\ y = -3$

Substituting into row 1, $2x - (-3) + 3(2) = 17$

$\therefore\ 2x = 8$

$\therefore\ x = 4$

$\therefore$ the unique solution is $x = 4,\ y = -3,\ z = 2.$

c The system has AM

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 1\\ 5 & 6 & 7 & 2\\ 8 & 9 & 10 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & 3 & 4 & 1\\ 0 & -3 & -6 & -1\\ 0 & -3 & -6 & 0 \end{array}\right) \qquad \begin{array}{l} 2R_2 - 5R_1 \to R_2\\ R_3 - 4R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 3 & 4 & 1\\ 0 & -3 & -6 & -1\\ 0 & 0 & 0 & 1 \end{array}\right) \qquad R_3 - R_2 \to R_3$$

Row 3 means that $0x + 0y + 0z = 1$ which is absurd.

$\therefore$ there are no solutions, and the system is inconsistent.

d The system has AM

$$\left(\begin{array}{ccc|c} 1 & -2 & 5 & 1\\ 2 & -1 & 8 & 2\\ -3 & 0 & -11 & -3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 5 & 1\\ 0 & 3 & -2 & 0\\ 0 & -6 & 4 & 0 \end{array}\right) \qquad \begin{array}{l} R_2 - 2R_1 \to R_2\\ R_3 + 3R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 5 & 1\\ 0 & 3 & -2 & 0\\ 0 & 0 & 0 & 0 \end{array}\right) \qquad R_3 + 2R_2 \to R_3$$

Row 3 indicates there are infinitely many solutions.
Let $z = t$.

Using row 2, $3y - 2t = 0$

$\therefore\ 3y = 2t$

$\therefore\ y = \frac{2}{3}t$

Substituting into row 1, $x - 2(\frac{2}{3}t) + 5t = 1$

$\therefore\ x = 1 - \frac{11}{3}t$

$\therefore$ the solutions have the form $x = 1 - \frac{11}{3}t,\ y = \frac{2}{3}t,\ z = t,\ t \in \mathbb{R}$.

e The system has AM

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4\\ 3 & 2 & 1 & 7\\ 5 & 2 & 3 & 11 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4\\ 0 & -4 & 4 & -5\\ 0 & -8 & 8 & -9 \end{array}\right) \qquad \begin{array}{l} R_2 - 3R_1 \to R_2\\ R_3 - 5R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4\\ 0 & -4 & 4 & -5\\ 0 & 0 & 0 & 1 \end{array}\right) \qquad R_3 - 2R_2 \to R_3$$

Row 3 means that $0x + 0y + 0z = 1$ which is absurd.

$\therefore$ there are no solutions, and the system is inconsistent.

f The system has AM

$$\left(\begin{array}{ccc|c} 2 & 4 & 1 & 1\\ 3 & -5 & -3 & 19\\ 5 & 13 & 7 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & 4 & 1 & 1\\ 0 & -22 & -9 & 35\\ 0 & 6 & 9 & -3 \end{array}\right) \qquad \begin{array}{l} 2R_2 - 3R_1 \to R_2\\ 2R_3 - 5R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & 4 & 1 & 1\\ 0 & -22 & -9 & 35\\ 0 & 0 & 24 & 24 \end{array}\right) \qquad \tfrac{11}{3}R_3 + R_2 \to R_3$$

Using row 3, $24z = 24$

$\therefore\ z = 1$

Substituting into row 2, $-22y - 9(1) = 35$

$\therefore\ 22y = -44$

$\therefore\ y = -2$

Substituting into row 1, $2x + 4(-2) + (1) = 1$

$\therefore\ 2x = 8$

$\therefore\ x = 4$

$\therefore$ the unique solution is $x = 4,\ y = -2,\ z = 1.$

2 **a** The system has AM

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 2 & -1 & 4 & 1\\ 1 & 7 & -1 & k \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 0 & -5 & 2 & -5\\ 0 & 5 & -2 & k-3 \end{array}\right) \qquad \begin{array}{l} R_2 - 2R_1 \to R_2\\ R_3 - R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3\\ 0 & -5 & 2 & -5\\ 0 & 0 & 0 & k-8 \end{array}\right) \qquad R_3 + R_2 \to R_3$$

b Using row 3, the system has no solutions if $k \neq 8$.

c The system has infinitely many solutions if the last row is all zeros. This occurs when $k = 8$.
In this case we let $z = t$.

$\therefore$ using row 2, $-5y + 2t = -5$

$\therefore\ 5y = 2t + 5$

$\therefore\ y = \frac{2}{5}t + 1$

Using row 1, $x + 2(\frac{2}{5}t + 1) + t = 3$

$\therefore\ x = 1 - \frac{9}{5}t$

$\therefore$ the solutions have the form $x = 1 - \frac{9}{5}t,\ y = \frac{2}{5}t + 1,\ z = t,\ t \in \mathbb{R}$.

d In row echelon form, row 3 reads $0x + 0y + 0z = k - 8$.
From **b** and **c** the system has no solutions if $k \neq 8$ or infinitely many solutions if $k = 8$.
$\therefore$ the system never has a unique solution.

3 **a** The system has AM

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 1 & -1 & 3 & -1 \\ 1 & -7 & k & -k \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 0 & -3 & 5 & -6 \\ 0 & -9 & k+2 & -k-5 \end{array}\right) \quad \begin{array}{l} \\ R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 0 & -3 & 5 & -6 \\ 0 & 0 & k-13 & -k+13 \end{array}\right) \quad \begin{array}{l} \\ \\ R_3 - 3R_2 \to R_3 \end{array}$$

b The system has infinitely many solutions if the last row is all zeros. This occurs when $k = 13$.

In this case we let $z = t$.

$\therefore$ using row 2, $-3y + 5t = -6$

$\therefore$ $3y = 5t + 6$

$\therefore$ $y = \frac{5}{3}t + 2$

Using row 1, $x + 2(\frac{5}{3}t + 2) - 2t = 5$

$\therefore$ $x = 1 - \frac{4}{3}t$

$\therefore$ the solutions have the form $x = 1 - \frac{4}{3}t$, $y = \frac{5}{3}t + 2$, $z = t$, $t \in \mathbb{R}$.

c When $k \neq 13$, row 3 gives $(k - 13)z = -(k - 13)$

$\therefore$ $z = -1$

Substituting into row 2, $-3y + 5(-1) = -6$

$\therefore$ $3y = 1$

$\therefore$ $y = \frac{1}{3}$

Substituting into row 1, $x + 2(\frac{1}{3}) - 2(-1) = 5$

$\therefore$ $x = \frac{7}{3}$

$\therefore$ the unique solution is $x = \frac{7}{3}$, $y = \frac{1}{3}$, $z = -1$ for all $k \neq 13$, $k \in \mathbb{R}$.

4 **a** The system has AM

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 2 & -1 & 1 & 7 \\ 3 & -5 & a & 16 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & -14 & a-9 & 19-3a \end{array}\right) \quad \begin{array}{l} \\ R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & 0 & a+1 & a+1 \end{array}\right) \quad \begin{array}{l} \\ \\ R_3 - 2R_2 \to R_3 \end{array}$$

b The system has infinitely many solutions if the last row is all zeros. This occurs when $a = -1$.

In this case we let $z = t$.

$\therefore$ using row 2, $-7y - 5t = 9 - 2(-1)$

$\therefore$ $7y = -5t - 11$

$\therefore$ $y = \dfrac{-5t - 11}{7}$

Substituting into row 1,

$$x + 3\left(\frac{-5t - 11}{7}\right) + 3t = (-1) - 1$$

$$\therefore \quad x + \frac{-15t - 33}{7} + \frac{21t}{7} = \frac{-14}{7}$$

$$\therefore \quad x = \frac{19 - 6t}{7}$$

$\therefore$ the solutions have the form

$x = \dfrac{19 - 6t}{7}$, $y = \dfrac{-5t - 11}{7}$, $z = t$, $t \in \mathbb{R}$.

c If $a \neq -1$, row 3 gives $(a + 1)z = a + 1$

$\therefore$ $z = 1$

Substituting into row 2, $-7y - 5(1) = 9 - 2a$

$\therefore$ $7y = 2a - 14$

$\therefore$ $y = \frac{2}{7}a - 2$

Substituting into row 1, $x + 3(\frac{2}{7}a - 2) + 3(1) = a - 1$

$\therefore$ $x + \frac{6}{7}a - 6 + 3 = a - 1$

$\therefore$ $x = \frac{1}{7}a + 2$

$\therefore$ the solutions have the form $x = \frac{1}{7}a + 2$, $y = \frac{2}{7}a - 2$, $z = 1$, $a \neq -1$, $a \in \mathbb{R}$.

EXERCISE 1B.3

1 **a** Not in reduced row echelon form as the row of all zeros is not at the bottom.

b Is in reduced row echelon form.

c Not in reduced row echelon form as the leading 1 in row 3 should have zeros above it in column 4.

2 **a** By inspection, the system has the unique solution $x_1 = 2$, $x_2 = -9$, $x_3 = 3$.

b By inspection, the system has no solutions as row 3 means $0x + 0y + 0z = 1$ which is absurd.

c The basic variables are x_1, x_2, and x_3, and x_4 is the free variable.

Let $x_4 = t$.

Using row 3, $x_3 + t = 6$ $\therefore$ $x_3 = 6 - t$

Using row 2, $x_2 + 2t = 4$ $\therefore$ $x_2 = 4 - 2t$

Using row 1, $x_1 + 2t = 5$ $\therefore$ $x_1 = 5 - 2t$

$\therefore$ the solutions have the form $x_1 = 5 - 2t$, $x_2 = 4 - 2t$, $x_3 = 6 - t$, $x_4 = t$, $t \in \mathbb{R}$.

3 **a** The system has AM

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & 12 \\ 1 & -1 & 1 & -8 \\ 4 & -2 & 1 & -8 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -6 \end{array}\right) \quad \{\text{using technology}\}$$

By inspection, the system has the unique solution $x_1 = 1$, $x_2 = 3$, $x_3 = -6$.

b The system has AM

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 2 & 4 \\ 1 & 1 & 0 & 4 & 9 \\ 0 & 1 & -1 & 1 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right) \quad \{\text{using technology}\}$$

The basic variables are x_1, x_2, and x_3, and x_4 is the free variable.

Let $x_4 = t$.

Using row 3, $x_3 + t = 6$ $\therefore$ $x_3 = 6 - t$

Using row 2, $x_2 + 2t = 4$ $\therefore$ $x_2 = 4 - 2t$

Using row 1, $x_1 + 2t = 5$ $\therefore$ $x_1 = 5 - 2t$

$\therefore$ the solutions have the form $x_1 = 5 - 2t$, $x_2 = 4 - 2t$, $x_3 = 6 - t$, $x_4 = t$, $t \in \mathbb{R}$.

c The system has AM

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & -1 & 4 & 7 \\ 3 & 3 & 10 & 15 \\ 6 & 9 & 19 & 9 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{11}{3} & 6 \\ 0 & 1 & -\frac{1}{3} & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \{\text{using technology}\}$$

By inspection, the system has no solutions as row 3 means $0x_1 + 0x_2 + 0x_3 = 1$ which is absurd.

d The system has AM

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -4 & 1 \\ 1 & 7 & 3 & 2 & 2 \\ 1 & 13 & 7 & 8 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -\frac{5}{3} & -5 & \frac{5}{6} \\ 0 & 1 & \frac{2}{3} & 1 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \{\text{using technology}\}$$

The basic variables are x_1 and x_2, and the free variables are x_3 and x_4.

Let $x_3 = s$ and $x_4 = t$.

Using row 2, $x_2 + \frac{2}{3}s + t = \frac{1}{6}$

$\therefore \; x_2 = \frac{1}{6} - \frac{2}{3}s - t$

Using row 1, $x_1 - \frac{5}{3}s - 5t = \frac{5}{6}$

$\therefore \; x_1 = \frac{5}{6} + \frac{5}{3}s + 5t$

$\therefore$ the solutions have the form $x_1 = \frac{5}{6} + \frac{5}{3}s + 5t$, $x_2 = \frac{1}{6} - \frac{2}{3}s - t$, $x_3 = s$, $x_4 = t$, where $s, t \in \mathbb{R}$.

e The system has AM

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & -2 & 3 & 1 \\ 3 & -3 & 2 & -4 & -9 & 3 \\ 2 & 2 & -1 & 2 & 6 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{array}\right) \quad \{\text{using technology}\}$$

The basic variables are x_1, x_2, and x_3, and the free variables are x_4 and x_5.

Let $x_4 = s$ and $x_5 = t$.

Using row 3, $x_3 - 2s = 0$

$\therefore \; x_3 = 2s$

Using row 2, $x_2 + 3t = 0$

$\therefore \; x_2 = -3t$

Using row 1, $x_1 = 1$

$\therefore$ the solutions have the form $x_1 = 1$, $x_2 = -3t$, $x_3 = 2s$, $x_4 = s$, $x_5 = t$, where $s, t \in \mathbb{R}$.

f The system has AM

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 1 & 2 \\ 1 & -1 & 1 & -1 & 1 & 3 \\ 3 & 1 & 3 & 3 & 3 & 7 \\ 2 & 0 & 2 & 1 & 2 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 1 & \frac{1}{2} & 1 & \frac{5}{2} \\ 0 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \{\text{using technology}\}$$

The basic variables are x_1 and x_2, and the free variables are x_3, x_4, and x_5.

Let $x_3 = r$, $x_4 = s$, $x_5 = t$.

Using row 2, $x_2 + \frac{3}{2}s = -\frac{1}{2}$

$\therefore \; x_2 = -\frac{1}{2} - \frac{3}{2}s$

Using row 1, $x_1 + r + \frac{1}{2}s + t = \frac{5}{2}$

$\therefore \; x_1 = \frac{5}{2} - r - \frac{1}{2}s - t$

$\therefore$ the solutions have the form $x_1 = \frac{5}{2} - r - \frac{1}{2}s - t$, $x_2 = -\frac{1}{2} - \frac{3}{2}s$, $x_3 = r$, $x_4 = s$, $x_5 = t$, where $r, s, t \in \mathbb{R}$.

4 **a** When $d = 1$, $h = 12$

$\therefore \; x_1(1)^3 + x_2(1)^2 + x_3(1) + x_4 = 12$

$\therefore \; x_1 + x_2 + x_3 + x_4 = 12 \quad .... \; (1)$

When $d = 2.5$, $h = 46$

$\therefore \; x_1(2.5)^3 + x_2(2.5)^2 + x_3(2.5) + x_4 = 46$

$\therefore \; \frac{125}{8}x_1 + \frac{25}{4}x_2 + \frac{5}{2}x_3 + x_4 = 46 \quad .... \; (2)$

b **i** The gradient of the hill is the rate of change of height over distance. This is modelled by the derivative function $h'(d)$, which is found by differentiating the height function $h(d)$ with respect to d.

$h(d) = x_1 d^3 + x_2 d^2 + x_3 d + x_4$

$\therefore \; h'(d) = 3x_1 d^2 + 2x_2 d + x_3$

ii When $d = 1$, $h'(d) = 0.1$

$\therefore \; 3x_1(1)^2 + 2x_2(1) + x_3 = 0.1$

$\therefore \; 3x_1 + 2x_2 + x_3 = 0.1 \quad .... \; (3)$

When $d = 2.5$, $h'(d) = 0$

$\therefore \; 3x_1(2.5)^2 + 2x_2(2.5) + x_3 = 0$

$\therefore \; \frac{75}{4}x_1 + 5x_2 + x_3 = 0 \quad .... \; (4)$

c From (1), (2), (3), and (4), the system has AM

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 12 \\ \frac{125}{8} & \frac{25}{4} & \frac{5}{2} & 1 & 46 \\ 3 & 2 & 1 & 0 & 0.1 \\ \frac{75}{4} & 5 & 1 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -\frac{2714}{135} \\ 0 & 1 & 0 & 0 & \frac{4748}{45} \\ 0 & 0 & 1 & 0 & -\frac{2711}{18} \\ 0 & 0 & 0 & 1 & \frac{4169}{54} \end{array}\right) \quad \{\text{using technology}\}$$

By inspection, the system has the unique solution $x_1 = -\frac{2714}{135}$, $x_2 = \frac{4748}{45}$, $x_3 = -\frac{2711}{18}$, $x_4 = \frac{4169}{54}$.

d $h(d) = -\frac{2714}{135}d^3 + \frac{4748}{45}d^2 - \frac{2711}{18}d + \frac{4169}{54}$

$\therefore$ when $d = 2$, $h(2) \approx 37.2$

$\therefore$ at the point 2 km from the ocean the height of the hill is about 37.2 m above sea level.

EXERCISE 1B.4

1 **a** The second equation is a multiple of the first. So the lines are coincident, and there are infinitely many solutions.

$\therefore$ the system has non-trivial solutions.

b The system is underspecified as it has more unknowns than equations.

$\therefore$ it has infinitely many solutions

$\therefore$ the system has non-trivial solutions.

2 **a** The system has AM

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & -1 & 5 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \quad \{\text{using technology}\}$$

Let $x_3 = t$.

$$\begin{array}{ll} \text{Using row 2,} & \text{Using row 1,} \\ x_2 - t = 0 & x_1 + 2t = 0 \\ \therefore\ x_2 = t & \therefore\ x_1 = -2t \end{array}$$

$\therefore$ the solutions have the form $x_1 = -2t,\ x_2 = t,$ $x_3 = t,\ t \in \mathbb{R}$.

b The system has AM

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \{\text{using technology}\}$$

$\therefore$ the only solution is the trivial solution $x_1 = x_2 = x_3 = 0$.

c The system has AM

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 2 & 1 & -1 & -2 & 0 \\ 3 & -1 & 2 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \end{array}\right) \quad \{\text{using technology}\}$$

Let $x_4 = t$.

$$\begin{array}{lll} \text{Using row 3,} & \text{Using row 2,} & \text{Using row 1,} \\ x_3 + \frac{2}{3}t = 0 & x_2 - \frac{2}{3}t = 0 & x_1 - \frac{1}{3}t = 0 \\ \therefore\ x_3 = -\frac{2}{3}t & \therefore\ x_2 = \frac{2}{3}t & \therefore\ x_1 = \frac{1}{3}t \end{array}$$

$\therefore$ the solutions have the form $x_1 = \frac{1}{3}t,\ x_2 = \frac{2}{3}t,$ $x_3 = -\frac{2}{3}t,\ t \in \mathbb{R}$.

3 The system has AM

$$\left(\begin{array}{cc|c} 1 & p-2 & 0 \\ p-2 & 1 & 0 \end{array}\right) \quad \{\text{swapping rows}\}$$

$$\sim \left(\begin{array}{cc|c} 1 & p-2 & 0 \\ 0 & 1-(p-2)^2 & 0 \end{array}\right) \quad R_2 - (p-2)R_1 \to R_2$$

$\therefore$ $(1-(p-2)^2)y = 0$

$\therefore$ if $(p-2)^2 \neq 1,\ y = 0$ and $x = 0$

So for non-trivial solutions $(p-2)^2 = 1$

$$\therefore\ p - 2 = \pm 1$$

$$\therefore\ p = 1 \text{ or } 3.$$

4 **a** If $x = x_1,\ y = y_1$ is a solution of $a_1x + b_1y = 0$
$a_2x + b_2y = 0$

then $a_1x_1 + b_1y_1 = 0$ .... (1)

and $a_2x_1 + b_2y_1 = 0$ .... (2)

Now if $x = cx_1$ and $y = cy_1$ then

$$\begin{array}{ll} a_1x + b_1y & \quad a_2x + b_2y \\ = a_1(cx_1) + b_1(cy_1) & \quad = a_2(cx_1) + b_2(cy_1) \\ = c(a_1x_1 + b_1y_1) & \quad = c(a_2x_1 + b_2y_1) \\ = c(0) \quad \{\text{from (1)}\} & \quad = c(0) \quad \{\text{from (2)}\} \\ = 0 & \quad = 0 \end{array}$$

(and)

$\therefore$ $x = cx_1,\ y = cy_1$ is a solution for all $c \in \mathbb{R}$.

b If $x = x_2,\ y = y_2$ is also a solution

then $a_1x_2 + b_1y_2 = 0$ .... (3)

and $a_2x_2 + b_2y_2 = 0$ .... (4)

Now if $x = x_1 + x_2$ and $y = y_1 + y_2$ then

$$\begin{aligned} a_1x + b_1y &= a_1(x_1 + x_2) + b_1(y_1 + y_2) \\ &= (a_1x_1 + b_1y_1) + (a_1x_2 + b_1y_2) \\ &= 0 + 0 \quad \{\text{from (1) and (3)}\} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} a_2x + b_2y &= a_2(x_1 + x_2) + b_2(y_1 + y_2) \\ &= (a_2x_1 + b_2y_1) + (a_2x_2 + b_2y_2) \\ &= 0 + 0 \quad \{\text{from (2) and (4)}\} \\ &= 0 \end{aligned}$$

$\therefore$ $x = x_1 + x_2,\ y = y_1 + y_2$ is a solution.

EXERCISE 1C.1

1 **a** $4\mathbf{A} = \begin{pmatrix} 4\times 1 & 4\times 2 \\ 4\times 6 & 4\times 3 \\ 4\times 5 & 4\times 4 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 24 & 12 \\ 20 & 16 \end{pmatrix}$

b $-2\mathbf{C} = \begin{pmatrix} -2\times -1 & -2\times 2 \\ -2\times -3 & -2\times 5 \\ -2\times 0 & -2\times 2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 6 & -10 \\ 0 & -4 \end{pmatrix}$

c $\mathbf{A} + 2\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 6 & 3 \\ 5 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ -6 & 10 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ 0 & 13 \\ 5 & 8 \end{pmatrix}$

d $\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 & 2 \\ -3 & 5 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 6 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -9 & 2 \\ -5 & -2 \end{pmatrix}$

e $-2\mathbf{A} + \frac{1}{2}\mathbf{C} = \begin{pmatrix} -2 & -4 \\ -12 & -6 \\ -10 & -8 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{3}{2} & \frac{5}{2} \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -\frac{5}{2} & -3 \\ -\frac{27}{2} & -\frac{7}{2} \\ -10 & -7 \end{pmatrix}$$

f $\frac{1}{3}\mathbf{A} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 2 & 1 \\ \frac{5}{3} & \frac{4}{3} \end{pmatrix}$

2 **a**

$$\begin{aligned} &\mathbf{A} + \mathbf{B} \\ &= (a_{ij}) + (b_{ij}) \\ &= (a_{ij} + b_{ij}) \\ &= (b_{ij} + a_{ij}) \\ &= (b_{ij}) + (a_{ij}) \\ &= \mathbf{B} + \mathbf{A} \end{aligned}$$

b

$$\begin{aligned} &k\mathbf{A} + k\mathbf{B} \\ &= (ka_{ij}) + (kb_{ij}) \\ &= (ka_{ij} + kb_{ij}) \\ &= k(a_{ij} + b_{ij}) \\ &= k(\mathbf{A} + \mathbf{B}) \end{aligned}$$

c

$$\begin{aligned} &\mathbf{B} - \mathbf{A} \\ &= (b_{ij} - a_{ij}) \\ &= (-(a_{ij} - b_{ij})) \\ &= -(a_{ij} - b_{ij}) \\ &= -(\mathbf{A} - \mathbf{B}) \end{aligned}$$

d

$$\begin{aligned} &(a + b)\mathbf{A} \\ &= ((a + b)a_{ij}) \\ &= (aa_{ij} + ba_{ij}) \\ &= (aa_{ij}) + (ba_{ij}) \\ &= a\mathbf{A} + b\mathbf{A} \quad \text{for all } a, b \in \mathbb{R} \end{aligned}$$

e

$$\begin{aligned} \underbrace{\mathbf{A} + \mathbf{A} + \mathbf{A} + ... + \mathbf{A}}_{k \text{ of these}} &= \underbrace{(a_{ij}) + (a_{ij}) + (a_{ij}) + ... + (a_{ij})}_{k \text{ of these}} \\ &= (a_{ij} + a_{ij} + a_{ij} + ... + a_{ij}) \\ &= (ka_{ij}) \\ &= k\mathbf{A},\ k \in \mathbb{Z}^+ \end{aligned}$$

3 **a** $\begin{pmatrix} x & x^2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} y & 9 \\ 3 & y+7 \end{pmatrix}$

$\therefore\ x = y,\ x^2 = 9,\ 3 = 3,\ \text{and}\ 4 = y + 7$

$\therefore\ x = y,\ x = \pm 3,\ \text{and}\ y = -3$

$\therefore\ x = -3\ \text{and}\ y = -3$

b $\begin{pmatrix} x & 2y \\ y & x \end{pmatrix} = \begin{pmatrix} -y & x \\ x & y \end{pmatrix}$

$\therefore\ x = -y,\ 2y = x,\ y = x,\ \text{and}\ x = y$

$\therefore\ y = -y = 2y\ \text{and}\ x = y$

$\therefore\ x = 0\ \text{and}\ y = 0$

4 **a** $\mathbf{K} = \begin{pmatrix} 9 & 8 \\ 12 & 14 \\ 7 & 5 \end{pmatrix}$ **b** $\mathbf{T} = \begin{pmatrix} 12 & 9 \\ 6 & 9 \\ 13 & 10 \end{pmatrix}$

c $\mathbf{K} + \mathbf{T} = \begin{pmatrix} 21 & 17 \\ 18 & 23 \\ 20 & 15 \end{pmatrix}$, $\mathbf{K} - \mathbf{T} = \begin{pmatrix} -3 & -1 \\ 6 & 5 \\ -6 & -5 \end{pmatrix}$

d $\mathbf{K} + \mathbf{T}$ is the total number of As, Bs, and Cs of both classes for each year.

$\mathbf{K} - \mathbf{T}$ is the difference in grades of the classes for each year.

5 **a** $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$

$$= \begin{pmatrix} -1 & -2 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 0 \end{pmatrix}$$

$\mathbf{A} + (\mathbf{B} + \mathbf{C})$

$$= \begin{pmatrix} -3 & 2 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 0 \end{pmatrix}$$

b
$$\begin{aligned}(\mathbf{A} + \mathbf{B}) + \mathbf{C} &= (a_{ij} + b_{ij}) + (c_{ij}) \\ &= (a_{ij} + b_{ij} + c_{ij}) \\ &= (a_{ij}) + (b_{ij} + c_{ij}) \\ &= \mathbf{A} + (\mathbf{B} + \mathbf{C})\end{aligned}$$

6 **a** The zero $m \times n$ matrix is $\mathbf{O} = (0)$

$$\begin{aligned}\mathbf{A} + \mathbf{O} &= (a_{ij}) + (0) \\ &= (a_{ij} + 0) \\ &= (0 + a_{ij}) \\ &= (0) + (a_{ij}) \\ &= \mathbf{O} + \mathbf{A}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\mathbf{A} + \mathbf{O} &= (a_{ij}) + (0) \\ &= (a_{ij} + 0) \\ &= (a_{ij}) \\ &= \mathbf{A}\end{aligned}$$

$\therefore\ \mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$

b
$$\begin{aligned}&\mathbf{A} + (-\mathbf{A}) \\ &= (a_{ij}) + (-a_{ij}) \\ &= (a_{ij} + (-a_{ij})) \\ &= ((-a_{ij}) + a_{ij}) \\ &= (-a_{ij}) + (a_{ij}) \\ &= (-\mathbf{A}) + \mathbf{A}\end{aligned} \qquad \text{and} \qquad \begin{aligned}&\mathbf{A} + (-\mathbf{A}) \\ &= (a_{ij}) + (-a_{ij}) \\ &= (a_{ij} + (-a_{ij})) \\ &= (a_{ij} - a_{ij}) \\ &= (0) \\ &= \mathbf{O}\end{aligned}$$

EXERCISE 1C.2

1 **a** $3\mathbf{A} + 4\mathbf{A} = 7\mathbf{A}$ **b** $\mathbf{C} - 5\mathbf{C} = -4\mathbf{C}$

c $2\mathbf{M} - 2\mathbf{M} = \mathbf{O}$ **d** $-\mathbf{X} + \mathbf{X} = \mathbf{O}$

e
$$\begin{aligned}&3(\mathbf{A} + \mathbf{B}) - \mathbf{B} \\ &= 3\mathbf{A} + 3\mathbf{B} - \mathbf{B} \\ &= 3\mathbf{A} + 2\mathbf{B}\end{aligned}$$

f
$$\begin{aligned}&2\mathbf{B} - (\mathbf{A} - \mathbf{B}) \\ &= 2\mathbf{B} - \mathbf{A} + \mathbf{B} \\ &= 3\mathbf{B} - \mathbf{A}\end{aligned}$$

g
$$\begin{aligned}&\mathbf{A} - (2\mathbf{A} + \mathbf{C}) \\ &= \mathbf{A} - 2\mathbf{A} - \mathbf{C} \\ &= -\mathbf{A} - \mathbf{C}\end{aligned}$$

h
$$\begin{aligned}&2(\mathbf{A} + \mathbf{B}) - (\mathbf{A} - \mathbf{B}) \\ &= 2\mathbf{A} + 2\mathbf{B} - \mathbf{A} + \mathbf{B} \\ &= \mathbf{A} + 3\mathbf{B}\end{aligned}$$

i
$$\begin{aligned}\mathbf{A} - 2\mathbf{D} - \tfrac{1}{2}(\mathbf{D} - \mathbf{A}) &= \mathbf{A} - 2\mathbf{D} - \tfrac{1}{2}\mathbf{D} + \tfrac{1}{2}\mathbf{A} \\ &= \tfrac{3}{2}\mathbf{A} - \tfrac{5}{2}\mathbf{D}\end{aligned}$$

2 **a** $\mathbf{X} + \mathbf{B} = 2\mathbf{A}$

$\therefore\ \mathbf{X} + \mathbf{B} + (-\mathbf{B}) = 2\mathbf{A} + (-\mathbf{B})$

$\therefore\ \mathbf{X} + \mathbf{O} = 2\mathbf{A} - \mathbf{B}$

$\therefore\ \mathbf{X} = 2\mathbf{A} - \mathbf{B}$

b $\mathbf{B} - \mathbf{X} = \mathbf{C}$

$\therefore\ \mathbf{B} - \mathbf{X} + \mathbf{X} = \mathbf{C} + \mathbf{X}$

$\therefore\ \mathbf{B} + \mathbf{O} = \mathbf{C} + \mathbf{X}$

$\therefore\ \mathbf{C} + \mathbf{X} = \mathbf{B}$

$\therefore\ \mathbf{C} + \mathbf{X} + (-\mathbf{C}) = \mathbf{B} + (-\mathbf{C})$

$\therefore\ \mathbf{X} + \mathbf{O} = \mathbf{B} - \mathbf{C}$

$\therefore\ \mathbf{X} = \mathbf{B} - \mathbf{C}$

c $\mathbf{B} + 2\mathbf{X} = \mathbf{C}$

$\therefore\ \mathbf{B} + 2\mathbf{X} + (-\mathbf{B}) = \mathbf{C} + (-\mathbf{B})$

$\therefore\ 2\mathbf{X} + \mathbf{O} = \mathbf{C} - \mathbf{B}$

$\therefore\ \tfrac{1}{2}(2\mathbf{X}) = \tfrac{1}{2}(\mathbf{C} - \mathbf{B})$

$\therefore\ \mathbf{X} = \tfrac{1}{2}(\mathbf{C} - \mathbf{B})$

$\therefore\ \mathbf{X} = \tfrac{1}{2}\mathbf{C} - \tfrac{1}{2}\mathbf{B}$

d $\tfrac{1}{2}\mathbf{X} + \mathbf{A} = 2\mathbf{C}$

$\therefore\ \tfrac{1}{2}\mathbf{X} + \mathbf{A} + (-\mathbf{A}) = 2\mathbf{C} + (-\mathbf{A})$

$\therefore\ \tfrac{1}{2}\mathbf{X} + \mathbf{O} = 2\mathbf{C} - \mathbf{A}$

$\therefore\ 2(\tfrac{1}{2}\mathbf{X}) = 2(2\mathbf{C} - \mathbf{A})$

$\therefore\ \mathbf{X} = 4\mathbf{C} - 2\mathbf{A}$

e $3(\mathbf{X} - \mathbf{B}) = 2\mathbf{B} + \mathbf{C}$

$\therefore\ 3\mathbf{X} - 3\mathbf{B} = 2\mathbf{B} + \mathbf{C}$

$\therefore\ 3\mathbf{X} - 3\mathbf{B} + 3\mathbf{B} = 2\mathbf{B} + \mathbf{C} + 3\mathbf{B}$

$\therefore\ 3\mathbf{X} + \mathbf{O} = 5\mathbf{B} + \mathbf{C}$

$\therefore\ \tfrac{1}{3}(3\mathbf{X}) = \tfrac{1}{3}(5\mathbf{B} + \mathbf{C})$

$\therefore\ \mathbf{X} = \tfrac{5}{3}\mathbf{B} + \tfrac{1}{3}\mathbf{C}$

f $\mathbf{C} - \tfrac{5}{2}\mathbf{X} = \mathbf{A} - \tfrac{1}{2}\mathbf{C}$

$\therefore\ \mathbf{C} - \tfrac{5}{2}\mathbf{X} + (-\mathbf{C}) = \mathbf{A} - \tfrac{1}{2}\mathbf{C} + (-\mathbf{C})$

$\therefore\ -\tfrac{5}{2}\mathbf{X} + \mathbf{O} = \mathbf{A} - \tfrac{3}{2}\mathbf{C}$

$\therefore\ -\tfrac{2}{5}(-\tfrac{5}{2}\mathbf{X}) = -\tfrac{2}{5}(\mathbf{A} - \tfrac{3}{2}\mathbf{C})$

$\therefore\ \mathbf{X} = -\tfrac{2}{5}\mathbf{A} + \tfrac{3}{5}\mathbf{C}$

3 **a** $3\mathbf{A} - 2\mathbf{X} = 3\mathbf{B}$

$\therefore\ 3\mathbf{A} - 2\mathbf{X} + (-3\mathbf{A}) = 3\mathbf{B} + (-3\mathbf{A})$

$\therefore\ -2\mathbf{X} + \mathbf{O} = 3\mathbf{B} - 3\mathbf{A}$

$\therefore\ -\tfrac{1}{2}(-2\mathbf{X}) = -\tfrac{1}{2}(3\mathbf{B} - 3\mathbf{A})$

$\therefore\ \mathbf{X} = \tfrac{3}{2}(\mathbf{A} - \mathbf{B})$

b $\mathbf{X} = \tfrac{3}{2}(\mathbf{A} - \mathbf{B})$

$$= \tfrac{3}{2}\begin{pmatrix} -2 & 1 & -1 \\ -2 & 1 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & \tfrac{3}{2} & -\tfrac{3}{2} \\ -3 & \tfrac{3}{2} & -6 \end{pmatrix}$$

EXERCISE 1D.1

1 $$\mathbf{A} \text{ is } 1\times \underbrace{3 \quad \text{and} \quad \mathbf{B} \text{ is } 2}_{\neq} \times 3$$

The number of columns of **A** is not equal to the number of rows of **B**.

$\therefore$ **AB** cannot be found.

2 **a** **AB** exists if $n = m$.

b If **AB** exists, it has order 3×2.

c The number of columns of **B** (2) does not equal the number of rows of **A** (3).

$\therefore$ **BA** cannot be found.

3 **a** $$\mathbf{A} \text{ is } 2\times \underbrace{2 \quad \text{and} \quad \mathbf{B} \text{ is } 1}_{\neq} \times 2$$

$\therefore$ **AB** cannot be found.

b $$\mathbf{B} \text{ is } 1\times \underbrace{2 \quad \text{and} \quad \mathbf{A} \text{ is } 2}_{\checkmark} \times 2 \qquad \therefore \ \mathbf{BA} \text{ is } 1 \times 2$$

$$\begin{aligned}\mathbf{BA} &= \begin{pmatrix} 6 & 5 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 6\times 3 + 5\times 1 & 6\times 2 + 5 \times 4 \end{pmatrix} \\ &= \begin{pmatrix} 23 & 32 \end{pmatrix}\end{aligned}$$

4 **a** $$\mathbf{A} \text{ is } 1\times \underbrace{3 \quad \text{and} \quad \mathbf{B} \text{ is } 3}_{\checkmark} \times 1 \qquad \therefore \ \mathbf{AB} \text{ is } 1 \times 1$$

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix} 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\ &= (3\times 2 + 1 \times 3 + 4 \times 1) \\ &= (13)\end{aligned}$$

b $$\mathbf{B} \text{ is } 3\times \underbrace{1 \quad \text{and} \quad \mathbf{A} \text{ is } 1}_{\checkmark} \times 3 \qquad \therefore \ \mathbf{BA} \text{ is } 3 \times 3$$

$$\begin{aligned}\mathbf{BA} &= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2\times 3 & 2\times 1 & 2\times 4 \\ 3\times 3 & 3 \times 1 & 3\times 4 \\ 1\times 3 & 1\times 1 & 1\times 4 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 2 & 8 \\ 9 & 3 & 12 \\ 3 & 1 & 4 \end{pmatrix}\end{aligned}$$

5 **a** $$\begin{aligned}&\begin{pmatrix} 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 8 & 10 \end{pmatrix}\end{aligned}$$

b $$\begin{aligned}&\begin{pmatrix} -1 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -1\times 1 + 0 \times -3 + 1 \times 2 \\ -2 \times 1 + 2 \times -3 + -1 \times 2 \\ 0 \times 1 + 3 \times -3 + 1 \times 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -10 \\ -7 \end{pmatrix}\end{aligned}$$

c $$\begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 & -1 \\ 0 & -2 & 3 & 1 \\ 3 & 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 4 & 0 \\ 5 & -4 & 7 & 6 \end{pmatrix}$$

d $$\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & 4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 1 & -6 \\ 6 & 3 & 0 & -9 \\ -2 & 3 & 1 & 3 \end{pmatrix}$$

6 **a** $$\mathbf{C} = \begin{pmatrix} 8.95 \\ 12.95 \\ 9.95 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 156 & 193 & 218 \\ 183 & 284 & 257 \end{pmatrix}$$

b $$\begin{aligned}\mathbf{NC} &= \begin{pmatrix} 156 & 193 & 218 \\ 183 & 284 & 257 \end{pmatrix}\begin{pmatrix} 8.95 \\ 12.95 \\ 9.95 \end{pmatrix} \\ &= \begin{pmatrix} 156 \times 8.95 + 193 \times 12.95 + 218 \times 9.95 \\ 183 \times 8.95 + 284 \times 12.95 + 257 \times 9.95 \end{pmatrix} \\ &= \begin{pmatrix} 6064.65 \\ 7872.80 \end{pmatrix}\end{aligned}$$

NC gives the income for each month.

$\therefore$ the restaurant had an income of \$6064.65 in the first month and \$7872.80 in the second month.

c $$\begin{aligned}\text{Total income} &= \$6064.65 + \$7872.80 \\ &= \$13\,937.45\end{aligned}$$

EXERCISE 1D.2

1 **a** $$\begin{pmatrix} 2 & 6 & 0 & 7 \\ 3 & 2 & 8 & 6 \\ 1 & 4 & 0 & 2 \\ 3 & 0 & 1 & 8 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 6 \\ 11 \end{pmatrix} = \begin{pmatrix} 115 \\ 136 \\ 46 \\ 106 \end{pmatrix}$$

b $$\begin{aligned}&5.22\begin{pmatrix} 1 & 0 & 6 & 8 & 9 \\ 2 & 7 & 4 & 5 & 0 \\ 8 & 2 & 4 & 4 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 5.22 & 0 & 31.32 & 41.76 & 46.98 \\ 10.44 & 36.54 & 20.88 & 26.1 & 0 \\ 41.76 & 10.44 & 20.88 & 20.88 & 31.32 \end{pmatrix}\end{aligned}$$

c $$2\begin{pmatrix} 13 & 12 & 4 \\ 11 & 12 & 8 \\ 7 & 9 & 7 \end{pmatrix} + 3\begin{pmatrix} 3 & 6 & 11 \\ 2 & 9 & 8 \\ 3 & 13 & 17 \end{pmatrix} = \begin{pmatrix} 35 & 42 & 41 \\ 28 & 51 & 40 \\ 23 & 57 & 65 \end{pmatrix}$$

d $$\begin{aligned}&0.4\begin{pmatrix} 13 & 12 & 4 \\ 11 & 12 & 8 \\ 7 & 9 & 7 \end{pmatrix} - 1.3\begin{pmatrix} 3 & 6 & 11 \\ 2 & 9 & 8 \\ 3 & 13 & 17 \end{pmatrix} \\ &= \begin{pmatrix} 1.3 & -3 & -12.7 \\ 1.8 & -6.9 & -7.2 \\ -1.1 & -13.3 & -19.3 \end{pmatrix}\end{aligned}$$

2 $$\text{Prices matrix} = \begin{pmatrix} 125 \\ 315 \\ 405 \\ 375 \end{pmatrix}$$

$$\begin{aligned}\text{Monthly income} &= \begin{pmatrix} 50 & 42 & 18 & 65 \\ 65 & 37 & 25 & 82 \\ 120 & 29 & 23 & 75 \\ 42 & 36 & 19 & 72 \end{pmatrix}\begin{pmatrix} 125 \\ 315 \\ 405 \\ 375 \end{pmatrix} \\ &= \begin{pmatrix} 51\,145 \\ 60\,655 \\ 61\,575 \\ 51\,285 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore \ \text{total income} &= \$51\,145 + \$60\,655 + \$61\,575 + \$51\,285 \\ &= \$224\,660\end{aligned}$$

EXERCISE 1D.3

1 **a** $\mathbf{X}(2\mathbf{X}+\mathbf{I})$
$= 2\mathbf{X}^2+\mathbf{XI}$
$= 2\mathbf{X}^2+\mathbf{X}$

b $(3\mathbf{I}+\mathbf{B})\mathbf{B}$
$= 3\mathbf{IB}+\mathbf{B}^2$
$= 3\mathbf{B}+\mathbf{B}^2$

c $\mathbf{D}(\mathbf{D}^2+3\mathbf{D}+2\mathbf{I})$
$= \mathbf{D}^3+3\mathbf{D}^2+2\mathbf{DI}$
$= \mathbf{D}^3+3\mathbf{D}^2+2\mathbf{D}$

d $(\mathbf{A}+\mathbf{B})(\mathbf{C}-\mathbf{D})$
$= (\mathbf{A}+\mathbf{B})\mathbf{C}+(\mathbf{A}+\mathbf{B})(-\mathbf{D})$
$= \mathbf{AC}+\mathbf{BC}-\mathbf{AD}-\mathbf{BD}$

e $(\mathbf{B}-\mathbf{A})(\mathbf{B}+\mathbf{A})$
$= (\mathbf{B}-\mathbf{A})\mathbf{B}+(\mathbf{B}-\mathbf{A})\mathbf{A}$
$= \mathbf{B}^2-\mathbf{AB}+\mathbf{BA}-\mathbf{A}^2$

f $(\mathbf{A}-2\mathbf{I})^2$
$= (\mathbf{A}-2\mathbf{I})(\mathbf{A}-2\mathbf{I})$
$= (\mathbf{A}-2\mathbf{I})\mathbf{A}+(\mathbf{A}-2\mathbf{I})(-2\mathbf{I})$
$= \mathbf{A}^2-2\mathbf{IA}-2\mathbf{AI}+4\mathbf{I}^2$
$= \mathbf{A}^2-2\mathbf{A}-2\mathbf{A}+4\mathbf{I}$
$= \mathbf{A}^2-4\mathbf{A}+4\mathbf{I}$

g $(5\mathbf{I}-2\mathbf{B})^2$
$= (5\mathbf{I}-2\mathbf{B})(5\mathbf{I}-2\mathbf{B})$
$= (5\mathbf{I}-2\mathbf{B})5\mathbf{I}+(5\mathbf{I}-2\mathbf{B})(-2\mathbf{B})$
$= 25\mathbf{I}^2-10\mathbf{BI}-10\mathbf{IB}+4\mathbf{B}^2$
$= 25\mathbf{I}-10\mathbf{B}-10\mathbf{B}+4\mathbf{B}^2$
$= 25\mathbf{I}-20\mathbf{B}+4\mathbf{B}^2$

h $(\mathbf{A}+\mathbf{B})^3$
$= (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})$
$= [(\mathbf{A}+\mathbf{B})\mathbf{A}+(\mathbf{A}+\mathbf{B})\mathbf{B}](\mathbf{A}+\mathbf{B})$
$= (\mathbf{A}^2+\mathbf{BA}+\mathbf{AB}+\mathbf{B}^2)(\mathbf{A}+\mathbf{B})$
$= (\mathbf{A}^2+\mathbf{BA}+\mathbf{AB}+\mathbf{B}^2)\mathbf{A}+(\mathbf{A}^2+\mathbf{BA}+\mathbf{AB}+\mathbf{B}^2)\mathbf{B}$
$= \mathbf{A}^3+\mathbf{BA}^2+\mathbf{ABA}+\mathbf{B}^2\mathbf{A}+\mathbf{A}^2\mathbf{B}+\mathbf{BAB}+\mathbf{AB}^2+\mathbf{B}^3$

2 **a** $\mathbf{A}^3 = \mathbf{A}\times\mathbf{A}^2$
$= \mathbf{A}(3\mathbf{A}+2\mathbf{I})$
$= 3\mathbf{A}^2+2\mathbf{AI}$
$= 3(3\mathbf{A}+2\mathbf{I})+2\mathbf{A}$
$= 9\mathbf{A}+6\mathbf{I}+2\mathbf{A}$
$= 11\mathbf{A}+6\mathbf{I}$

b $\mathbf{A}^4 = \mathbf{A}\times\mathbf{A}^3$
$= \mathbf{A}(11\mathbf{A}+6\mathbf{I})$
$= 11\mathbf{A}^2+6\mathbf{AI}$
$= 11(3\mathbf{A}+2\mathbf{I})+6\mathbf{A}$
$= 33\mathbf{A}+22\mathbf{I}+6\mathbf{A}$
$= 39\mathbf{A}+22\mathbf{I}$

c $\mathbf{A}^8 = \mathbf{A}^4\times\mathbf{A}^4$
$= (39\mathbf{A}+22\mathbf{I})(39\mathbf{A}+22\mathbf{I})$
$= (39\mathbf{A}+22\mathbf{I})39\mathbf{A}+(39\mathbf{A}+22\mathbf{I})22\mathbf{I}$
$= 1521\mathbf{A}^2+858\mathbf{IA}+858\mathbf{AI}+484\mathbf{I}^2$
$= 1521(3\mathbf{A}+2\mathbf{I})+858\mathbf{A}+858\mathbf{A}+484\mathbf{I}$
$= 4563\mathbf{A}+3042\mathbf{I}+1716\mathbf{A}+484\mathbf{I}$
$= 6279\mathbf{A}+3526\mathbf{I}$

3 **a** $\mathbf{A}(2\mathbf{A}+3\mathbf{I})$
$= 2\mathbf{A}^2+3\mathbf{AI}$
$= 2\mathbf{I}+3\mathbf{A}$

b $(\mathbf{A}-\mathbf{I})^2$
$= (\mathbf{A}-\mathbf{I})(\mathbf{A}-\mathbf{I})$
$= (\mathbf{A}-\mathbf{I})\mathbf{A}+(\mathbf{A}-\mathbf{I})(-\mathbf{I})$
$= \mathbf{A}^2-\mathbf{IA}-\mathbf{AI}+\mathbf{I}^2$
$= \mathbf{I}-\mathbf{A}-\mathbf{A}+\mathbf{I}$
$= 2\mathbf{I}-2\mathbf{A}$

c $\mathbf{A}(\mathbf{A}+5\mathbf{I})^2$
$= \mathbf{A}(\mathbf{A}+5\mathbf{I})(\mathbf{A}+5\mathbf{I})$
$= (\mathbf{A}^2+5\mathbf{AI})(\mathbf{A}+5\mathbf{I})$
$= (\mathbf{I}+5\mathbf{A})(\mathbf{A}+5\mathbf{I})$
$= (\mathbf{I}+5\mathbf{A})\mathbf{A}+(\mathbf{I}+5\mathbf{A})5\mathbf{I}$
$= \mathbf{IA}+5\mathbf{A}^2+5\mathbf{I}^2+25\mathbf{AI}$
$= \mathbf{A}+5\mathbf{I}+5\mathbf{I}+25\mathbf{A}$
$= 26\mathbf{A}+10\mathbf{I}$

4 **a** Consider $\mathbf{A} = \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$

$\therefore\ \mathbf{A}^2 = \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = \mathbf{O}$

So, we have found $\mathbf{A}^2 = \mathbf{O}$ where $\mathbf{A} \neq \mathbf{O}$
$\therefore\ \mathbf{A}^2 = \mathbf{O} \not\Rightarrow \mathbf{A} = \mathbf{O}$

b Consider $\mathbf{A} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$

$\therefore\ \mathbf{A}^2 = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} = \mathbf{A}$

So, we have found $\mathbf{A}^2 = \mathbf{A}$ where $\mathbf{A}$ is not $\mathbf{O}$ or $\mathbf{I}$.
$\therefore\ \mathbf{A}^2 = \mathbf{A} \not\Rightarrow \mathbf{A} = \mathbf{O}$ or $\mathbf{I}$.

5 Let $\mathbf{A} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$.

If $\mathbf{A}^2 = \mathbf{A}$, then $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$

$\therefore\ \begin{pmatrix}a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$

Equating corresponding elements:

$a^2+bc = a \quad \therefore \quad bc = a(1-a)$ .... (1)
$ab+bd = b \quad \therefore \quad b(a+d-1) = 0$ .... (2)
$ac+cd = c \quad \therefore \quad c(a+d-1) = 0$ .... (3)
$bc+d^2 = d \quad \therefore \quad bc = d(1-d)$ .... (4)

If $\underline{b = c = 0}$ then from (1) and (4), $a = 0$ or 1
and $d = 0$ or 1

$\therefore$ $\mathbf{A}$ is $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, or $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

If $\underline{b \text{ and } c \text{ are not both } 0}$ then $a+d-1 = 0$
$\therefore\ d = 1-a$

$\therefore$ $\mathbf{A}$ is of the form $\begin{pmatrix}a & b\\ c & 1-a\end{pmatrix}$ where $bc = a(1-a)$.

6 The last step is invalid as
$\mathbf{A}(\mathbf{A}-2\mathbf{I}) = \mathbf{O} \not\Rightarrow \mathbf{A} = \mathbf{O}$ or $\mathbf{A} = 2\mathbf{I}$.

For example, $\mathbf{A} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $\mathbf{B} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

have $\mathbf{AB} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = \mathbf{O}$

but $\mathbf{A} \neq \mathbf{O}$ and $\mathbf{B} \neq \mathbf{O}$.

7 In the binomial expansion for real numbers $ab = ba$ is essential.
But for matrices $\mathbf{AB} \neq \mathbf{BA}$ {non-commutative} in general.
$\therefore$ $(\mathbf{A}+\mathbf{B})^n$ cannot be found using the binomial expansion.
However, $\mathbf{A}$ and $\mathbf{I}$ commute as $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ {identity law}.
$\therefore$ expansions of the form $(\mathbf{A}+k\mathbf{I})^n$, $n \in \mathbb{Z}^+$ can be found using the binomial expansion.

EXERCISE 1E

1 $\mathbf{A} = \begin{pmatrix}1 & 3 & 2\\ 0 & 4 & 1\end{pmatrix}$, $\mathbf{B} = \begin{pmatrix}2 & 0 & 3\\ 1 & -2 & -1\end{pmatrix}$

a $\mathbf{A}^T = \begin{pmatrix}1 & 0\\ 3 & 4\\ 2 & 1\end{pmatrix}$ $\quad\therefore\ (\mathbf{A}^T)^T = \begin{pmatrix}1 & 3 & 2\\ 0 & 4 & 1\end{pmatrix} = \mathbf{A}$

b $\mathbf{A}+\mathbf{B} = \begin{pmatrix}3 & 3 & 5\\ 1 & 2 & 0\end{pmatrix}$ $\quad\therefore\ (\mathbf{A}+\mathbf{B})^T = \begin{pmatrix}3 & 1\\ 3 & 2\\ 5 & 0\end{pmatrix}$

c $\mathbf{A}^T+\mathbf{B}^T=\begin{pmatrix}1&0\\3&4\\2&1\end{pmatrix}+\begin{pmatrix}2&1\\0&-2\\3&-1\end{pmatrix}=\begin{pmatrix}3&1\\3&2\\5&0\end{pmatrix}$

d $3\mathbf{A}=\begin{pmatrix}3&9&6\\0&12&3\end{pmatrix}\quad\therefore\quad(3\mathbf{A})^T=\begin{pmatrix}3&0\\9&12\\6&3\end{pmatrix}$

e $3\mathbf{A}^T=3\begin{pmatrix}1&0\\3&4\\2&1\end{pmatrix}=\begin{pmatrix}3&0\\9&12\\6&3\end{pmatrix}$

f $(\mathbf{A}-\mathbf{B})^T=\begin{pmatrix}-1&3&-1\\-1&6&2\end{pmatrix}^T=\begin{pmatrix}-1&-1\\3&6\\-1&2\end{pmatrix}$

g $\mathbf{A}^T-\mathbf{B}^T=\begin{pmatrix}1&0\\3&4\\2&1\end{pmatrix}-\begin{pmatrix}2&1\\0&-2\\3&-1\end{pmatrix}=\begin{pmatrix}-1&-1\\3&6\\-1&2\end{pmatrix}$

h $(-2\mathbf{B})^T=\begin{pmatrix}-4&0&-6\\-2&4&2\end{pmatrix}^T=\begin{pmatrix}-4&-2\\0&4\\-6&2\end{pmatrix}$

i $-2\mathbf{B}^T=-2\begin{pmatrix}2&1\\0&-2\\3&-1\end{pmatrix}=\begin{pmatrix}-4&-2\\0&4\\-6&2\end{pmatrix}$

2 **a** $\mathbf{A}=\begin{pmatrix}1&0&1\\2&1&3\\-1&4&1\end{pmatrix},\quad\mathbf{B}=\begin{pmatrix}-1&2&1\\0&3&2\\4&0&5\end{pmatrix}$

$\therefore\quad\mathbf{AB}=\begin{pmatrix}3&2&6\\10&7&19\\5&10&12\end{pmatrix}$ {using technology}

b $(\mathbf{AB})^T=\begin{pmatrix}3&10&5\\2&7&10\\6&19&12\end{pmatrix}$

c
$$\begin{aligned}\mathbf{A}^T\mathbf{B}^T&=\begin{pmatrix}1&2&-1\\0&1&4\\1&3&1\end{pmatrix}\begin{pmatrix}-1&0&4\\2&3&0\\1&2&5\end{pmatrix}\\&=\begin{pmatrix}2&4&-1\\6&11&20\\-1&20&9\end{pmatrix}\end{aligned}$$

d
$$\begin{aligned}\mathbf{B}^T\mathbf{A}^T&=\begin{pmatrix}-1&0&4\\2&3&0\\1&2&5\end{pmatrix}\begin{pmatrix}1&2&-1\\0&1&4\\1&3&1\end{pmatrix}\\&=\begin{pmatrix}3&10&5\\2&7&10\\6&19&12\end{pmatrix}\end{aligned}$$

3 **a** Let $\mathbf{A}=(a_{ij})$,
then $\mathbf{A}^T=(a_{ji})$
and $(\mathbf{A}^T)^T=(a_{ij})$
$=\mathbf{A}$

b Let $\mathbf{A}=(a_{ij})$, $\mathbf{B}=(b_{ij})$
$$\begin{aligned}\therefore\quad\mathbf{A}+\mathbf{B}&=(a_{ij})+(b_{ij})\\ \therefore\quad\mathbf{A}+\mathbf{B}&=(a_{ij}+b_{ij})\\ \therefore\quad(\mathbf{A}+\mathbf{B})^T&=(a_{ji}+b_{ji})\\&=(a_{ji})+(b_{ji})\\&=\mathbf{A}^T+\mathbf{B}^T\end{aligned}$$

c Let $\mathbf{A}=(a_{ij})$, $\quad\therefore\quad s\mathbf{A}=s(a_{ij})$
$$\begin{aligned}&=(sa_{ij})\\ \therefore\quad(s\mathbf{A})^T&=(sa_{ji})=s\mathbf{A}^T\end{aligned}$$

4 **a**
$$\begin{aligned}(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3)^T&=[(\mathbf{A}_1\mathbf{A}_2)\mathbf{A}_3]^T\\&=\mathbf{A}_3{}^T(\mathbf{A}_1\mathbf{A}_2)^T\qquad\{(\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T\}\\&=\mathbf{A}_3{}^T\mathbf{A}_2{}^T\mathbf{A}_1{}^T\end{aligned}$$

b **Proof:** (By the principle of mathematical induction)

P_n is: $(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\,....\,\mathbf{A}_n)^T=\mathbf{A}_n^T\mathbf{A}_{n-1}^T\,....\,\mathbf{A}_3{}^T\mathbf{A}_2{}^T\mathbf{A}_1{}^T$ for all $n\in\mathbb{Z}^+$.

(1) If $n=1$, $\mathbf{A}_1^T=\mathbf{A}_1^T$ is true.
$\therefore$ P_1 is true.

(2) If P_k is true, then
$(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\,....\,\mathbf{A}_k)^T=\mathbf{A}_k^T\,....\,\mathbf{A}_3{}^T\mathbf{A}_2{}^T\mathbf{A}_1{}^T$
$$\begin{aligned}\therefore\quad&(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\,....\,\mathbf{A}_k\mathbf{A}_{k+1})^T\\&=([\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\,....\,\mathbf{A}_k]\mathbf{A}_{k+1})^T\\&=\mathbf{A}_{k+1}^T[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\,....\,\mathbf{A}_k]^T\\&=\mathbf{A}_{k+1}^T\mathbf{A}_k^T\mathbf{A}_{k-1}^T\,....\,\mathbf{A}_3{}^T\mathbf{A}_2{}^T\mathbf{A}_1{}^T\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true.
Since P_1 is true, P_n is true for all $n\in\mathbb{Z}^+$.
{Principle of mathematical induction}

5 **a** If $\mathbf{A}$ is symmetric, then $\mathbf{A}^T=\mathbf{A}$

$\therefore$ $(\mathbf{A}^T)^T=\mathbf{A}=\mathbf{A}^T$ {Property 1}

$\therefore$ $\mathbf{A}^T$ is also symmetric {its transpose is equal to itself}

b If $\mathbf{A}$ and $\mathbf{B}$ are symmetric, then $\mathbf{A}^T=\mathbf{A}$ and $\mathbf{B}^T=\mathbf{B}$
$$\begin{aligned}\therefore\quad(\mathbf{A}+\mathbf{B})^T&=\mathbf{A}^T+\mathbf{B}^T\qquad\{\text{Property 2}\}\\&=\mathbf{A}+\mathbf{B}\end{aligned}$$
$\therefore$ $\mathbf{A}+\mathbf{B}$ is symmetric {its transpose is equal to itself}

c If $\mathbf{A}$ and $\mathbf{B}$ are symmetric, then $\mathbf{A}^T=\mathbf{A}$ and $\mathbf{B}^T=\mathbf{B}$.
$$\begin{aligned}\text{Now }\mathbf{AB}\text{ is symmetric}\quad&\Leftrightarrow\quad(\mathbf{AB})^T=\mathbf{AB}\\&\Leftrightarrow\quad\mathbf{B}^T\mathbf{A}^T=\mathbf{AB}\qquad\{\text{Property 4}\}\\&\Leftrightarrow\quad\mathbf{BA}=\mathbf{AB}\end{aligned}$$

6 $(\Rightarrow)$ If $\mathbf{A}$ is skew symmetric then $\mathbf{A}^T=-\mathbf{A}$
$$\begin{aligned}\therefore\quad(a_{ij})^T&=-(a_{ij})\\ \therefore\quad a_{ji}&=-a_{ij}\\ \therefore\quad a_{ij}&=-a_{ji}\end{aligned}$$

$(\Leftarrow)$ If $a_{ij}=-a_{ji}$
$$\begin{aligned}\therefore\quad a_{ij}+a_{ji}&=0\\ \therefore\quad(a_{ij}+a_{ji})&=0\\ \therefore\quad\mathbf{A}+\mathbf{A}^T&=\mathbf{0}\\ \therefore\quad\mathbf{A}^T&=-\mathbf{A}\end{aligned}$$

7 **a**
$$\begin{aligned}(\mathbf{AA}^T)^T&=(\mathbf{A}^T)^T\mathbf{A}^T\qquad\{\text{Property 4}\}\\&=\mathbf{AA}^T\qquad\{\text{Property 1}\}\end{aligned}$$
$\therefore$ $\mathbf{AA}^T$ is symmetric.
$$\begin{aligned}(\mathbf{A}^T\mathbf{A})^T&=\mathbf{A}^T(\mathbf{A}^T)^T\qquad\{\text{Property 4}\}\\&=\mathbf{A}^T\mathbf{A}\qquad\{\text{Property 1}\}\end{aligned}$$
$\therefore$ $\mathbf{A}^T\mathbf{A}$ is symmetric.
$$\begin{aligned}(\mathbf{A}+\mathbf{A}^T)^T&=\mathbf{A}^T+(\mathbf{A}^T)^T\qquad\{\text{Property 2}\}\\&=\mathbf{A}^T+\mathbf{A}\qquad\{\text{Property 1}\}\\&=\mathbf{A}+\mathbf{A}^T\end{aligned}$$
$\therefore$ $\mathbf{A}+\mathbf{A}^T$ is symmetric.

b
$$\begin{aligned}(\mathbf{A}-\mathbf{A}^T)^T&=\mathbf{A}^T-(\mathbf{A}^T)^T\qquad\{\text{Property 2}\}\\&=\mathbf{A}^T-\mathbf{A}\qquad\{\text{Property 1}\}\\&=-(\mathbf{A}-\mathbf{A}^T)\end{aligned}$$
$\therefore$ $\mathbf{A}-\mathbf{A}^T$ is skew symmetric.

8 **a** Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

b Example:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

9 **a** If $\mathbf{A}$ is symmetric, $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$.

Now
$$\begin{aligned}(\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P})^{\mathrm{T}} &= \mathbf{P}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}(\mathbf{P}^{\mathrm{T}})^{\mathrm{T}} \\ &= \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}\end{aligned}$$

$\therefore$ $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$ is symmetric.

b If $\mathbf{A}$ is skew symmetric, $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$

$$\begin{aligned}\therefore\quad (\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P})^{\mathrm{T}} &= \mathbf{P}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}(\mathbf{P}^{\mathrm{T}})^{\mathrm{T}} \\ &= \mathbf{P}^{\mathrm{T}}(-\mathbf{A})\mathbf{P} \\ &= -\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}\end{aligned}$$

$\therefore$ $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$ is skew symmetric.

EXERCISE 1F.1

1
$$\begin{pmatrix} 2 & -4 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} -\frac{5}{2} & 2 \\ -\frac{3}{2} & 1 \end{pmatrix} = \begin{pmatrix} -5+6 & 4-4 \\ -\frac{15}{2}+\frac{15}{2} & 6-5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore\quad \begin{pmatrix} 2 & -4 \\ 3 & -5 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{5}{2} & 2 \\ -\frac{3}{2} & 1 \end{pmatrix}$$

2 **a** Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$

$\therefore$ $\det\mathbf{A} = 3(1) - 2(4) = -5$

$$\therefore\quad \mathbf{A}^{-1} = \tfrac{1}{-5}\begin{pmatrix} 1 & -2 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

b Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$

$\therefore$ $\det\mathbf{A} = 2(3) - (-1) = 7$

$$\therefore\quad \mathbf{A}^{-1} = \tfrac{1}{7}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}$$

c Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$\therefore$ $\det\mathbf{A} = 1(0) - 1(1) = -1$

$$\therefore\quad \mathbf{A}^{-1} = \tfrac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

d Let $\mathbf{A} = \begin{pmatrix} 5 & 0 \\ 3 & 0 \end{pmatrix}$

$\therefore$ $\det\mathbf{A} = 5(0) - 3(0) = 0$

$\therefore$ $\mathbf{A}^{-1}$ does not exist.

e Let $\mathbf{A} = \begin{pmatrix} a & a \\ -a & 1 \end{pmatrix}$

$$\begin{aligned}\therefore\quad \det\mathbf{A} &= a(1) - a(-a) \\ &= a + a^2 \\ &= a(1+a)\end{aligned}$$

$\therefore$ $\mathbf{A}^{-1}$ exists provided $a(1+a) \neq 0$, that is, provided $a \neq 0$ or -1.

If $a \neq 0$ or -1,
$$\mathbf{A}^{-1} = \frac{1}{a(1+a)}\begin{pmatrix} 1 & -a \\ a & a \end{pmatrix} = \begin{pmatrix} \dfrac{1}{a(1+a)} & \dfrac{-1}{1+a} \\ \dfrac{1}{1+a} & \dfrac{1}{1+a} \end{pmatrix}$$

3 **a**
$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$$

$\therefore$ $\det\mathbf{A} = (-1)(-5) - (2)(3) = -1$

b
$$-\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix}$$

$\therefore$ $\det(-\mathbf{A}) = 1(5) - (-2)(-3) = -1$

c
$$2\mathbf{A} = \begin{pmatrix} -2 & 4 \\ 6 & -10 \end{pmatrix}$$

$\therefore$ $\det(2\mathbf{A}) = (-2)(-10) - 4(6) = -4$

4 Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\therefore$ $k\mathbf{A} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

and
$$\begin{aligned}\det(k\mathbf{A}) &= (ka)(kd) - (kb)(kc) \\ &= k^2ad - k^2bc \\ &= k^2(ad - bc) \\ &= k^2\det(\mathbf{A})\end{aligned}$$

5 $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$

$\therefore$ $|\,\mathbf{A}\,| = ad - bc$ and $|\,\mathbf{B}\,| = wz - xy$

Now
$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix}$$

$$\begin{aligned}\therefore\quad |\,\mathbf{AB}\,| &= (aw+by)(cx+dz) - (ax+bz)(cw+dy) \\ &= \cancel{acwx} + adwz + bcxy + \cancel{bdyz} \\ &\qquad \cancel{-\,acwx} - adxy - bcwz \cancel{-\,bdyz} \\ &= ad(wz - xy) + bc(xy - wz) \\ &= ad(wz - xy) - bc(wz - xy) \\ &= (wz - xy)(ad - bc) \\ &= |\,\mathbf{B}\,|\,|\,\mathbf{A}\,| \\ &= |\,\mathbf{A}\,|\,|\,\mathbf{B}\,|\end{aligned}$$

6 **a**
$$\mathbf{AB} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -4 & 6 \\ 1 & -1 \end{pmatrix} \quad \{2\times\underset{\uparrow}{3}\ \text{by}\ \underset{\uparrow}{3}\times 2\}$$

✓

$$\begin{aligned}&= \begin{pmatrix} -1+0+2 & 2+0-2 \\ 1-4+3 & -2+6-3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathbf{I}\end{aligned}$$

b As $\mathbf{BA}$ is 3×3 and $\mathbf{AB}$ is 2×2, $\mathbf{AB} \neq \mathbf{BA}$ and $\mathbf{A}$, $\mathbf{B}$ cannot be inverses.
The inverse of $\mathbf{A}$ must satisfy $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

c As $\mathbf{A}$ must have the same number of rows as columns, $\mathbf{A}$ must be square.

7 **a** $\mathbf{A} = \begin{pmatrix} 2 & k \\ 3 & -6 \end{pmatrix}$

$$\begin{aligned}\therefore\quad \det\mathbf{A} &= 2(-6) - k(3) \\ &= -12 - 3k \\ &= -3(k+4)\end{aligned}$$

$$\mathbf{A}^{-1} = \frac{1}{-3(k+4)}\begin{pmatrix} -6 & -k \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \dfrac{2}{k+4} & \dfrac{k}{3(k+4)} \\ \dfrac{1}{k+4} & \dfrac{-2}{3(k+4)} \end{pmatrix}$$

$\therefore$ $\mathbf{A}^{-1}$ exists provided $-3(k+4) \neq 0$

$\therefore$ $k \neq -4$

b $\mathbf{A} = \begin{pmatrix} k-5 & -3 \\ 2 & k \end{pmatrix}$

$$\begin{aligned} \therefore\ \det \mathbf{A} &= (k-5)(k) - (-3)(2) \\ &= k^2 - 5k + 6 \\ &= (k-3)(k-2) \end{aligned}$$

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{(k-3)(k-2)} \begin{pmatrix} k & 3 \\ -2 & k-5 \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{k}{(k-3)(k-2)} & \dfrac{3}{(k-3)(k-2)} \\ \dfrac{-2}{(k-3)(k-2)} & \dfrac{k-5}{(k-3)(k-2)} \end{pmatrix} \end{aligned}$$

$\therefore\ \mathbf{A}^{-1}$ exists provided $(k-3)(k-2) \neq 0$

$\therefore\ k \neq 3$ or 2

c $\mathbf{A} = \begin{pmatrix} k & 12 \\ k+1 & k+5 \end{pmatrix}$

$$\begin{aligned} \therefore\ \det \mathbf{A} &= k(k+5) - 12(k+1) \\ &= k^2 + 5k - 12k - 12 \\ &= k^2 - 7k - 12 \end{aligned}$$

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{k^2-7k-12} \begin{pmatrix} k+5 & -12 \\ -k-1 & k \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{k+5}{k^2-7k-12} & \dfrac{-12}{k^2-7k-12} \\ \dfrac{-k-1}{k^2-7k-12} & \dfrac{k}{k^2-7k-12} \end{pmatrix} \end{aligned}$$

$\therefore\ \mathbf{A}^{-1}$ exists provided $k^2 - 7k - 12 \neq 0$.

Now $k^2 - 7k - 12 = 0$ when $k = \dfrac{7 \pm \sqrt{49+48}}{2}$

$= \dfrac{7 \pm \sqrt{97}}{2}$

$\therefore\ \mathbf{A}^{-1}$ exists provided $k \neq \dfrac{7 \pm \sqrt{97}}{2}$

EXERCISE 1F.2

1 **a** If $\mathbf{AXB} = \mathbf{C}$ we premultiply both sides by $\mathbf{A}^{-1}$ and postmultiply both sides by $\mathbf{B}^{-1}$.

$$\begin{aligned} \therefore\ \mathbf{A}^{-1}(\mathbf{AXB})\mathbf{B}^{-1} &= \mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ \therefore\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{X}(\mathbf{B}\mathbf{B}^{-1}) &= \mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ \therefore\ \mathbf{I}\,\mathbf{X}\,\mathbf{I} &= \mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ \therefore\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1} \end{aligned}$$

b If $\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \mathbf{X} \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$

then $\mathbf{X} = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}^{-1}$ {using **a**}

$$\begin{aligned} \therefore\ \mathbf{X} &= 1\begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} 1 \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 16 & 18 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -20 & -22 \\ -6 & -7 \end{pmatrix} \end{aligned}$$

2 **a** If $\mathbf{X} = \mathbf{AY}$ and $\mathbf{Y} = \mathbf{BZ}$ then $\mathbf{X} = \mathbf{AY}$

$= \mathbf{ABZ}$

b Using **a**, $\mathbf{A}^{-1}\mathbf{X} = \mathbf{A}^{-1}\mathbf{ABZ}$ {premultiplying by $\mathbf{A}^{-1}$}

$$\begin{aligned} \therefore\ \mathbf{A}^{-1}\mathbf{X} &= \mathbf{IBZ} \\ \therefore\ \mathbf{A}^{-1}\mathbf{X} &= \mathbf{BZ} \\ \therefore\ \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X} &= \mathbf{B}^{-1}\mathbf{BZ} \quad \{\text{premultiplying by } \mathbf{B}^{-1}\} \\ \therefore\ \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X} &= \mathbf{IZ} \\ \therefore\ \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X} &= \mathbf{Z} \\ \therefore\ \mathbf{Z} &= \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X} \end{aligned}$$

3 $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$

$$\begin{aligned} \therefore\ \mathbf{A}^2 &= \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -12 \\ -12 & 29 \end{pmatrix} \\ &= p\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} p+q & -2p \\ -2p & 5p+q \end{pmatrix} \end{aligned}$$

$\therefore\ p+q = 5,\ -2p = -12,$ and $5p+q = 29$

$\therefore\ p = 6$

Substituting in $p+q=5$

gives $6+q=5$

$\therefore\ q = -1$

Substituting in $5p+q=29$ {to check consistency}

gives $5(6)+q=29$

$\therefore\ q=-1$ ✓

$\therefore\ \mathbf{A}^2 = 6\mathbf{A} - \mathbf{I}$ which is of the form $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$ where $p=6$ and $q=-1$.

$$\begin{aligned} \therefore\ \mathbf{A}^2\mathbf{A}^{-1} &= 6\mathbf{A}\mathbf{A}^{-1} - \mathbf{A}^{-1} \quad \{\text{postmultiplying by } \mathbf{A}^{-1}\} \\ \therefore\ \mathbf{A}(\mathbf{A}\mathbf{A}^{-1}) &= 6\mathbf{I} - \mathbf{A}^{-1} \\ \therefore\ \mathbf{AI} &= 6\mathbf{I} - \mathbf{A}^{-1} \\ \therefore\ \mathbf{A} &= 6\mathbf{I} - \mathbf{A}^{-1} \\ \therefore\ \mathbf{A}^{-1} &= 6\mathbf{I} - \mathbf{A} \\ \therefore\ \mathbf{A}^{-1} &= -\mathbf{A} + 6\mathbf{I} \end{aligned}$$

which is of the form $\mathbf{A}^{-1} = r\mathbf{A} + s\mathbf{I}$ where $r=-1$ and $s=6$.

4 **a**

$$\begin{aligned} \mathbf{A}^2 &= 2\mathbf{A} + \mathbf{I} \\ \therefore\ (\mathbf{AA})\mathbf{A}^{-1} &= 2\mathbf{A}\mathbf{A}^{-1} + \mathbf{I}\mathbf{A} \quad \{\text{postmultiplying by } \mathbf{A}^{-1}\} \\ \therefore\ \mathbf{A}(\mathbf{A}\mathbf{A}^{-1}) &= 2\mathbf{I} + \mathbf{A}^{-1} \\ \therefore\ \mathbf{AI} &= 2\mathbf{I} + \mathbf{A}^{-1} \\ \therefore\ \mathbf{A} &= 2\mathbf{I} + \mathbf{A}^{-1} \\ \therefore\ \mathbf{A}^{-1} &= \mathbf{A} - 2\mathbf{I} \end{aligned}$$

b

$$\begin{aligned} 3\mathbf{A} &= 2\mathbf{I} - \mathbf{A}^2 \\ \therefore\ 3\mathbf{A}\mathbf{A}^{-1} &= 2\mathbf{I}\mathbf{A}^{-1} - (\mathbf{AA})\mathbf{A}^{-1} \end{aligned}$$

{postmultiplying by $\mathbf{A}^{-1}$}

$$\begin{aligned} \therefore\ 3\mathbf{I} &= 2\mathbf{A}^{-1} - \mathbf{A}(\mathbf{A}\mathbf{A}^{-1}) \\ \therefore\ 3\mathbf{I} &= 2\mathbf{A}^{-1} - \mathbf{AI} \\ \therefore\ 3\mathbf{I} &= 2\mathbf{A}^{-1} - \mathbf{A} \\ \therefore\ 2\mathbf{A}^{-1} &= 3\mathbf{I} + \mathbf{A} \\ \therefore\ \mathbf{A}^{-1} &= \tfrac{3}{2}\mathbf{I} + \tfrac{1}{2}\mathbf{A} \end{aligned}$$

c

$$\begin{aligned} 2\mathbf{A}^2 - 3\mathbf{A} - \mathbf{I} &= \mathbf{O} \\ \therefore\ 2(\mathbf{AA})\mathbf{A}^{-1} - 3\mathbf{AA}^{-1} - \mathbf{IA}^{-1} &= \mathbf{O} \\ &\quad \{\text{post multiplying by } \mathbf{A}^{-1}\} \\ \therefore\ 2\mathbf{A}(\mathbf{AA}^{-1}) - 3\mathbf{I} - \mathbf{A}^{-1} &= \mathbf{O} \\ \therefore\ 2\mathbf{AI} - 3\mathbf{I} - \mathbf{A}^{-1} &= \mathbf{O} \\ \therefore\ 2\mathbf{A} - 3\mathbf{I} - \mathbf{A}^{-1} &= \mathbf{O} \\ \therefore\ \mathbf{A}^{-1} &= 2\mathbf{A} - 3\mathbf{I} \end{aligned}$$

5 If $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$,

$$\begin{aligned} \text{then}\quad \mathbf{A}^2 &= \mathbf{AA} \\ &= (\mathbf{AB})\mathbf{A} \\ &= \mathbf{A}(\mathbf{BA}) \qquad \{\text{associative rule}\} \\ &= \mathbf{AB} \\ \therefore\ \mathbf{A}^2 &= \mathbf{A} \end{aligned}$$

Note: As we do not know that inverses exist, we **cannot** conclude that $\mathbf{AB} = \mathbf{A} \Rightarrow \mathbf{B} = \mathbf{I}$.

6 We require the condition that:

A is non-singular or $|\mathbf{A}| \neq 0$

If this is so,

$$\begin{aligned} \mathbf{AB} = \mathbf{AC} &\Rightarrow \mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}(\mathbf{AC}) \\ &\Rightarrow (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{C} \\ &\Rightarrow \mathbf{IB} = \mathbf{IC} \\ &\Rightarrow \mathbf{B} = \mathbf{C} \end{aligned}$$

7 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$

$$\begin{aligned} \text{then}\quad \mathbf{X}^3 &= (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP}) \\ &= (\mathbf{P}^{-1}\mathbf{A})(\mathbf{PP}^{-1})\mathbf{A}(\mathbf{PP}^{-1})\mathbf{AP} \qquad \{\text{associative rule}\} \\ &= \mathbf{P}^{-1}\mathbf{AIAIAP} \\ &= \mathbf{P}^{-1}\mathbf{AAAP} \\ &= \mathbf{P}^{-1}\mathbf{A}^3\mathbf{P} \\ &= \mathbf{P}^{-1}\mathbf{IP} \\ &= \mathbf{P}^{-1}\mathbf{P} \\ &= \mathbf{I} \end{aligned}$$

8 If $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{O}$ and $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ then

$$\begin{aligned} &a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} \\ &= a(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP}) + b(\mathbf{P}^{-1}\mathbf{AP}) + c\mathbf{I} \\ &= a\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})\mathbf{AP} + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I} \\ &= a\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I} \\ &= \mathbf{P}^{-1}(a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I})\mathbf{P} \\ &= \mathbf{P}^{-1}\mathbf{OP} \\ &= \mathbf{O} \end{aligned}$$

9 $\mathbf{A}^2 + s\mathbf{A} + t\mathbf{I} = \mathbf{O}$

$$\text{then}\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + s\begin{pmatrix} a & b \\ c & d \end{pmatrix} + t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} + \begin{pmatrix} sa+t & sb \\ sc & sd+t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} a^2 + bc + sa + t &= 0 \quad \text{.... (1)} \\ ab + bd + sb &= 0 \quad \text{.... (2)} \\ ac + cd + sc &= 0 \quad \text{.... (3)} \\ bc + d^2 + sd + t &= 0 \quad \text{.... (4)} \end{aligned}$$

$$\begin{aligned} (1) - (4) \text{ gives}\quad a^2 - d^2 + s(a-d) &= 0 \\ \therefore\ (a+d)(a-d) + s(a-d) &= 0 \\ \therefore\ (a-d)(a+d+s) &= 0 \\ \therefore\ s &= -(a+d) \\ &\{\text{as } a \text{ is not necessarily equal to } d\} \end{aligned}$$

In (2) $b(a+d) + sb = 0$ ✓

In (3) $c(a+d) + sc = 0$ ✓

Substituting $s = -(a+d)$ in (1) gives

$$\begin{aligned} a^2 + bc - (a+d)a + t &= 0 \\ \therefore\ a^2 + bc - a^2 - ad + t &= 0 \\ \therefore\ t &= ad - bc \end{aligned}$$

10

$$\begin{aligned} \mathbf{AB}^{-1} &= \mathbf{B}^{-1}\mathbf{A} \\ \Rightarrow \mathbf{B}(\mathbf{AB}^{-1}) &= \mathbf{B}(\mathbf{B}^{-1}\mathbf{A}) \qquad \{\text{premultiplying by } \mathbf{B}\} \\ \Rightarrow \mathbf{BAB}^{-1} &= (\mathbf{BB}^{-1})\mathbf{A} \\ \Rightarrow \mathbf{BAB}^{-1} &= \mathbf{IA}^{-1} = \mathbf{A} \\ \Rightarrow (\mathbf{BAB}^{-1})\mathbf{B} &= \mathbf{AB} \qquad \{\text{postmultiplying by } \mathbf{B}\} \\ \Rightarrow \mathbf{BA}(\mathbf{B}^{-1}\mathbf{B}) &= \mathbf{AB} \\ \Rightarrow \mathbf{BAI} &= \mathbf{AB} \\ \Rightarrow \mathbf{AB} &= \mathbf{BA} \end{aligned}$$

11 **a** As $\mathbf{A}^{-1} = \mathbf{A}^T$

$\mathbf{A}^T\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

and $\mathbf{AA}^T = \mathbf{AA}^{-1} = \mathbf{I}$

So, as $\mathbf{A}^T\mathbf{A} = \mathbf{AA}^T = \mathbf{I}$, $\mathbf{A}^T$ and **A** are inverses

$\therefore\ \mathbf{A}^{-1} = \mathbf{A}^T$ (and $(\mathbf{A}^T)^{-1} = \mathbf{A}$)

b For **A**, **B** orthogonal, $\mathbf{A}^{-1} = \mathbf{A}^T$ and $\mathbf{B}^{-1} = \mathbf{B}^T$

$$\begin{aligned} \therefore\ (\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ &= \mathbf{B}^T\mathbf{A}^T \\ &= (\mathbf{AB})^T \end{aligned}$$

$\Rightarrow$ **AB** is orthogonal.

c For **A** orthogonal, $\mathbf{A}^{-1} = \mathbf{A}^T$

$$\begin{aligned} \therefore\ (\mathbf{A}^{-1})^{-1} &= (\mathbf{A}^T)^{-1} \\ &= (\mathbf{A}^{-1})^T \qquad \{\text{Property 4}\} \end{aligned}$$

$\Rightarrow$ $\mathbf{A}^{-1}$ is orthogonal.

EXERCISE 1F.3

1 **a**

$$\begin{aligned} \begin{vmatrix} 2 & 3 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 5 \end{vmatrix} &= 2\begin{vmatrix} 2 & 1 \\ 0 & 5 \end{vmatrix} - 3\begin{vmatrix} -1 & 1 \\ 2 & 5 \end{vmatrix} + 0\begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} \\ &= 2(10-0) - 3(-5-2) + 0 \\ &= 20 + 21 \\ &= 41 \quad ✓ \end{aligned}$$

b

$$\begin{aligned} &\begin{vmatrix} -1 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{vmatrix} \\ &= -1\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} \\ &= -1(0-0) - 2(1-0) - 3(2-0) \\ &= -8 \quad ✓ \end{aligned}$$

c

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} &= 2\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} \\ &= 2(3-2) - 1(-3-4) + 3(-1-2) \\ &= 0 \quad ✓ \end{aligned}$$

d

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} &= 1\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ &= 1(6-0) \\ &= 6 \quad ✓ \end{aligned}$$

e
$$\begin{aligned}\begin{vmatrix}0&0&2\\0&1&0\\3&0&0\end{vmatrix}&=0\begin{vmatrix}1&0\\0&0\end{vmatrix}-0\begin{vmatrix}0&0\\3&0\end{vmatrix}+2\begin{vmatrix}0&1\\3&0\end{vmatrix}\\&=2(0-3)\\&=-6\ \checkmark\end{aligned}$$

f
$$\begin{aligned}\begin{vmatrix}4&1&3\\-1&0&2\\-1&1&1\end{vmatrix}&=4\begin{vmatrix}0&2\\1&1\end{vmatrix}-1\begin{vmatrix}-1&2\\-1&1\end{vmatrix}+3\begin{vmatrix}-1&0\\-1&1\end{vmatrix}\\&=4(0-2)-1(-1-(-2))+3(-1-0)\\&=-12\ \checkmark\end{aligned}$$

g
$$\begin{vmatrix}1&0&1\\0&-1&1\\-2&1&0\end{vmatrix}$$
$$\begin{aligned}&=1\begin{vmatrix}-1&1\\1&0\end{vmatrix}-0\begin{vmatrix}0&1\\-2&0\end{vmatrix}+1\begin{vmatrix}0&-1\\-2&1\end{vmatrix}\\&=1(0-1)+1(0-2)\\&=-3\ \checkmark\end{aligned}$$

h
$$\begin{vmatrix}3&-1&2\\1&-4&1\\-3&1&-1\end{vmatrix}$$
$$\begin{aligned}&=3\begin{vmatrix}-4&1\\1&-1\end{vmatrix}-(-1)\begin{vmatrix}1&1\\-3&-1\end{vmatrix}+2\begin{vmatrix}1&-4\\-3&1\end{vmatrix}\\&=3(4-1)+(-1-(-3))+2(1-12)\\&=9+2-22\\&=-11\ \checkmark\end{aligned}$$

2 **a**
$$\begin{aligned}\begin{vmatrix}x&2&9\\3&1&2\\-1&0&x\end{vmatrix}&=x\begin{vmatrix}1&2\\0&x\end{vmatrix}-2\begin{vmatrix}3&2\\-1&x\end{vmatrix}+9\begin{vmatrix}3&1\\-1&0\end{vmatrix}\\&=x(x)-2(3x--2)+9(1)\\&=x^2-6x-4+9\\&=x^2-6x+5\\&=(x-5)(x-1)\end{aligned}$$

The matrix is singular when its determinant is 0, which occurs when $x=1$ or 5.

b This means that the matrix has in inverse for all x in $\mathbb{R}$, $x\neq 1$ or 5.

3 **a**
$$\begin{aligned}\begin{vmatrix}a&0&0\\0&b&0\\0&0&c\end{vmatrix}&=a\begin{vmatrix}b&0\\0&c\end{vmatrix}-0\begin{vmatrix}0&0\\0&c\end{vmatrix}+0\begin{vmatrix}0&b\\0&0\end{vmatrix}\\&=a(bc-0)\\&=abc\end{aligned}$$

b
$$\begin{aligned}\begin{vmatrix}0&x&y\\-x&0&z\\-y&-z&0\end{vmatrix}&=0\begin{vmatrix}0&z\\-z&0\end{vmatrix}-x\begin{vmatrix}-x&z\\-y&0\end{vmatrix}+y\begin{vmatrix}-x&0\\-y&-z\end{vmatrix}\\&=-x(0--yz)+y(xz)\\&=-xyz+xyz\\&=0\end{aligned}$$

c
$$\begin{aligned}\begin{vmatrix}a&b&c\\b&c&a\\c&a&b\end{vmatrix}&=a\begin{vmatrix}c&a\\a&b\end{vmatrix}-b\begin{vmatrix}b&a\\c&b\end{vmatrix}+c\begin{vmatrix}b&c\\c&a\end{vmatrix}\\&=a(cb-a^2)-b(b^2-ac)+c(ba-c^2)\\&=abc-a^3-b^3+abc+abc-c^3\\&=3abc-a^3-b^3-c^3\end{aligned}$$

4
$$\begin{aligned}\begin{vmatrix}a^2&1&1\\0&a&b\\1&0&0\end{vmatrix}&=a^2\begin{vmatrix}a&b\\0&0\end{vmatrix}-1\begin{vmatrix}0&b\\1&0\end{vmatrix}+1\begin{vmatrix}0&a\\1&0\end{vmatrix}\\&=a^2(0)-(0-b)+1(0-a)\\&=b-a\end{aligned}$$

$\therefore\ \begin{pmatrix}a^2&1&1\\0&a&b\\1&0&0\end{pmatrix}$ has an inverse for all $a, b\in\mathbb{R}$ where $b\neq a$.

5
$$\begin{aligned}\begin{vmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{vmatrix}&=1\begin{vmatrix}b&b^2\\c&c^2\end{vmatrix}-a\begin{vmatrix}1&b^2\\1&c^2\end{vmatrix}+a^2\begin{vmatrix}1&b\\1&c\end{vmatrix}\\&=bc^2-b^2c-a(c^2-b^2)+a^2(c-b)\\&=bc(c-b)-a(c+b)(c-b)+a^2(c-b)\\&=(c-b)\left[bc-ac-ab+a^2\right]\\&=(c-b)(a-b)(a-c)\\&=(a-b)(b-c)(c-a)\end{aligned}$$

EXERCISE 1F.4

1 **(Examples only)**

Property 1: If $\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}c&d\\a&b\end{pmatrix}$

then $|\mathbf{B}|=bc-ad=-(ad-bc)$

$\therefore\ |\mathbf{B}|=-|\mathbf{A}|$

Property 2: If $\mathbf{A}=\begin{pmatrix}a&b\\0&0\end{pmatrix}$, $|\mathbf{A}|=0-0=0$.

Property 3: If $\mathbf{A}=\begin{pmatrix}a&b\\a&b\end{pmatrix}$, $|\mathbf{A}|=ab-ab=0$.

Property 4: If $\mathbf{B}=\begin{pmatrix}a&b\\kc&kd\end{pmatrix}$,
$$\begin{aligned}|\mathbf{B}|&=kad-kbc\\&=k(ad-bc)\\&=k|\mathbf{A}|\end{aligned}$$

Property 5: If $\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and

$\mathbf{B}=\begin{pmatrix}a&b\\c+ka&d+kb\end{pmatrix}$,

then
$$\begin{aligned}|\mathbf{B}|&=a(d+kb)-b(c+ka)\\&=ad+\cancel{kab}-bc\cancel{-kab}\\&=ad-bc\\&=|\mathbf{A}|\end{aligned}$$

Property 6: If $\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}e&f\\g&h\end{pmatrix}$

then $\mathbf{AB}=\begin{pmatrix}ae+bg&af+bh\\ce+dg&cf+dh\end{pmatrix}$

$$\begin{aligned}\therefore\quad&|\mathbf{AB}|\\&=(ae+bg)(cf+dh)-(af+bh)(ce+dg)\\&=\cancel{acef}+adeh+bcfg+\cancel{bdgh}\\&\qquad\cancel{-acef}-adfg-bceh\cancel{-bdgh}\\&=ad(eh-fg)+bc(fg-eh)\\&=ad(eh-fg)-bc(eh-fg)\\&=(ad-bc)(eh-fg)\\&=|\mathbf{A}||\mathbf{B}|\end{aligned}$$

Property 7: If $\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, then $\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}a&c\\b&d\end{pmatrix}$

$\therefore\ \left|\mathbf{A}^{\mathrm{T}}\right|=\begin{vmatrix}a&c\\b&d\end{vmatrix}=ad-bc=|\mathbf{A}|$

2 **(Examples only)**

Property 1:

Let $\mathbf{A}=\begin{pmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} a_1 & b_1 & c_1\\ a_3 & b_3 & c_3\\ a_2 & b_2 & c_2\end{pmatrix}$

where $\mathbf{B}$ is obtained from $\mathbf{A}$ by interchanging rows 2 and 3, then

$$\begin{aligned}|\mathbf{B}| &= a_1\begin{vmatrix} b_3 & c_3\\ b_2 & c_2\end{vmatrix} - b_1\begin{vmatrix} a_3 & c_3\\ a_2 & c_2\end{vmatrix} + c_1\begin{vmatrix} a_3 & b_3\\ a_2 & b_2\end{vmatrix}\\ &= a_1(b_3c_2-b_2c_3)-b_1(a_3c_2-a_2c_3)\\ &\quad +c_1(a_3b_2-a_2b_3)\\ &= a_1b_3c_2-a_1b_2c_3-a_3b_1c_2+a_2b_1c_3\\ &\quad +a_3b_2c_1-a_2b_3c_1\\ &= -a_1(b_2c_3-b_3c_2)+b_1(a_2c_3-a_3c_2)\\ &\quad -c_1(a_2b_3-a_3b_2)\\ &= -|\mathbf{A}|\end{aligned}$$

$\therefore\ |\mathbf{B}|=-|\mathbf{A}|$

Property 2:

Let $\mathbf{A}=\begin{pmatrix} 0 & 0 & 0\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}$,

then $|\mathbf{A}|=0\begin{vmatrix} b_2 & c_2\\ b_3 & c_3\end{vmatrix}-0\begin{vmatrix} a_2 & c_2\\ a_3 & c_3\end{vmatrix}+0\begin{vmatrix} a_2 & b_2\\ a_3 & b_3\end{vmatrix}$

$\therefore\ |\mathbf{A}|=0$

Property 3:

Let $\mathbf{A}=\begin{pmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_2 & b_2 & c_2\end{pmatrix}$ where rows 2 and 3 are identical

$$\begin{aligned}\text{then}\quad |\mathbf{A}| &= a_1\begin{vmatrix} b_2 & c_2\\ b_2 & c_2\end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2\\ a_2 & c_2\end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2\\ a_2 & b_2\end{vmatrix}\\ &= a_1(b_2c_2-b_2c_2)-b_1(a_2c_2-a_2c_2)\\ &\quad +c_1(a_2b_2-a_2b_2)\\ &= 0\end{aligned}$$

Property 4:

Let $\mathbf{A}=\begin{pmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}$, then $\mathbf{B}=\begin{pmatrix} a_1 & b_1 & c_1\\ ka_2 & kb_2 & kc_2\\ a_3 & b_3 & c_3\end{pmatrix}$

where $\mathbf{B}$ is obtained from $\mathbf{A}$ by multiplying row 2 by k, $k\in\mathbb{R}$.

$$\begin{aligned}\therefore\ |\mathbf{B}| &= a_1\begin{vmatrix} kb_2 & kc_2\\ b_3 & c_3\end{vmatrix} - b_1\begin{vmatrix} ka_2 & kc_2\\ a_3 & c_3\end{vmatrix} + c_1\begin{vmatrix} ka_2 & kb_2\\ a_3 & b_3\end{vmatrix}\\ &= a_1(kb_2c_3-kc_2b_3)-b_1(ka_2c_3-kc_2a_3)\\ &\quad +c_1(ka_2b_3-kb_2a_3)\\ &= ka_1(b_2c_3-c_2b_3)-kb_1(a_2c_3-c_2a_3)\\ &\quad +kc_1(a_2b_3-b_2a_3)\\ &= k|\mathbf{A}|\end{aligned}$$

$\therefore\ |\mathbf{B}|=k|\mathbf{A}|$

Property 5:

Let $\mathbf{A}=\begin{pmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}$, then

$\mathbf{B}=\begin{pmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3+ka_2 & b_3+kb_2 & c_3+kc_2\end{pmatrix}$

where $\mathbf{B}$ is obtained from $\mathbf{A}$ by adding R_3 and kR_2.

$$\begin{aligned}\therefore\ |\mathbf{B}| &= (a_3+ka_2)\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix} - (b_3+kb_2)\begin{vmatrix} a_1 & c_1\\ a_2 & c_2\end{vmatrix}\\ &\quad + (c_3+kc_2)\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}\end{aligned}$$

{on expanding by row 3}

$$\begin{aligned}\Rightarrow\ |\mathbf{B}| &= a_3\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1\\ a_2 & c_2\end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}\\ &\quad + k\left(a_2\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix} - b_2\begin{vmatrix} a_1 & c_1\\ a_2 & c_2\end{vmatrix} + c_2\begin{vmatrix} a_1 & b_1\\ a_2 & b_2\end{vmatrix}\right)\\ &= |\mathbf{A}| + k\,(\cancel{a_2b_1c_2}\ \cancel{-a_2b_2c_1}\ \cancel{-a_1b_2c_2}+\cancel{a_2b_2c_1}\\ &\qquad +\cancel{a_1b_2c_2}\ \cancel{-a_2b_1c_2})\\ &= |\mathbf{A}|+k(0)\\ &= |\mathbf{A}|\end{aligned}$$

3 Let $\mathbf{A}=\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$

$\therefore\ k\mathbf{A}=\begin{pmatrix} ka & kb & kc\\ kd & ke & kf\\ kg & kh & ki\end{pmatrix}$

$$\begin{aligned}\therefore\ |k\mathbf{A}| &= ka(keki-kfkh)-kb(kdki-kfkg)\\ &\quad +kc(kdkh-kekg)\\ &= k^3a(ei-fh)-k^3b(di-fg)+k^3c(dh-eg)\\ &= k^3|\mathbf{A}|\end{aligned}$$

4 Let $\mathbf{A}=\begin{pmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}$

$\therefore\ \mathbf{A}^{\mathrm{T}}=\begin{pmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}$

$$\begin{aligned}\therefore\ |\mathbf{A}^{\mathrm{T}}| &= a_1\begin{vmatrix} b_2 & b_3\\ c_2 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3\\ c_1 & c_3\end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2\\ c_1 & c_2\end{vmatrix}\\ &= a_1\begin{vmatrix} b_2 & c_2\\ b_3 & c_3\end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1\\ b_3 & c_3\end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1\\ b_2 & c_2\end{vmatrix}\end{aligned}$$

$\{|\mathbf{B}^{\mathrm{T}}|=|\mathbf{B}|$ for 2×2 matrices$\}$

$=|\mathbf{A}|$ {by definition}

5 $\mathbf{A}=\begin{pmatrix} 1 & 3 & 0\\ 2 & -1 & 1\\ 4 & 1 & -2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 2 & -1 & 1\\ 1 & 2 & 3\\ 0 & 1 & 4\end{pmatrix}$

$\therefore\ |\mathbf{A}|=25$ and $|\mathbf{B}|=15$ {using technology}

Now $\mathbf{AB}=\begin{pmatrix} 5 & 5 & 10\\ 3 & -3 & 3\\ 9 & -4 & -1\end{pmatrix}$ {using technology}

where $|\mathbf{AB}|=375$ {using technology}

and as $|\mathbf{A}|\,|\mathbf{B}|=25\times 15=375$

$|\mathbf{AB}|=|\mathbf{A}|\,|\mathbf{B}|$ has been verified.

6 If $\mathbf{A}$ is non-singular, $\mathbf{A}^{-1}$ exists and $\mathbf{AA}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$

$\therefore\ |\mathbf{AA}^{-1}|=1$ $\{|\mathbf{I}|=1\}$

$\Rightarrow\ |\mathbf{A}|\,|\mathbf{A}^{-1}|=1$ $\{|\mathbf{AB}|=|\mathbf{A}|\,|\mathbf{B}|\}$

$\Rightarrow\ |\mathbf{A}^{-1}|=\dfrac{1}{|\mathbf{A}|}$ which exists as $|\mathbf{A}|\neq 0$

7 If $\mathbf{A}$ is orthogonal, $\mathbf{A}^T = \mathbf{A}^{-1}$

$$\begin{aligned}&\Rightarrow\ |\,\mathbf{A}^T\,| = |\,\mathbf{A}^{-1}\,|\\ &\Rightarrow\ |\,\mathbf{A}\,| = \frac{1}{|\,\mathbf{A}\,|} \qquad \{\,|\,\mathbf{A}^T\,| = |\,\mathbf{A}\,| \text{ and property 6}\}\\ &\Rightarrow\ |\,\mathbf{A}\,|^2 = 1\\ &\Rightarrow\ |\,\mathbf{A}\,| = \pm 1\end{aligned}$$

8 **a** $f(x) = \begin{vmatrix} 1 & x & x^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ and $\mathbf{A} = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$

Now $f(b) = f(c) = 0$ {two identical rows in each case}

$\therefore$ $(x-b)$ and $(x-c)$ are factors of $f(x)$
{factor theorem of polynomials}

b From **a**, $(x-b)(x-c)$ is a factor of $f(x)$

$\therefore$ $(a-b)(a-c)$ is a factor of $f(a)$

$\therefore$ $(a-b)(a-c)$ is a factor of $|\,\mathbf{A}\,|$ $\{f(a) = |\,\mathbf{A}\,|\}$

c Consider $g(x) = \begin{vmatrix} 1 & a & a^2 \\ 1 & x & x^2 \\ 1 & c & c^2 \end{vmatrix}$.

Now $g(a) = g(c) = 0$

$\therefore$ $(x-a)(x-c)$ is a factor of $g(x)$

$\therefore$ $(b-a)(b-c)$ is a factor of $g(b)$

$\therefore$ $(b-a)(b-c)$ is a factor of $|\,\mathbf{A}\,|$ $\{g(b) = |\,\mathbf{A}\,|\}$

9 $$(\mathbf{A}^T - 3\mathbf{I})^{-1} = \begin{pmatrix} 2 & 4 & 1 \\ 1 & 0 & 1 \\ 3 & 3 & 2 \end{pmatrix}$$

$$\therefore\ \mathbf{A}^T - 3\mathbf{I} = \begin{pmatrix} 2 & 4 & 1 \\ 1 & 0 & 1 \\ 3 & 3 & 2 \end{pmatrix}^{-1}$$

$$\therefore\ \mathbf{A}^T - 3\mathbf{I} = \begin{pmatrix} -3 & -5 & 4 \\ 1 & 1 & -1 \\ 3 & 6 & -4 \end{pmatrix} \qquad \{\text{using technology}\}$$

$$\therefore\ \mathbf{A}^T = \begin{pmatrix} 0 & -5 & 4 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{pmatrix}$$

$$\therefore\ \mathbf{A} = \begin{pmatrix} 0 & 1 & 3 \\ -5 & 4 & 6 \\ 4 & -1 & -1 \end{pmatrix}$$

EXERCISE 1G

1 **a** In matrix form, the system is $\begin{pmatrix} 2 & 4 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \end{pmatrix}$.

If $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 5 & -1 \end{pmatrix}$

then $|\,\mathbf{A}\,| = (2)(-1) - (4)(5) = -22$

$$\therefore\ \mathbf{A}^{-1} = -\tfrac{1}{22}\begin{pmatrix} -1 & -4 \\ -5 & 2 \end{pmatrix} = \tfrac{1}{22}\begin{pmatrix} 1 & 4 \\ 5 & -2 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} -6 \\ 7 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{22}\begin{pmatrix} 1 & 4 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} -6 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\therefore$ $x = 1$, $y = -2$

b In matrix form, the system is $\begin{pmatrix} 1 & -3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \end{pmatrix}$.

If $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ -3 & -2 \end{pmatrix}$

then $|\,\mathbf{A}\,| = (1)(-2) - (-3)(-3) = -11$

$$\therefore\ \mathbf{A}^{-1} = -\tfrac{1}{11}\begin{pmatrix} -2 & 3 \\ 3 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 13 \\ 5 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = -\tfrac{1}{11}\begin{pmatrix} -2 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 13 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$\therefore$ $x = 1$, $y = -4$

c In matrix form, the system is $\begin{pmatrix} 5 & 2 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \end{pmatrix}$.

If $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ -1 & -3 \end{pmatrix}$

then $|\,\mathbf{A}\,| = (5)(-3) - (2)(-1) = -13$

$$\therefore\ \mathbf{A}^{-1} = -\tfrac{1}{13}\begin{pmatrix} -3 & -2 \\ 1 & 5 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 15 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = -\tfrac{1}{13}\begin{pmatrix} -3 & -2 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 15 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

$\therefore$ $x = 3$, $y = -6$

d In matrix form, the system is $\begin{pmatrix} -2 & 5 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 20 \end{pmatrix}$.

If $\mathbf{A} = \begin{pmatrix} -2 & 5 \\ 3 & -2 \end{pmatrix}$

then $|\,\mathbf{A}\,| = (-2)(-2) - (5)(3) = -11$

$$\therefore\ \mathbf{A}^{-1} = -\tfrac{1}{11}\begin{pmatrix} -2 & -5 \\ -3 & -2 \end{pmatrix} = \tfrac{1}{11}\begin{pmatrix} 2 & 5 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 4 \\ 20 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{11}\begin{pmatrix} 2 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 20 \end{pmatrix} = \begin{pmatrix} \frac{108}{11} \\ \frac{52}{11} \end{pmatrix}$$

$\therefore$ $x = \frac{108}{11}$, $y = \frac{52}{11}$

2 **a** In matrix form, the system is $\mathbf{AX} = \mathbf{B}$

which is $\begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$ where

$\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$.

b $|\,\mathbf{A}\,| = (-3)(-2) - (1)(6) = 0$

$\therefore$ $\mathbf{A}$ is singular, and $\mathbf{A}^{-1}$ does not exist.

$\therefore$ the system does not have a unique solution.

3 **a** In matrix form, the system is $\mathbf{AX} = \mathbf{B}$

which is $\begin{pmatrix} 2 & -k \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ where

$\mathbf{A} = \begin{pmatrix} 2 & -k \\ -2 & 3 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$.

b $|\,\mathbf{A}\,| = (2)(3) - (-k)(-2) = 6 - 2k$

$|\,\mathbf{A}\,| = 0$ if $k = 3$

$\therefore$ $\mathbf{A}^{-1}$ exists if $k \neq 3$

$\therefore$ the system has a unique solution if $k \neq 3$.

4 **a** Yes, this matrix equation represents a system of 4 linear equations in 4 unknowns.

If $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 0 \\ -3 & 14 \end{pmatrix}$ have order 2×2 then $\mathbf{X}$ has order 2×2.

b $|\,\mathbf{A}\,| = (3)(4) - (-2)(1) = 14$

$$\therefore \quad \mathbf{A}^{-1} = \tfrac{1}{14}\begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\therefore \quad \mathbf{X} = \tfrac{1}{14}\begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ -3 & 14 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$

5 $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions of $\mathbf{AX} = \mathbf{B}$

$\therefore \quad \mathbf{AX}_1 = \mathbf{B}$ .... (1) and $\mathbf{AX}_2 = \mathbf{B}$ .... (2)

Now $\mathbf{X}_3 = t\mathbf{X}_1 + (1-t)\mathbf{X}_2$

$$\begin{aligned} \therefore \quad \mathbf{AX}_3 &= \mathbf{A}(t\mathbf{X}_1 + (1-t)\mathbf{X}_2) \\ &= t\mathbf{AX}_1 + (1-t)\mathbf{AX}_2 \\ &= t\mathbf{B} + (1-t)\mathbf{B} \qquad \{\text{using (1) and (2)}\} \\ &= t\mathbf{B} + \mathbf{B} - t\mathbf{B} \\ &= \mathbf{B} \end{aligned}$$

$\therefore$ $\mathbf{X}_3$ is also a solution of $\mathbf{AX} = \mathbf{B}$ for all $t \in \mathbb{R}$.

As $t \in \mathbb{R}$, $\mathbf{X}_3$ represents infinitely many solutions. So, if $\mathbf{AX} = \mathbf{B}$ has two solutions then it has infinitely many solutions.

6 **a** In matrix form, the system is

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ k & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 14 \end{pmatrix}$$

If $\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ k & 1 & 2 \end{pmatrix}$ then

$$\begin{aligned} |\,\mathbf{A}\,| &= 1\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ k & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -1 \\ k & 1 \end{vmatrix} \\ &= 1(-2 - -1) - 2(4 - -k) - 3(2 - -k) \\ &= -1 - 2(4+k) - 3(2+k) \\ &= -1 - 8 - 2k - 6 - 3k \\ &= -15 - 5k \end{aligned}$$

$\therefore$ $|\,\mathbf{A}\,| = 0$ if $k = -3$ $\therefore$ $\mathbf{A}^{-1}$ exists if $k \neq -3$

$\therefore$ the system has a unique solution for $k \in \mathbb{R}$, $k \neq -3$.

b In matrix form, the system is

$$\begin{pmatrix} 2 & -1 & -4 \\ 3 & -k & 1 \\ 5 & -1 & k \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$$

If $\mathbf{A} = \begin{pmatrix} 2 & -1 & -4 \\ 3 & -k & 1 \\ 5 & -1 & k \end{pmatrix}$ then

$$\begin{aligned} |\,\mathbf{A}\,| &= 2\begin{vmatrix} -k & 1 \\ -1 & k \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 5 & k \end{vmatrix} - 4\begin{vmatrix} 3 & -k \\ 5 & -1 \end{vmatrix} \\ &= 2(-k^2 - -1) + 1(3k - 5) - 4(-3 - -5k) \\ &= 2(1 - k^2) + (3k - 5) - 4(5k - 3) \\ &= 2 - 2k^2 + 3k - 5 - 20k + 12 \\ &= -2k^2 - 17k + 9 \\ &= (-2k + 1)(k + 9) \end{aligned}$$

$\therefore$ $|\,\mathbf{A}\,| = 0$ if $k = \frac{1}{2}$ or -9

$\therefore$ $\mathbf{A}^{-1}$ exists if $k \neq \frac{1}{2}$ or -9

$\therefore$ the system has a unique solution for $k \in \mathbb{R}$, $k \neq \frac{1}{2}$ or -9.

7 **a** In matrix form, the system is

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ -8 \\ 13 \end{pmatrix}$$

The system has the form $\mathbf{AX} = \mathbf{B}$, so $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

$$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 14 \\ -8 \\ 13 \end{pmatrix} = \begin{pmatrix} 2.3 \\ 1.3 \\ -4.5 \end{pmatrix}$$

{using technology}

$\therefore$ $x = 2.3$, $y = 1.3$, $z = -4.5$

b In matrix form, the system is

$$\begin{pmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 17 \end{pmatrix}$$

The system has the form $\mathbf{AX} = \mathbf{B}$, so $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

$$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ -6 \\ 17 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{95}{21} \\ \frac{2}{21} \end{pmatrix}$$

{using technology}

$\therefore$ $x = -\frac{1}{3}$, $y = -\frac{95}{21}$, $z = \frac{2}{21}$

c In matrix form, the system is

$$\begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \\ 0 \end{pmatrix}$$

The system has the form $\mathbf{AX} = \mathbf{B}$, so $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

$$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 15 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

{using technology}

$\therefore$ $x = 2$, $y = 4$, $z = -1$

d In matrix form, the system is

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ 7 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 23 \\ -23 \\ 62 \end{pmatrix}$$

The system has the form $\mathbf{AX} = \mathbf{B}$, so $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

$$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ 7 & 1 & -4 \end{pmatrix}^{-1}\begin{pmatrix} 23 \\ -23 \\ 62 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$$

{using technology}

$\therefore$ $x = 4$, $y = 6$, $z = -7$

e In matrix form, the system is

$$\begin{pmatrix} 10 & -1 & 4 \\ 7 & 3 & -5 \\ 13 & -17 & 23 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 89 \\ -309 \end{pmatrix}$$

The system has the form $\mathbf{AX} = \mathbf{B}$, so $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

$$\begin{aligned} \therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 10 & -1 & 4 \\ 7 & 3 & -5 \\ 13 & -17 & 23 \end{pmatrix}^{-1}\begin{pmatrix} -9 \\ 89 \\ -309 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 11 \\ -7 \end{pmatrix} \qquad \{\text{using technology}\} \end{aligned}$$

$\therefore$ $x = 3$, $y = 11$, $z = -7$

f In matrix form, the system is

$$\begin{pmatrix} 1.3 & 2.7 & -3.1 \\ 2.8 & -0.9 & 5.6 \\ 6.1 & 1.4 & -3.2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8.2 \\ 17.3 \\ -0.6 \end{pmatrix}$$

The system has the form $\mathbf{AX} = \mathbf{B}$, so $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1.3 & 2.7 & -3.1 \\ 2.8 & -0.9 & 5.6 \\ 6.1 & 1.4 & -3.2 \end{pmatrix}^{-1}\begin{pmatrix} 8.2 \\ 17.3 \\ -0.6 \end{pmatrix}$$

$$\approx \begin{pmatrix} 0.326 \\ 7.65 \\ 4.16 \end{pmatrix} \qquad \{\text{using technology}\}$$

$\therefore\ x \approx 0.326,\ \ y \approx 7.65,\ \ z \approx 4.16$

8 If $\mathbf{A}$ is non-singular the matrix method is excellent and a unique solution is easily obtained.

However, when $|\mathbf{A}| = 0$ the matrix method does not enable us to find out whether there are no solutions or infinitely many. The form of the infinitely many solutions is also not attainable.

9 **a** $2x + 3y + 8z = 352$

$x + 5y + 4z = 274$

$x + 2y + 11z = 351$

b In matrix form, the system is

$$\begin{pmatrix} 2 & 3 & 8 \\ 1 & 5 & 4 \\ 1 & 2 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 352 \\ 274 \\ 351 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 3 & 8 \\ 1 & 5 & 4 \\ 1 & 2 & 11 \end{pmatrix}^{-1}\begin{pmatrix} 352 \\ 274 \\ 351 \end{pmatrix} = \begin{pmatrix} 42 \\ 28 \\ 23 \end{pmatrix}$$

{using technology}

$\therefore$ the salaries for managers, clerks, and labourers are €42 000, €28 000, and €23 000 respectively.

c Total salary bill $= 3x + 8y + 37z$

$= 3(42) + 8(28) + 37(23)$

$= 1201$ thousand euros

$=$ €1 201 000

10 **a** Let x, y, and z represent the costs per kilogram (in dollars) of cashews, macadamias, and Brazil nuts respectively.

$\therefore\ 0.5x + 0.3y + 0.2z = 12.5$

$0.2x + 0.4y + 0.4z = 12.4$

$0.6x + 0.1y + 0.3z = 11.7$

In matrix form, the system is

$$\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.6 & 0.1 & 0.3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12.5 \\ 12.4 \\ 11.7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.6 & 0.1 & 0.3 \end{pmatrix}^{-1}\begin{pmatrix} 12.5 \\ 12.4 \\ 11.7 \end{pmatrix}$$

$$= \begin{pmatrix} 12 \\ 15 \\ 10 \end{pmatrix} \qquad \{\text{using technology}\}$$

$\therefore$ the costs per kilogram of cashews, macadamias, and Brazil nuts are \$12, \$15, and \$10 respectively.

b Cost per kilogram of 400 g of cashews, 200 g of macadamias, and 400 g of Brazil nuts

$= 0.4x + 0.2y + 0.4z$

$= 0.4(12) + 0.2(15) + 0.4(10)$

$=$ \$11.80

11 **a** $P(0) = b + \dfrac{c}{4} = 160\,000$

$P(1) = a + b + \dfrac{c}{5} = 198\,000$

$P(2) = 2a + b + \dfrac{c}{6} = 240\,000$

In matrix form, the system is

$$\begin{pmatrix} 0 & 1 & \frac{1}{4} \\ 1 & 1 & \frac{1}{5} \\ 2 & 1 & \frac{1}{6} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 160\,000 \\ 198\,000 \\ 240\,000 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 1 & \frac{1}{4} \\ 1 & 1 & \frac{1}{5} \\ 2 & 1 & \frac{1}{6} \end{pmatrix}^{-1}\begin{pmatrix} 160\,000 \\ 198\,000 \\ 240\,000 \end{pmatrix}$$

$$= \begin{pmatrix} 50\,000 \\ 100\,000 \\ 240\,000 \end{pmatrix} \qquad \{\text{using technology}\}$$

$\therefore\ a = 50\,000,\ \ b = 100\,000,\ \ c = 240\,000$

b $t = -1$ corresponds to 2009.

$\therefore\ P(-1) = 50\,000(-1) + 100\,000 + \dfrac{240\,000}{(-1) + 4}$

$= 130\,000$

$\therefore$ the profit of £130 000 in 2009 fits the model.

c Predicted profit for 2013 is

$P(3) = 50\,000(3) + 100\,000 + \dfrac{240\,000}{(3) + 4}$

$= \dfrac{1\,990\,000}{7}$

$\approx$ £284 000

Predicted profit for 2015 is

$P(5) = 50\,000(5) + 100\,000 + \dfrac{240\,000}{(5) + 4}$

$= \dfrac{1\,130\,000}{3}$

$\approx$ £377 000

12 **a** Let o, a, p, c, and l represent the cost per item (in dollars) of oranges, apples, pears, cabbages, and lettuces respectively.

In matrix form, the system is

$$\underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 5 & 2 & 2 \end{pmatrix}}\underset{\mathbf{X}}{\begin{pmatrix} o \\ a \\ p \\ c \\ l \end{pmatrix}} = \underset{\mathbf{B}}{\begin{pmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 10.9 \end{pmatrix}}$$

b Using technology, $|\mathbf{A}| = 0$

$\therefore$ $\mathbf{A}^{-1}$ does not exist and $\mathbf{X}$ cannot be found using this information.

c If the last line is amended,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 9.2 \end{pmatrix}$$

$|\mathbf{A}| = 6$ {using technology}

$\therefore$ $\mathbf{A}^{-1}$ exists

$\therefore$ $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ and the system can be solved

$$\therefore \begin{pmatrix} o \\ a \\ p \\ c \\ l \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 9.2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5 \\ 0.8 \\ 0.7 \\ 2 \\ 1.5 \end{pmatrix} \quad \{\text{using technology}\}$$

∴ oranges cost \$0.50 each, apples cost \$0.80 each, pears cost \$0.70 each, cabbages cost \$2.00 each, and lettuces cost \$1.50 each.

EXERCISE 1H

1 **a** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ **e** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$ **f** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$

g $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}$ **h** $\begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2 **a** $3R_3 \to R_3$ **b** $R_1 \leftrightarrow R_2$ **c** $R_3 - 2R_1 \to R_3$
d $-2R_2 \to R_2$ **e** $R_2 + 3R_3 \to R_2$

3 **a** $\mathbf{A} = \begin{pmatrix} -1 & 1 & 4 \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 7 \\ 0 & -5 & -4 \end{pmatrix} \qquad \begin{matrix} R_2 + R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{7}{3} \\ 0 & -5 & -4 \end{pmatrix} \qquad \tfrac{1}{3}R_2 \to R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{7}{3} \\ 0 & 0 & \frac{23}{3} \end{pmatrix} \qquad R_3 + 5R_2 \to R_3$$

b $\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$

$\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \quad \mathbf{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix},$

$\mathbf{E}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}$

and $\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{7}{3} \\ 0 & 0 & \frac{23}{3} \end{pmatrix}$, as required.

4 **a** **i** For the given system,
$(\mathbf{A} \mid \mathbf{I} \mid \mathbf{B})$

$$= \left(\begin{array}{cc|ccc|c} 1 & 3 & 1 & 0 & 0 & 4 \\ 2 & -1 & 0 & 1 & 0 & 3 \\ 3 & -5 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|ccc|c} 1 & 3 & 1 & 0 & 0 & 4 \\ 0 & -7 & -2 & 1 & 0 & -5 \\ 0 & -14 & -3 & 0 & 1 & -10 \end{array}\right) \qquad \begin{matrix} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{matrix}$$

$$\sim \left(\begin{array}{cc|ccc|c} 1 & 3 & 1 & 0 & 0 & 4 \\ 0 & 1 & \frac{2}{7} & -\frac{1}{7} & 0 & \frac{5}{7} \\ 0 & -14 & -3 & 0 & 1 & -10 \end{array}\right) \qquad -\tfrac{1}{7}R_2 \to R_2$$

$$\sim \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}}_{\mathbf{WA}} \middle| \underbrace{\begin{matrix} \frac{1}{7} & \frac{3}{7} & 0 \\ \frac{2}{7} & -\frac{1}{7} & 0 \\ 1 & -2 & 1 \end{matrix}}_{\mathbf{W}} \middle| \underbrace{\begin{matrix} \frac{13}{7} \\ \frac{5}{7} \\ 0 \end{matrix}}_{\mathbf{WB}} \right) \qquad \begin{matrix} R_1 - 3R_2 \to R_1 \\ \\ R_3 + 14R_2 \to R_3 \end{matrix}$$

ii *Check*: $\mathbf{W}(\mathbf{A} \mid \mathbf{B})$

$$= \begin{pmatrix} \frac{1}{7} & \frac{3}{7} & 0 \\ \frac{2}{7} & -\frac{1}{7} & 0 \\ 1 & -2 & 1 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & -1 & 3 \\ 3 & -5 & 2 \end{array}\right)$$

$$= \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}}_{\mathbf{WA}} \middle| \underbrace{\begin{matrix} \frac{13}{7} \\ \frac{5}{7} \\ 0 \end{matrix}}_{\mathbf{WB}} \right)$$

b **i** For the given system,
$(\mathbf{A} \mid \mathbf{I} \mid \mathbf{B})$

$$= \left(\begin{array}{ccc|cc|c} 1 & 1 & 2 & 1 & 0 & 1 \\ 2 & 0 & -3 & 0 & 1 & 4 \end{array}\right)$$

$$\sim \left(\underbrace{\begin{matrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{7}{2} \end{matrix}}_{\mathbf{WA}} \middle| \underbrace{\begin{matrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{matrix}}_{\mathbf{W}} \middle| \underbrace{\begin{matrix} 2 \\ -1 \end{matrix}}_{\mathbf{WB}} \right)$$

{using technology}

ii *Check*: $\mathbf{W}(\mathbf{A} \mid \mathbf{B})$

$$= \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 0 & -3 & 4 \end{array}\right)$$

$$\sim \left(\underbrace{\begin{matrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{7}{2} \end{matrix}}_{\mathbf{WA}} \middle| \underbrace{\begin{matrix} 2 \\ -1 \end{matrix}}_{\mathbf{WB}} \right)$$

c **i** For the given system,
$(\mathbf{A} \mid \mathbf{I} \mid \mathbf{B})$

$$= \left(\begin{array}{cccc|ccc|c} 1 & 1 & 2 & 3 & 1 & 0 & 0 & 4 \\ 2 & 1 & 5 & 1 & 0 & 1 & 0 & 7 \\ 0 & 1 & 1 & 6 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\sim \left(\underbrace{\begin{matrix} 1 & 0 & 0 & -\frac{7}{2} \\ 0 & 1 & 0 & \frac{11}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{matrix}}_{\mathbf{WA}} \middle| \underbrace{\begin{matrix} 2 & -\frac{1}{2} & -\frac{3}{2} \\ 1 & -\frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{matrix}}_{\mathbf{W}} \middle| \underbrace{\begin{matrix} -\frac{3}{2} \\ \frac{5}{2} \\ \frac{3}{2} \end{matrix}}_{\mathbf{WB}} \right)$$

{using technology}

ii *Check*: $\mathbf{W}(\mathbf{A} \mid \mathbf{B})$

$$= \begin{pmatrix} 2 & -\frac{1}{2} & -\frac{3}{2} \\ 1 & -\frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 1 & 7 \\ 0 & 1 & 1 & 6 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{11}{2} & \frac{5}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} \end{array}\right) \quad \underbrace{}_{\mathbf{WA}} \; \underbrace{}_{\mathbf{WB}}$$

5 **a** $(\mathbf{A} \mid \mathbf{I}) = \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right)$

$$\sim \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -9 & -2 & 1 \end{array}\right) \quad R_2 - 2R_1 \to R_2$$

$$\sim \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & \frac{2}{9} & -\frac{1}{9} \end{array}\right) \quad -\tfrac{1}{9}R_2 \to R_2$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{9} & \frac{4}{9} \\ 0 & 1 & \frac{2}{9} & -\frac{1}{9} \end{array}\right) \quad R_1 - 4R_2 \to R_1$$

$$= (\mathbf{I} \mid \mathbf{A}^{-1})$$

$$\therefore \quad \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{9} & \frac{4}{9} \\ \frac{2}{9} & -\frac{1}{9} \end{pmatrix}$$

b $(\mathbf{A} \mid \mathbf{I}) = \left(\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right)$

$$\sim \left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 4 & 5 & 0 & 1 \end{array}\right) \quad \tfrac{1}{3}R_1 \to R_1$$

$$\sim \left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{19}{3} & -\frac{4}{3} & 1 \end{array}\right) \quad R_2 - 4R_1 \to R_2$$

$$\sim \left(\begin{array}{cc|cc} 1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{4}{19} & \frac{3}{19} \end{array}\right) \quad \tfrac{3}{19}R_2 \to R_2$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{5}{19} & \frac{1}{19} \\ 0 & 1 & -\frac{4}{19} & \frac{3}{19} \end{array}\right) \quad R_1 + \tfrac{1}{3}R_2 \to R_1$$

$$= (\mathbf{I} \mid \mathbf{A}^{-1})$$

$$\therefore \quad \mathbf{A}^{-1} = \begin{pmatrix} \frac{5}{19} & \frac{1}{19} \\ -\frac{4}{19} & \frac{3}{19} \end{pmatrix}$$

c $(\mathbf{A} \mid \mathbf{I})$

$$= \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 8 & -3 & -3 & 1 & 0 \\ 0 & 3 & -1 & -2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 - 3R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 3 & -1 & -2 & 0 & 1 \end{array}\right) \quad \tfrac{1}{8}R_2 \to R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{8} & -\frac{7}{8} & -\frac{3}{8} & 1 \end{array}\right) \quad \begin{array}{l} R_1 + 2R_2 \to R_1 \\ \\ R_3 - 3R_2 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 0 & 1 & -7 & -3 & 8 \end{array}\right) \quad 8R_3 \to R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 4 & -10 \\ 0 & 1 & 0 & -3 & -1 & 3 \\ 0 & 0 & 1 & -7 & -3 & 8 \end{array}\right) \quad \begin{array}{l} R_1 - \frac{5}{4}R_3 \to R_1 \\ R_2 + \frac{3}{8}R_3 \to R_2 \end{array}$$

$$= (\mathbf{I} \mid \mathbf{A}^{-1})$$

$$\therefore \quad \mathbf{A}^{-1} = \begin{pmatrix} 9 & 4 & -10 \\ -3 & -1 & 3 \\ -7 & -3 & 8 \end{pmatrix}$$

d $(\mathbf{A} \mid \mathbf{I})$

$$= \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 3 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 7 & -7 & -2 & 1 & 0 \\ 0 & 5 & -7 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -\frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 5 & -7 & -3 & 0 & 1 \end{array}\right) \quad \tfrac{1}{7}R_2 \to R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{3}{7} & \frac{2}{7} & 0 \\ 0 & 1 & -1 & -\frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & -2 & -\frac{11}{7} & -\frac{5}{7} & 1 \end{array}\right) \quad \begin{array}{l} R_1 + 2R_2 \to R_1 \\ \\ R_3 - 5R_2 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{3}{7} & \frac{2}{7} & 0 \\ 0 & 1 & -1 & -\frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{11}{14} & \frac{5}{14} & -\frac{1}{2} \end{array}\right) \quad -\tfrac{1}{2}R_3 \to R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{14} & -\frac{1}{14} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{14} & \frac{5}{14} & -\frac{1}{2} \end{array}\right) \quad \begin{array}{l} R_1 - R_3 \to R_1 \\ R_2 + R_3 \to R_2 \end{array}$$

$$= (\mathbf{I} \mid \mathbf{A}^{-1})$$

$$\therefore \quad \mathbf{A}^{-1} = \begin{pmatrix} -\frac{5}{14} & -\frac{1}{14} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{11}{14} & \frac{5}{14} & -\frac{1}{2} \end{pmatrix}$$

6 **a** **i**
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 7 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 7 \\ 0 & 0 \end{vmatrix}$$
$$= 1(7-0) + 0 + 0$$
$$= 7$$

ii
$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$
$$= 0 + 0 + 1(0-1)$$
$$= -1$$

iii
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ -5 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & -5 \end{vmatrix}$$
$$= 1(1+0) + 0 + 0$$
$$= 1$$

b **i** $\mathbf{E}_s$ is the elementary matrix corresponding to swapping two rows.

$\therefore\ \mathbf{E}_s\mathbf{A}$ is obtained by swapping two rows of $\mathbf{A}$.

$$\therefore\ |\mathbf{E}_s\mathbf{A}| = -|\mathbf{A}| \quad \{\text{property 1 of determinants}\}$$
$$\therefore\ |\mathbf{E}_s||\mathbf{A}| = -|\mathbf{A}| \quad \{\text{property 6 of determinants}\}$$
$$\therefore\ |\mathbf{E}_s| = -1 \quad \{\text{for } |\mathbf{A}| \neq 0\}$$

ii $\mathbf{E}_k$ is the elementary matrix corresponding to multiplying a row by a non-zero constant k.

$\therefore\ \mathbf{E}_k\mathbf{A}$ is obtained by multiplying one row of $\mathbf{A}$ by k.

$$\therefore\ |\mathbf{E}_k\mathbf{A}| = k|\mathbf{A}| \quad \{\text{property 4 of determinants}\}$$
$$\therefore\ |\mathbf{E}_k||\mathbf{A}| = k|\mathbf{A}| \quad \{\text{property 6 of determinants}\}$$
$$\therefore\ |\mathbf{E}_k| = k \quad \{\text{for } |\mathbf{A}| \neq 0\}$$

iii $\mathbf{E}_a$ is the elementary matrix corresponding to adding a multiple of one row to another.

$\therefore\ \mathbf{E}_a\mathbf{A}$ is obtained by adding a multiple of one row of $\mathbf{A}$ to another row.

$$\therefore\ |\mathbf{E}_a\mathbf{A}| = |\mathbf{A}| \quad \{\text{property 5 of determinants}\}$$
$$\therefore\ |\mathbf{E}_a||\mathbf{A}| = |\mathbf{A}| \quad \{\text{property 6 of determinants}\}$$
$$\therefore\ |\mathbf{E}_a| = 1 \quad \{\text{for } |\mathbf{A}| \neq 0\}$$

c The check matrix $\mathbf{W}$ is a product of elementary matrices $\mathbf{E}_i$

$\therefore\ |\mathbf{W}|$ is a product of 1s, (-1)s, and ks, where $k \neq 0$

$\therefore\ |\mathbf{W}| \neq 0 \quad \therefore\ \mathbf{W}^{-1}$ exists.

d Using the Gaussian elimination method in **Example 23**:

i $$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

iii $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}$$

iv $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

v $$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

vi $$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

vii $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

viii $$\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

e The inverse of an elementary matrix for swapping two rows is itself.

The inverse of an elementary matrix for multiplying a row by a non-zero constant k contains $\frac{1}{k}$ instead of k and all other elements of the matrix are unchanged.

The inverse of an elementary matrix for adding a of one row to another row contains $-a$ instead of a and all other elements of the matrix are unchanged.

7 **a** $$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} \quad \tfrac{1}{3}R_2 \to R_2 \qquad \mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} \quad R_1 - 2R_3 \to R_1 \qquad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_2 - \tfrac{1}{3}R_3 \to R_2 \qquad \mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \mathbf{I}_3$$

$$\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \mathbf{I}_3$$

b $$\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}_3$$
$$\therefore\ \mathbf{E}_3^{-1}\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{E}_3^{-1}\mathbf{I}_3 \quad \{\text{premultiply by } \mathbf{E}_3^{-1}\}$$
$$\therefore\ \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{E}_3^{-1}$$
$$\therefore\ \mathbf{E}_2^{-1}\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{E}_2^{-1}\mathbf{E}_3^{-1} \quad \{\text{premultiply by } \mathbf{E}_2^{-1}\}$$
$$\therefore\ \mathbf{E}_1\mathbf{A} = \mathbf{E}_2^{-1}\mathbf{E}_3^{-1}$$
$$\therefore\ \mathbf{E}_1^{-1}\mathbf{E}_1\mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1} \quad \{\text{premultiply by } \mathbf{E}_1^{-1}\}$$
$$\therefore\ \mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}$$
$$\therefore\ \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

8 $$(\mathbf{A} \mid \mathbf{I})$$
$$= \left(\begin{array}{ccc|ccc} a & b & a & 1 & 0 & 0 \\ 0 & c & a & 0 & 1 & 0 \\ 0 & 0 & d & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & 1 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{a}{c} & 0 & \frac{1}{c} & 0 \\ 0 & 0 & d & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \frac{1}{a}R_1 \to R_1 \\ \frac{1}{c}R_2 \to R_2 \\ \ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1-\frac{b}{c} & \frac{1}{a} & \frac{-b}{ac} & 0 \\ 0 & 1 & \frac{a}{c} & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{d} \end{array}\right) \quad \begin{array}{l} R_1 - \frac{b}{a}R_2 \to R_1 \\ \ \\ \frac{1}{d}R_3 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & \frac{-b}{ac} & \frac{b-c}{cd} \\ 0 & 1 & 0 & 0 & \frac{1}{c} & \frac{-a}{cd} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{d} \end{array}\right) \quad \begin{array}{l} R_1 - \left(1-\frac{b}{c}\right)R_3 \to R_1 \\ R_2 - \frac{a}{c}R_3 \to R_2 \\ \ \end{array}$$

$$= (\mathbf{I} \mid \mathbf{A}^{-1})$$

$$\therefore\quad \mathbf{A}^{-1}=\begin{pmatrix}\frac{1}{a} & \frac{-b}{ac} & \frac{b-c}{cd}\\ 0 & \frac{1}{c} & \frac{-a}{cd}\\ 0 & 0 & \frac{1}{d}\end{pmatrix}$$

9 **a**

$(\mathbf{A}\mid\mathbf{I})$

$$=\left(\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0\\ 0 & d & e & 0 & 1 & 0\\ 0 & 0 & f & 0 & 0 & 1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & \frac{1}{a} & 0 & 0\\ 0 & 1 & \frac{e}{d} & 0 & \frac{1}{d} & 0\\ 0 & 0 & f & 0 & 0 & 1\end{array}\right)\qquad \begin{array}{l}\frac{1}{a}R_1\to R_1\\ \frac{1}{d}R_2\to R_2\\ \\ \end{array}$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{cd-be}{ad} & \frac{1}{a} & \frac{-b}{ad} & 0\\ 0 & 1 & \frac{e}{d} & 0 & \frac{1}{d} & 0\\ 0 & 0 & 1 & 0 & 0 & \frac{1}{f}\end{array}\right)\qquad \begin{array}{l}R_1-\frac{b}{a}R_2\to R_1\\ \\ \frac{1}{f}R_3\to R_3\end{array}$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & \frac{-b}{ad} & \frac{be-cd}{adf}\\ 0 & 1 & 0 & 0 & \frac{1}{d} & \frac{-e}{df}\\ 0 & 0 & 1 & 0 & 0 & \frac{1}{f}\end{array}\right)\qquad \begin{array}{l}R_1-\left(\frac{cd-be}{ad}\right)R_3\to R_1\\ R_2-\frac{e}{d}R_3\to R_2\\ \\ \end{array}$$

$=(\mathbf{I}\mid\mathbf{A}^{-1})$

$$\therefore\quad \mathbf{A}^{-1}=\begin{pmatrix}\frac{1}{a} & \frac{-b}{ad} & \frac{be-cd}{adf}\\ 0 & \frac{1}{d} & \frac{-e}{df}\\ 0 & 0 & \frac{1}{f}\end{pmatrix}$$

b If $\mathbf{E}$ is the elementary matrix then

$$\begin{aligned} \mathbf{EA}&=\mathbf{B} && \{\text{given}\}\\ \therefore\quad (\mathbf{EA})\mathbf{C}&=\mathbf{BC} && \{\text{postmultiply by } \mathbf{C} \text{ of appropriate size}\}\\ \therefore\quad \mathbf{E}(\mathbf{AC})&=\mathbf{BC} \end{aligned}$$

$\therefore$ $\mathbf{BC}$ is obtained from $\mathbf{AC}$ under the same elementary operation.

EXERCISE 11.1

1 **a** $\mathbf{u}$, $\mathbf{v}$ are $n\times 1$ matrices, so we let $\mathbf{u}=(u_{i1})$ and $\mathbf{v}=(v_{i1})$, $i=1, 2, 3, ..., n$.

$$\begin{aligned}\text{Now}\quad \mathbf{u}+\mathbf{v}&=(u_{i1})+(v_{i1})\\ &=(u_{i1}+v_{i1}) \quad \{\text{addition}\}\\ &=(v_{i1}+u_{i1})\\ &=\mathbf{v}+\mathbf{u}\quad \text{for all } \mathbf{u}, \mathbf{v}\in\mathbb{R}^n\end{aligned}$$

b $\mathbf{u}$, $\mathbf{0}$ are $n\times 1$ matrices, so we let $\mathbf{u}=(u_{i1})$ and $\mathbf{0}=(0_{i1})$, $i=1, 2, 3, ..., n$.

$$\begin{aligned}\text{Now}\quad &\mathbf{u}+\mathbf{0} & \text{and}\quad &\mathbf{0}+\mathbf{u}\\ &=(u_{i1})+(0_{i1}) & &=(0_{i1})+(u_{i1})\\ &=(u_{i1}+0_{i1}) & &=(0_{i1}+u_{i1})\\ &\quad\{\text{addition}\} & &\quad\{\text{addition}\}\\ &=(u_{i1}) & &=(u_{i1})\\ &=\mathbf{u} & &=\mathbf{u}\end{aligned}$$

$\therefore$ $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ for all $\mathbf{u}\in\mathbb{R}^n$.

c $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are $n\times 1$ matrices, so we let $\mathbf{u}=(u_{i1})$, $\mathbf{v}=(v_{i1})$, and $\mathbf{w}=(w_{i1})$, $i=1, 2, 3, ..., n$.

$$\begin{aligned}\text{Now}\quad (\mathbf{u}+\mathbf{v})+\mathbf{w}&=((u_{i1})+(v_{i1}))+(w_{i1})\\ &=(u_{i1}+v_{i1}+w_{i1})\quad \{\text{addition}\}\\ &=(u_{i1})+(v_{i1}+w_{i1})\\ &=\mathbf{u}+(\mathbf{v}+\mathbf{w})\quad \text{for all } \mathbf{u}, \mathbf{v}, \text{and } \mathbf{w}\in\mathbb{R}^n\end{aligned}$$

2 **a** $\mathbf{u}$ is an $n\times 1$ matrix, so we let $\mathbf{u}=(u_{i1})$, $i=1, 2, 3, ..., n$.

$$\begin{aligned}\text{Now}\quad c_1(c_2\mathbf{u})&=c_1(c_2(u_{i1}))\\ &=c_1(c_2u_{i1})\\ &=c_1c_2(u_{i1})\\ &=c_1c_2\mathbf{u}\quad \text{for all } \mathbf{u}\in\mathbb{R}^n \text{ and } c_1, c_2\in\mathbb{R}\end{aligned}$$

b $\mathbf{u}$ is an $n\times 1$ matrix, so we let $\mathbf{u}=(u_{i1})$, $i=1, 2, 3, ..., n$.

$$\begin{aligned}\text{Now}\quad &(c_1+c_2)\mathbf{u}\\ &=(c_1+c_2)(u_{i1})\\ &=((c_1+c_2)u_{i1})\\ &=(c_1u_{i1}+c_2u_{i1})\\ &=(c_1u_{i1})+(c_2u_{i1})\\ &=c_1(u_{i1})+c_2(u_{i1})\\ &=c_1\mathbf{u}+c_2\mathbf{u}\quad \text{for all } \mathbf{u}\in\mathbb{R}^n \text{ and } c_1, c_2\in\mathbb{R}\end{aligned}$$

3 **a**

$$\begin{pmatrix}3\\-1\end{pmatrix}\bullet\begin{pmatrix}2\\6\end{pmatrix}=6-6 \qquad \begin{pmatrix}3\\-1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\end{pmatrix}=-3-2$$
$$=0 \qquad\qquad =-5$$

$$\begin{pmatrix}2\\6\end{pmatrix}\bullet\begin{pmatrix}-1\\2\end{pmatrix}=-2+12$$
$$=10$$

$\therefore$ the vectors are *not* mutually orthogonal.

b

$$\begin{pmatrix}3\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\3\\0\end{pmatrix}=3-3+0=0$$

$$\begin{pmatrix}3\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\-5\end{pmatrix}=9+1-10=0$$

$$\begin{pmatrix}1\\3\\0\end{pmatrix}\bullet\begin{pmatrix}3\\-1\\-5\end{pmatrix}=3-3+0=0$$

$\therefore$ the vectors are mutually orthogonal.

c

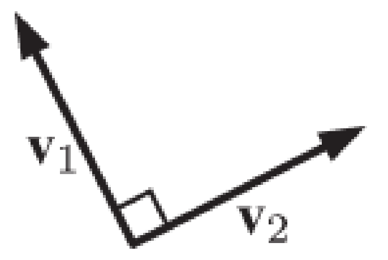
$$\begin{pmatrix}4\\0\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\3\\4\end{pmatrix}=-4+0+4=0$$

$$\begin{pmatrix}4\\0\\1\end{pmatrix}\bullet\begin{pmatrix}2\\2\\-1\end{pmatrix}=8+0-1=7$$

$$\begin{pmatrix}-1\\3\\4\end{pmatrix}\bullet\begin{pmatrix}2\\2\\-1\end{pmatrix}=-2+6-4=0$$

$\therefore$ the vectors are *not* mutually orthogonal.

4 Let vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ be orthogonal in $\mathbb{R}^2$.



In $\mathbb{R}^2$, any vector $\mathbf{v}_3$ that is orthogonal to $\mathbf{v}_1$ must either have the same direction as $\mathbf{v}_2$, or the opposite direction to $\mathbf{v}_2$.

In either case, $\mathbf{v}_2$ and $\mathbf{v}_3$ are not orthogonal.

Hence, a set of 3 vectors in $\mathbb{R}^2$ cannot be mutually orthogonal.

5 **a** If $\begin{pmatrix}4\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}1\\3\\k\end{pmatrix}=0$

$$\begin{aligned}\text{then}\quad 4-3+2k&=0\\ \therefore\quad k&=-\tfrac{1}{2}\end{aligned}$$

$\therefore$ the vectors are orthogonal if $k=-\frac{1}{2}$.

b If $\begin{pmatrix}3\\1\\k\end{pmatrix}\bullet\begin{pmatrix}1\\k-1\\1\end{pmatrix}=0$

$$\begin{aligned}\text{then}\quad 3+(k-1)+k&=0\\ \therefore\quad 2+2k&=0\\ \therefore\quad k&=-1\end{aligned}$$

If $\begin{pmatrix}3\\1\\k\end{pmatrix}\bullet\begin{pmatrix}k\\-4\\-7\end{pmatrix}=0$

$$\begin{aligned}\text{then}\quad 3k-4-7k&=0\\ \therefore\quad -4k-4&=0\\ \therefore\quad k&=-1\end{aligned}$$

If $\begin{pmatrix}1\\k-1\\1\end{pmatrix}\bullet\begin{pmatrix}k\\-4\\-7\end{pmatrix}=0$

$$\begin{aligned}\text{then}\quad k-4(k-1)-7&=0\\ \therefore\quad -3k-3&=0\\ \therefore\quad k&=-1\end{aligned}$$

$\therefore$ the vectors are mutually orthogonal if $k=-1$.

6 Suppose $c_1\begin{pmatrix}1\\0\\-3\end{pmatrix}+c_2\begin{pmatrix}-2\\1\\3\end{pmatrix}=\begin{pmatrix}8\\-3\\-15\end{pmatrix}$

for some $c_1, c_2\in\mathbb{R}$

$$\begin{aligned}\therefore\quad c_1-2c_2&=8\\ c_2&=-3\\ -3c_1+3c_2&=-15\end{aligned}$$

$\therefore$ $c_1=2$ and $c_2=-3$

$$\therefore\quad \begin{pmatrix}8\\-3\\-15\end{pmatrix}=2\begin{pmatrix}1\\0\\-3\end{pmatrix}-3\begin{pmatrix}-2\\1\\3\end{pmatrix}$$

7 **a** Suppose $c_1\begin{pmatrix}1\\0\\-1\end{pmatrix}+c_2\begin{pmatrix}-1\\2\\3\end{pmatrix}+c_3\begin{pmatrix}1\\4\\-1\end{pmatrix}=\begin{pmatrix}-3\\-2\\9\end{pmatrix}$

for some $c_1, c_2, c_3\in\mathbb{R}$

$$\begin{aligned}\therefore\quad c_1-\ \ c_2+\ \ c_3&=-3\\ 2c_2+4c_3&=-2\\ -c_1+3c_2-\ \ c_3&=9\end{aligned}$$

which has augmented matrix $\left(\begin{array}{ccc|c}1&-1&1&-3\\0&2&4&-2\\-1&3&-1&9\end{array}\right)$

which has reduced row echelon form

$$\left(\begin{array}{ccc|c}1&0&0&2\\0&1&0&3\\0&0&1&-2\end{array}\right)\quad\{\text{using technology}\}$$

$\therefore$ $c_1=2$, $c_2=3$, and $c_3=-2$

$$\therefore\quad \begin{pmatrix}-3\\-2\\9\end{pmatrix}=2\begin{pmatrix}1\\0\\-1\end{pmatrix}+3\begin{pmatrix}-1\\2\\3\end{pmatrix}-2\begin{pmatrix}1\\4\\-1\end{pmatrix}$$

b
$$\begin{aligned}\begin{pmatrix}-3\\-2\\9\end{pmatrix}&=\big(\mathbf{v}_1\,\big|\,\mathbf{v}_2\,\big|\,\mathbf{v}_3\big)\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}\\ &=\begin{pmatrix}1&-1&1\\0&2&4\\-1&3&-1\end{pmatrix}\begin{pmatrix}2\\3\\-2\end{pmatrix}\end{aligned}$$

8 **a** Suppose $c_1\begin{pmatrix}1\\-1\\1\end{pmatrix}+c_2\begin{pmatrix}-1\\1\\-1\end{pmatrix}+c_3\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}-3\\1\\-2\end{pmatrix}$

for some $c_1, c_2, c_3\in\mathbb{R}$

$$\begin{aligned}\therefore\quad c_1-c_2+2c_3&=-3\\ -c_1+c_2\qquad&=1\\ c_1-c_2+\ \ c_3&=-2\end{aligned}$$

which has augmented matrix $\left(\begin{array}{ccc|c}1&-1&2&-3\\-1&1&0&1\\1&-1&1&-2\end{array}\right)$

which has reduced row echelon form

$$\left(\begin{array}{ccc|c}1&-1&0&-1\\0&0&1&-1\\0&0&0&0\end{array}\right)\quad\{\text{using technology}\}$$

$\therefore$ $c_1-c_2=-1$, and $c_3=-1$

One solution is $c_1=0$, $c_2=1$, $c_3=-1$.

Thus $\begin{pmatrix}-3\\1\\-2\end{pmatrix}=0\begin{pmatrix}1\\-1\\1\end{pmatrix}+1\begin{pmatrix}-1\\1\\-1\end{pmatrix}-1\begin{pmatrix}2\\0\\1\end{pmatrix}$

b
$$\begin{aligned}\begin{pmatrix}-3\\1\\-2\end{pmatrix}&=\big(\mathbf{v}_1\,\big|\,\mathbf{v}_2\,\big|\,\mathbf{v}_3\big)\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}\\ &=\begin{pmatrix}1&-1&2\\-1&1&0\\1&-1&1\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix}\end{aligned}$$

9 Suppose $c_1\begin{pmatrix}2\\1\\-2\end{pmatrix}+c_2\begin{pmatrix}-2\\-1\\0\end{pmatrix}+c_3\begin{pmatrix}4\\2\\-3\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$

for some $c_1, c_2, c_3\in\mathbb{R}$

$$\begin{aligned}\therefore\quad 2c_1-2c_2+4c_3&=a\\ c_1-\ \ c_2+2c_3&=b\\ -2c_1\qquad\ -3c_3&=c\end{aligned}$$

which has augmented matrix

$$\left(\begin{array}{ccc|c}2&-2&4&a\\1&-1&2&b\\-2&0&-3&c\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&-1&2&b\\2&-2&4&a\\-2&0&-3&c\end{array}\right)\qquad R_2\leftrightarrow R_1$$

$$\sim\left(\begin{array}{ccc|c}1&-1&2&b\\0&0&0&a-2b\\0&-2&1&2b+c\end{array}\right)\qquad \begin{array}{l}R_2-2R_1\to R_2\\ R_3+2R_1\to R_3\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&-1&2&b\\0&-2&1&2b+c\\0&0&0&a-2b\end{array}\right)\qquad R_3\leftrightarrow R_2$$

If $a\neq 2b$, the system is inconsistent and $\therefore$ has no solutions.
So, if $a=2b$, the system has at least one solution.

$\therefore$ $\begin{pmatrix}a\\b\\c\end{pmatrix}$ is a linear combination of the given 3 vectors.

10 **a** Suppose $c_1\begin{pmatrix}2\\1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\1\\-1\end{pmatrix}+c_3\begin{pmatrix}-1\\2\\-8\end{pmatrix}=\begin{pmatrix}3\\-2\\0\end{pmatrix}$

for some $c_1, c_2, c_3 \in \mathbb{R}$

$$\begin{aligned}\therefore\quad 2c_1 + c_2 - \ \ c_3 &= 3\\ c_1 + c_2 + 2c_3 &= -2\\ c_1 - c_2 - 8c_3 &= 0\end{aligned}$$

which has augmented matrix $\left(\begin{array}{ccc|c}2&1&-1&3\\1&1&2&-2\\1&-1&-8&0\end{array}\right)$

which has reduced row echelon form

$\left(\begin{array}{ccc|c}1&0&-3&0\\0&1&5&0\\0&0&0&1\end{array}\right)$ {using technology}

The last row indicates $0 = 1$

$\therefore$ the system is inconsistent

$\therefore$ no solutions for c_1, c_2, c_3 exist.

$\therefore$ $\begin{pmatrix}3\\-2\\0\end{pmatrix}$ cannot be written as a linear combination of these vectors.

b The three vectors lie in a plane, and the point does not lie on that plane.

11 Suppose $\begin{pmatrix}a\\b\\c\end{pmatrix}=k_1\begin{pmatrix}1\\1\\1\end{pmatrix}+k_2\begin{pmatrix}0\\1\\1\end{pmatrix}+k_3\begin{pmatrix}0\\0\\2\end{pmatrix}$

$$\begin{aligned}\therefore\quad k_1 \qquad\qquad &= a\\ k_1 + k_2 \qquad &= b\\ k_1 + k_2 + 2k_3 &= c\end{aligned}$$

Thus $k_1 = a,\quad k_2 = b - a,\quad k_3 = \dfrac{c-a-(b-a)}{2}$

$\therefore\quad k_3 = \dfrac{c-b}{2}$

So, as k_1, k_2, and k_3 can be determined uniquely for a given vector $\begin{pmatrix}a\\b\\c\end{pmatrix}$, every vector of $\mathbb{R}^3$ can be written as a linear combination of the 3 given vectors.

For example, $\begin{pmatrix}2\\5\\-7\end{pmatrix}=2\begin{pmatrix}1\\1\\1\end{pmatrix}+3\begin{pmatrix}0\\1\\1\end{pmatrix}-6\begin{pmatrix}0\\0\\2\end{pmatrix}$

{as $k_1 = 2,\quad k_2 = 5 - 2 = 3,\quad k_3 = \dfrac{-7-5}{2} = -6$}

12 For any two distinct basic unit vectors

$$\mathbf{e}_i=\begin{pmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix}\begin{array}{l}\\ \\ \\ \leftarrow i\text{th position}\\ \\ \\ \end{array}\quad\text{and}\quad \mathbf{e}_j=\begin{pmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix}\begin{array}{l}\\ \\ \\ \leftarrow j\text{th position}\\ \\ \\ \end{array}$$

$$\begin{aligned}\mathbf{e}_i \bullet \mathbf{e}_j &= 0(0) + 0(0) + ... + \underset{\substack{\uparrow\\ i\text{th position}}}{1(0)} + ... + \underset{\substack{\uparrow\\ j\text{th position}}}{0(1)} + ... + 0(0)\\ &= 0\end{aligned}$$

$\therefore$ the set of basic unit vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ is mutually orthogonal.

EXERCISE 11.2

1 **a** Let W be a subspace of $\mathbb{R}^n$.

If $\mathbf{w} \in W$, then $0\mathbf{w} \in W$ as W is closed under scalar multiplication.

$\Rightarrow\quad \mathbf{0} \in W$

Thus, every subspace of $\mathbb{R}^n$ contains the zero vector $\mathbf{0}$.

b Consider $W = \{\mathbf{0}\}$

As $\mathbf{0} \in W \quad\Rightarrow\quad W$ is non-empty.

$$\begin{aligned}&(1)\ \text{If } \mathbf{u}, \mathbf{v} \in W &&\Rightarrow\quad \mathbf{u} = \mathbf{0} \ \text{ and } \ \mathbf{v} = \mathbf{0}\\ & &&\Rightarrow\quad \mathbf{u} + \mathbf{v} = \mathbf{0}\\ & &&\Rightarrow\quad \mathbf{u} + \mathbf{v} \in W\\ &(2)\ \text{If } c \in \mathbb{R} \text{ and } \mathbf{u} \in W &&\Rightarrow\quad c\mathbf{u} = c\mathbf{0}\\ & &&\Rightarrow\quad c\mathbf{u} = \mathbf{0}\\ & &&\Rightarrow\quad c\mathbf{u} \in W\end{aligned}$$

From (1) and (2), W is closed under vector addition and scalar multiplication.

$\therefore\quad W = \{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

2 $W = \left\{\begin{pmatrix}x\\x-3\\z\end{pmatrix},\quad x, z \in \mathbb{R}\right\}$

Let $\mathbf{u} = \begin{pmatrix}u_1\\u_1-3\\u_3\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}v_1\\v_1-3\\v_3\end{pmatrix} \in W$

$\therefore\quad \mathbf{u} + \mathbf{v} = \begin{pmatrix}u_1+v_1\\u_1+v_1-6\\u_3+v_3\end{pmatrix} \notin W$

{For $\mathbf{u} + \mathbf{v}$ to be in W, the 2nd element would have to be $u_1 + v_1 - 3$.}

$\therefore\quad W$ is not closed under vector addition.

$\Rightarrow\quad W$ is not a subspace of $\mathbb{R}^3$.

3 Consider $W_1 = \left\{\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\middle|\ a = 2,\ b = 0,\ c + d = 2\right\}$

$\therefore\quad W_1 = \left\{\begin{pmatrix}2\\0\\c\\2-c\end{pmatrix}\middle|\ c \in \mathbb{R}\right\}$

Let $\mathbf{u} = \begin{pmatrix}2\\0\\u_3\\2-u_3\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}2\\0\\v_3\\2-v_3\end{pmatrix} \in W_1$

$\therefore\quad \mathbf{u} + \mathbf{v} = \begin{pmatrix}4\\0\\u_3+v_3\\4-(u_3+v_3)\end{pmatrix} \notin W_1$

{For $\mathbf{u} + \mathbf{v}$ to be in W, the 1st element would have to be 2, and the 4th element would have to be $2 - (u_3 - v_3)$.}

$\therefore\quad W_1$ is not closed under vector addition.

$\Rightarrow\quad W_1$ is not a subspace of $\mathbb{R}^4$.

Consider $W_2 = \left\{\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\middle|\ a = 0,\ c + d = 0\right\}$

$\therefore\quad W_2 = \left\{\begin{pmatrix}0\\b\\c\\-c\end{pmatrix}\middle|\ b, c \in \mathbb{R}\right\}$

Let $\mathbf{u} = \begin{pmatrix} 0 \\ u_2 \\ u_3 \\ -u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ v_2 \\ v_3 \\ -v_3 \end{pmatrix} \in W_2$

$$\therefore \quad \mathbf{u} + \mathbf{v} = \begin{pmatrix} 0 \\ u_2 + v_2 \\ u_3 + v_3 \\ -(u_3 + v_3) \end{pmatrix} \notin W_2$$

and if $c \in \mathbb{R}$ then $c\mathbf{u} = c \begin{pmatrix} 0 \\ u_2 \\ u_3 \\ -u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ cu_2 \\ cu_3 \\ -cu_3 \end{pmatrix} \in W_2$

$\therefore$ W_2 is closed under vector addition and scalar multiplication and as W_2 is non-empty,

$\Rightarrow$ W_2 is a subspace of $\mathbb{R}^4$.

4 If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then $\mathbf{u} + \mathbf{v} \in \mathbb{R}^n$ and $c\mathbf{u} \in \mathbb{R}^n$ where $c \in \mathbb{R}$. Thus $\mathbb{R}^n$ is closed under vector addition and scalar multiplication, and as $\mathbb{R}^n$ is non-empty, $\mathbb{R}^n$ is a subspace of itself.

5 Consider $W = \left\{ \begin{pmatrix} x \\ 2x + 1 \\ 0 \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ 2u_1 + 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ 2v_1 + 1 \\ 0 \end{pmatrix} \in W$

$$\therefore \quad \mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ 2(u_1 + v_1) + 2 \\ 0 \end{pmatrix} \notin W$$

{For $\mathbf{u} + \mathbf{v}$ to be in W, the 2nd element would have to be $2(u_1 + v_1) + 1$.}

$\therefore$ W is not closed under vector addition.

$\Rightarrow$ W is not a subspace of $\mathbb{R}^3$.

6 Consider $W = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \middle| \; \mathbf{A}\mathbf{x} = \mathbf{0} \right\}$

As $\mathbf{0} \in W$ $\Rightarrow$ W is non-empty {$\mathbf{A}\mathbf{0} = \mathbf{0}$}

Let $\mathbf{v}_1, \mathbf{v}_2 \in W$ $\therefore$ $\mathbf{A}\mathbf{v}_1 = \mathbf{0}$ and $\mathbf{A}\mathbf{v}_2 = \mathbf{0}$

$$\begin{aligned} \therefore \quad \mathbf{A}(\mathbf{v}_1 + \mathbf{v}_2) &= \mathbf{A}\mathbf{v}_1 + \mathbf{A}\mathbf{v}_2 \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0} \in W \end{aligned}$$

and if $c \in \mathbb{R}$ then $\mathbf{A}(c\mathbf{v}_1) = c\mathbf{A}\mathbf{v}_1 = c(\mathbf{0}) = \mathbf{0} \in W$

Thus, for $\mathbf{v}_1, \mathbf{v}_2 \in W$, $\mathbf{v}_1 + \mathbf{v}_2 \in W$ and $c\mathbf{v}_1 \in W$.

$\therefore$ W is a non-empty closed subset under vector addition and scalar multiplication.

$\Rightarrow$ W is a subspace of $\mathbb{R}^n$.

7 **a** Every subspace of $\mathbb{R}^2$ is either:
(1) $\{\mathbf{0}\}$, (2) a line through O, or (3) $\mathbb{R}^2$.

Proof:

(1) We proved in **1 a** that $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$ and $\therefore$ of $\mathbb{R}^2$.

(2) Suppose W is a subspace of $\mathbb{R}^2$ where $W \neq \{\mathbf{0}\}$.
Let W contain a non-zero vector $\mathbf{u}$.
$\Rightarrow$ W contains all vectors $c_1\mathbf{u}$, $c_1 \in \mathbb{R}$
{closure under scalar multiplication}
and if these are the only vectors in W then
$W = \{c_1\mathbf{u} \mid c_1 \in \mathbb{R}\}$ which is a straight line through O(0, 0) with direction vector $\mathbf{u}$.

(3) Suppose now that W contains another non-zero vector $\mathbf{v}$, $\mathbf{v} \in \mathbb{R}^3$.
$\Rightarrow$ W contains all vectors $c_2\mathbf{v}$, $c_2 \in \mathbb{R}$
$\Rightarrow$ W contains all vectors $c_1\mathbf{u} + c_2\mathbf{v}$
$\Rightarrow$ W is $\mathbb{R}^2$.
{any vector in $\mathbb{R}^2$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$}

b Every subspace of $\mathbb{R}^3$ is either (1) $\{\mathbf{0}\}$, (2) a line through O, (3) a plane through O, or (4) $\mathbb{R}^3$.

Proof:

(1) We proved in **1 a** that $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$ and $\therefore$ of $\mathbb{R}^3$.

(2) Suppose W is a subspace of $\mathbb{R}^3$ where $W \neq \{\mathbf{0}\}$.
Let W contain a non-zero vector $\mathbf{u} \in \mathbb{R}^3$.
$\Rightarrow$ W contains all vectors $c_1\mathbf{u}$, $c_1 \in \mathbb{R}$
{closure under scalar multiplication}
and if these are the only vectors in W then
$W = \{c_1\mathbf{u} \mid c_1 \in \mathbb{R}\}$ which is a straight line passing through the origin O(0, 0, 0) with direction vector $\mathbf{u}$.

(3) Suppose W contains another non-zero vector $\mathbf{v}$, $\mathbf{v} \in \mathbb{R}^3$.
$\Rightarrow$ W contains all vectors $c_2\mathbf{v}$, $c_2 \in \mathbb{R}$.
$\Rightarrow$ W contains all vectors $c_1\mathbf{u} + c_2\mathbf{v}$
$\Rightarrow$ $W = \{c_1\mathbf{u} + c_2\mathbf{v} \mid c_1, c_2 \in \mathbb{R}\}$
$\Rightarrow$ W is a plane through O(0, 0, 0).

(4) Finally, if W contains another non-zero vector $\mathbf{w}$, $\mathbf{w} \in \mathbb{R}^3$.
$\Rightarrow$ W contains all vectors $c_3\mathbf{w}$, $c_3 \in \mathbb{R}$
$\Rightarrow$ W contains all vectors $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$
$\Rightarrow$ W is $\mathbb{R}^3$.
{any vector in $\mathbb{R}^3$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$}

EXERCISE 11.3

1 **a**
$$\begin{aligned} W &= \text{lin}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \\ &= \left\{ c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| \; c_1, c_2 \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} c_1 + c_2 \\ c_2 \end{pmatrix} \middle| \; c_1, c_2 \in \mathbb{R} \right\} \\ &= \mathbb{R}^2 \end{aligned}$$

b
$$\begin{aligned} W &= \text{lin}\left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\} \\ &= \left\{ c_1 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \middle| \; c_1 \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 2c_1 \end{pmatrix} \middle| \; c_1 \in \mathbb{R} \right\} \end{aligned}$$

which is the z-axis in 3-dimensional Cartesian space.

c
$$\begin{aligned} W &= \text{lin}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \\ &= \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \middle| \; c_1 \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} c_1 \\ 2c_1 \\ 3c_1 \end{pmatrix} \middle| \; c_1 \in \mathbb{R} \right\} \end{aligned}$$

which is a straight line passing through the origin O(0, 0, 0).

d $$W = \text{lin}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\3\end{pmatrix}\right\}$$
$$= \left\{c_1\begin{pmatrix}1\\2\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\3\end{pmatrix} \,\middle|\, c_1,\, c_2 \in \mathbb{R}\right\}$$
$$= \left\{\begin{pmatrix}c_1\\2c_1+c_2\\c_1+3c_2\end{pmatrix} \,\middle|\, c_1,\, c_2 \in \mathbb{R}\right\}$$

Let $x = c_1$, $y = 2c_1 + c_2$, and $z = c_1 + 3c_2$
then $y = 2x + c_2$ and $z = x + 3c_2$.

$$\therefore\ y - 2x = \frac{z-x}{3}$$
$$\therefore\ 3y - 6x = z - x$$
$$\therefore\ 5x - 3y + z = 0$$
$$\therefore\ W = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, 5x - 3y + z = 0\right\}$$

which is a plane in $\mathbb{R}^3$ passing through $O(0, 0, 0)$.

e $$W = \text{lin}\,\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$$
$$= \text{lin}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$
$$= \left\{c_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + c_4\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \,\middle|\, c_1,\, c_2,\, c_3,\, c_4 \in \mathbb{R}\right\}$$
$$= \left\{\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} \,\middle|\, c_1,\, c_2,\, c_3,\, c_4 \in \mathbb{R}\right\}$$
$$= \mathbb{R}^4$$

2 **a** Suppose $\begin{pmatrix}x\\y\end{pmatrix} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}3\\4\end{pmatrix}$ for some $c_1,\, c_2 \in \mathbb{R}$

$$\therefore\ \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&3\\2&4\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

This is a linear system of the form $\mathbf{x} = \mathbf{Ac}$ where
$|\mathbf{A}| = 4 - 6 = -2$.

Since $|\mathbf{A}| \neq 0$, $\mathbf{A}^{-1}$ exists, and so a non-trivial solution exists for $\mathbf{c}$.

$\Rightarrow$ W spans $\mathbb{R}^2$.

b If $\mathbf{x} = \begin{pmatrix}7\\8\end{pmatrix}$, then $\begin{pmatrix}7\\8\end{pmatrix} = \begin{pmatrix}1&3\\2&4\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$

$$\therefore\ \begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1&3\\2&4\end{pmatrix}^{-1}\begin{pmatrix}7\\8\end{pmatrix}$$
$$= \tfrac{1}{-2}\begin{pmatrix}4&-3\\-2&1\end{pmatrix}\begin{pmatrix}7\\8\end{pmatrix}$$
$$= \tfrac{1}{-2}\begin{pmatrix}4\\-6\end{pmatrix}$$
$$= \begin{pmatrix}-2\\3\end{pmatrix}$$
$$\therefore\ c_1 = -2, \quad c_2 = 3$$
$$\therefore\ \begin{pmatrix}7\\8\end{pmatrix} = -2\begin{pmatrix}1\\2\end{pmatrix} + 3\begin{pmatrix}3\\4\end{pmatrix}$$

3 Suppose $\begin{pmatrix}x\\y\\z\end{pmatrix} = c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\\0\end{pmatrix} + c_3\begin{pmatrix}-2\\3\\1\end{pmatrix}$ for some $c_1,\, c_2,\, c_3 \in \mathbb{R}$

$$\therefore\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1&1&-2\\0&-1&3\\1&0&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$$

This is a linear system of the form $\mathbf{x} = \mathbf{Ac}$ where

$$|\mathbf{A}| = 1\begin{vmatrix}-1&3\\0&1\end{vmatrix} - 1\begin{vmatrix}0&3\\1&1\end{vmatrix} - 2\begin{vmatrix}0&-1\\1&0\end{vmatrix}$$
$$= -1 + 3 - 2$$
$$= 0$$

Since $|\mathbf{A}| = 0$, $\mathbf{A}^{-1}$ does not exist.

$\Rightarrow$ W does not span $\mathbb{R}^3$.

4 **a** Suppose $\begin{pmatrix}x\\y\\z\end{pmatrix} = c_1\begin{pmatrix}1\\2\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\2\end{pmatrix} + c_3\begin{pmatrix}2\\0\\1\end{pmatrix}$ for some $c_1,\, c_2,\, c_3 \in \mathbb{R}$

$$\therefore\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1&0&2\\2&1&0\\0&2&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$$

This is a linear system of the form $\mathbf{x} = \mathbf{Ac}$ where

$$|\mathbf{A}| = 1\begin{vmatrix}1&0\\2&1\end{vmatrix} - 0 + 2\begin{vmatrix}2&1\\0&2\end{vmatrix}$$
$$= 1 + 8$$
$$= 9$$

Since $|\mathbf{A}| \neq 0$, $\mathbf{A}^{-1}$ exists, and so a non-trivial solution exists for $\mathbf{c}$.

$\Rightarrow$ W spans $\mathbb{R}^3$.

b If $\mathbf{x} = \begin{pmatrix}6\\7\\8\end{pmatrix}$ then $\begin{pmatrix}6\\7\\8\end{pmatrix} = \begin{pmatrix}1&0&2\\2&1&0\\0&2&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$

$$\therefore\ \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}1&0&2\\2&1&0\\0&2&1\end{pmatrix}^{-1}\begin{pmatrix}6\\7\\8\end{pmatrix} = \begin{pmatrix}2\\3\\2\end{pmatrix}$$

{using technology}

$$\therefore\ c_1 = 2, \quad c_2 = 3, \quad \text{and} \quad c_3 = 2$$
$$\therefore\ \begin{pmatrix}6\\7\\8\end{pmatrix} = 2\begin{pmatrix}1\\2\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\2\end{pmatrix} + 2\begin{pmatrix}2\\0\\1\end{pmatrix}$$

5 $$\text{lin}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\3\end{pmatrix}\right\}$$
$$= \left\{c_1\begin{pmatrix}1\\2\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\2\\3\end{pmatrix} \,\middle|\, c_1,\, c_2 \in \mathbb{R}\right\}$$
$$= \left\{\begin{pmatrix}c_1-c_2\\2c_1+2c_2\\c_1+3c_2\end{pmatrix} \,\middle|\, c_1,\, c_2 \in \mathbb{R}\right\}$$

Let $x = c_1 - c_2$, $y = 2c_1 + 2c_2$, and $z = c_1 + 3c_2$

Solving

$$\begin{array}{rl} c_1 - c_2 &= x \\ c_1 + c_2 &= \dfrac{y}{2} \\ \hline 2c_1 \quad &= x + \dfrac{y}{2} \end{array}$$

$$\therefore \quad c_1 = \frac{2x+y}{4}$$

and

$$\begin{aligned} c_2 &= \frac{y}{2} - c_1 \\ &= \frac{y}{2} - \frac{2x+y}{4} \\ &= \frac{2y-2x-y}{4} \\ &= \frac{y-2x}{4} \end{aligned}$$

$$\begin{aligned} z &= c_1 + 3c_2 \\ \therefore \quad z &= \frac{2x+y}{4} + 3\left(\frac{y-2x}{4}\right) \\ \therefore \quad 4z &= 2x + y + 3y - 6x \\ \therefore \quad 4z &= -4x + 4y \end{aligned}$$

$\therefore$ the equation of the plane is $x - y + z = 0$.

6 **a** Let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \\ 2x_1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \\ 2x_2 \end{pmatrix}$ be in S, and let $c \in \mathbb{R}$

$$\therefore \quad \mathbf{u} + \mathbf{v} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2(x_1 + x_2) \end{pmatrix} \quad \text{which } \in S$$

$$\text{and} \quad c\mathbf{u} = c\begin{pmatrix} x_1 \\ y_1 \\ 2x_1 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cy_1 \\ 2cx_1 \end{pmatrix} \quad \text{which is also } \in S$$

Thus, S is non-empty, and is closed under vector addition and scalar multiplication.

$\Rightarrow$ S is a subspace of $\mathbb{R}^3$.

b Let $\begin{pmatrix} x \\ y \\ 2x \end{pmatrix} = c_1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ for some $c_1, c_2 \in \mathbb{R}$

$$\therefore \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 2x \end{pmatrix}$$

which has augmented matrix

$$\left(\begin{array}{cc|c} 0 & 1 & x \\ 1 & 0 & y \\ 1 & 3 & 2x \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 3 & 2x \\ 0 & 1 & x \\ 1 & 0 & y \end{array}\right) \qquad \begin{array}{l} R_3 \to R_1 \\ R_1 \to R_2 \\ R_2 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & -x \\ 0 & 1 & x \\ 0 & -3 & y - 2x \end{array}\right) \qquad \begin{array}{l} R_1 - 3R_2 \to R_1 \\ \\ R_3 - R_1 \to R_3 \end{array}$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & -x \\ 0 & 1 & x \\ 0 & 0 & y + x \end{array}\right) \qquad R_3 + 3R_2 \to R_3$$

From row 3, solutions for c_1 and c_2 only exist for $x + y = 0$.

Vectors of the form $\begin{pmatrix} x \\ y \\ 2x \end{pmatrix}$ where $x + y \neq 0$ cannot be written in the form $c_1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$.

$\therefore$ W_1 does not span S.

c Let $\begin{pmatrix} x \\ y \\ 2x \end{pmatrix} = c_1\begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

for some $c_1, c_2 \in \mathbb{R}$

$$\therefore \quad \begin{pmatrix} -1 & 1 \\ 3 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 2x \end{pmatrix}$$

which has augmented matrix

$$\left(\begin{array}{cc|c} -1 & 1 & x \\ 3 & -1 & y \\ -2 & 2 & 2x \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -1 & -x \\ 3 & -1 & y \\ -2 & 2 & 2x \end{array}\right) \qquad -R_1 \to R_1$$

$$\sim \left(\begin{array}{cc|c} 1 & -1 & -x \\ 0 & 2 & y + 3x \\ 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} R_2 - 3R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{array}$$

$\therefore$ $2c_2 = y + 3x$ and $c_1 - c_2 = -x$

$\therefore$ $c_1 = \dfrac{x+y}{2}$, $c_2 = \dfrac{3x+y}{2}$

For any x and y, there exists a unique solution to c_1 and c_2 such that

$$\begin{pmatrix} x \\ y \\ 2x \end{pmatrix} = c_1\begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$\Rightarrow$ W_2 does span S.

7 **Proof:**

Let $\mathbf{u}$ and $\mathbf{w}$ be in W.

$\therefore$ $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + ... + c_r\mathbf{v}_r$

and $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + ... + k_r\mathbf{v}_r$

where c_i and $k_i \in \mathbb{R}$

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r \in W$, W is not empty.

Also $\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{v}_1 + (c_2 + k_2)\mathbf{v}_2 + ... + (c_r + k_r)\mathbf{v}_r$

$\therefore$ $\mathbf{u} + \mathbf{w} \in W$

and

$$\begin{aligned} c\mathbf{u} &= c(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_r\mathbf{v}_r) \\ &= cc_1\mathbf{v}_1 + cc_2\mathbf{v}_2 + ... + cc_r\mathbf{v}_r \end{aligned}$$

$\therefore$ $c\mathbf{u} \in W$

$\therefore$ W is closed under vector addition and scalar multiplication.

Hence, W is a subspace of $\mathbb{R}^n$.

Let $\mathbf{z} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + ... + m_r\mathbf{v}_r$ be in W,

and let V be any subspace of $\mathbb{R}^n$ containing $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r$.

$\therefore$ $m_1\mathbf{v}_1, m_2\mathbf{v}_2, m_3\mathbf{v}_3, ..., m_r\mathbf{v}_r \in V$

{V is closed under vector scalar multiplication}

and $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + ... + m_r\mathbf{v}_r \in V$

{V is closed under vector addition}

$\therefore$ $\mathbf{z} \in V$

Thus every element of W is also an element of V

$\therefore$ $W \subseteq V$

$\therefore$ W is the smallest subspace containing $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_r$.

EXERCISE 11.4

1 **a** If $x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \mathbf{0}$ then

$$x_1\begin{pmatrix}1\\0\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\0\end{pmatrix}+x_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\therefore \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\therefore x_1=x_2=x_3=0$$

$\therefore$ the system has only the trivial solution

$\therefore$ $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are linearly independent.

b If $x_1\begin{pmatrix}1\\0\\0\end{pmatrix}+x_2\begin{pmatrix}1\\1\\0\end{pmatrix}+x_3\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\begin{aligned}\text{then}\quad x_1+x_2+x_3&=0\\ x_2+x_3&=0\\ x_3&=0\end{aligned}$$

$$\therefore x_3=0,\ x_2=0,\ x_1=0$$

$\therefore$ the system has only the trivial solution

$\therefore$ $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ are linearly independent.

c If $x_1\begin{pmatrix}0\\1\\0\end{pmatrix}+x_2\begin{pmatrix}-1\\0\\-2\end{pmatrix}+x_3\begin{pmatrix}2\\3\\4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\begin{aligned}\text{then}\quad -x_2+2x_3&=0\\ x_1\qquad +3x_3&=0\\ -2x_2+4x_3&=0\end{aligned}$$

which has augmented matrix

$$\left(\begin{array}{ccc|c}0&-1&2&0\\1&0&3&0\\0&-2&4&0\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&0&3&0\\0&1&-2&0\\0&0&0&0\end{array}\right)\qquad\{\text{using technology}\}$$

$\therefore$ $x_1+3x_3=0$ and $x_2-2x_3=0$

If $x_3=t$ then $x_2=2t$ and $x_1=-3t$

$\therefore$ $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=t\begin{pmatrix}-3\\2\\1\end{pmatrix}$, $t\in\mathbb{R}$, which is a non-trivial solution.

$\therefore$ $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

2 Consider $x_1\begin{pmatrix}t\\1\\1\end{pmatrix}+x_2\begin{pmatrix}1\\t\\1\end{pmatrix}+x_3\begin{pmatrix}1\\1\\t\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\therefore \begin{pmatrix}t&1&1\\1&t&1\\1&1&t\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

The system has AM

$$\left(\begin{array}{ccc|c}t&1&1&0\\1&t&1&0\\1&1&t&0\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&1&t&0\\1&t&1&0\\t&1&1&0\end{array}\right)\qquad R_1\leftrightarrow R_3$$

$$\sim\left(\begin{array}{ccc|c}1&1&t&0\\0&t-1&1-t&0\\0&1-t&1-t^2&0\end{array}\right)\qquad\begin{array}{l}R_2-R_1\to R_2\\R_3-tR_1\to R_3\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&1&t&0\\0&t-1&1-t&0\\0&0&-t^2-t+2&0\end{array}\right)\qquad R_3+R_2\to R_3$$

The vectors are linearly independent if the system has only the trivial solution $x_1=x_2=x_3=0$.

From row 2 and row 3, this is when

$$\begin{aligned}t-1\neq 0\quad&\text{and}\quad -t^2-t+2\neq 0\\ \therefore\ t\neq 1\quad&\text{and}\quad (t+2)(t-1)\neq 0\\ &\qquad\therefore\ t\neq -2 \text{ or } 1\end{aligned}$$

$\therefore$ $\begin{pmatrix}t\\1\\1\end{pmatrix}, \begin{pmatrix}1\\t\\1\end{pmatrix}, \begin{pmatrix}1\\1\\t\end{pmatrix}$ are linearly dependent when $t=-2$ or 1.

3 **a** As $0\mathbf{u}+1\mathbf{v}+(-1)\mathbf{v}=\mathbf{0}$,
$\mathbf{u}$, $\mathbf{v}$, $\mathbf{v}$ are linearly dependent.

b As $-1\mathbf{u}+1\mathbf{v}+1(\mathbf{u}-\mathbf{v})=\mathbf{0}$,
$\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}-\mathbf{v}$ are linearly dependent.

c As $-2\mathbf{u}+1(\mathbf{u}+\mathbf{v})+1(\mathbf{u}-\mathbf{v})=\mathbf{0}$,
$\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, and $\mathbf{u}-\mathbf{v}$ are linearly dependent.

4 Consider $\mathbf{u}, \mathbf{v}\in\mathbb{R}^2$.

$\mathbf{u}$, $\mathbf{v}$ are linearly dependent

$\Leftrightarrow$ $a\mathbf{u}+b\mathbf{v}=\mathbf{0}$ for some $a, b\in\mathbb{R}$, both are not zero

$\Leftrightarrow$ $a\mathbf{u}=-b\mathbf{v}$

$\Leftrightarrow$ $\mathbf{u}=-\dfrac{b}{a}\mathbf{v}$ if $a\neq 0$

$\Leftrightarrow$ $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$ (if $b\neq 0$)
or $\mathbf{u}=\mathbf{0}$ (if $b=0$)

5 Consider $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}\in\mathbb{R}^3$, then as $1\mathbf{0}+0\mathbf{u}+0\mathbf{v}=\mathbf{0}$ where the scalars are not all zero

$\Rightarrow$ $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ are linearly dependent.

6 If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent then
$x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3=\mathbf{0}$ where x_1, x_2, x_3 are not all zero

$\Rightarrow$ $x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3+0\mathbf{v}_4=\mathbf{0}$ where $x_1, x_2, x_3, 0$ are not all zero

$\Rightarrow$ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

7 **a** If $\mathbf{u}=\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}-1\\1\\3\end{pmatrix}$ then

$$\begin{aligned}\mathbf{u}\times\mathbf{v}&=\begin{pmatrix}2\times 3-1\times 1\\1\times -1-1\times 3\\1\times 1-2\times -1\end{pmatrix}\\ &=\begin{pmatrix}5\\-4\\3\end{pmatrix}\end{aligned}$$

Consider $$x_1\begin{pmatrix}1\\2\\1\end{pmatrix}+x_2\begin{pmatrix}-1\\1\\3\end{pmatrix}+x_3\begin{pmatrix}5\\-4\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1&-1&5\\2&1&-4\\1&3&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{aligned}\Rightarrow x_1-x_2+5x_3&=0\\2x_1+x_2-4x_3&=0\\x_1+3x_2+3x_3&=0\end{aligned}$$

which has augmented matrix

$$\left(\begin{array}{ccc|c}1&-1&5&0\\2&1&-4&0\\1&3&3&0\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)\quad\{\text{using technology}\}$$

$\Rightarrow$ $x_1=x_2=x_3=x_4=0$ which has the trivial solution.

Thus $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ are linearly independent.

b If $\mathbf{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$

then $\mathbf{v}=\begin{pmatrix}ku_1\\ku_2\\ku_3\end{pmatrix}$, $k\in\mathbb{R}$

$$\therefore\ \mathbf{u}\times\mathbf{v}=\begin{pmatrix}u_2ku_3-u_3ku_2\\u_3ku_1-u_1ku_3\\u_1ku_2-u_2ku_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\therefore$ using **5**, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ include $\mathbf{0}$ so they are linearly dependent.

$\therefore$ no, the result in **a** is not true in general.

8 **Proof:**

$(\Rightarrow)$ If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors of $\mathbb{R}^3$ which are linearly dependent, then $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$ for some c_1, c_2, c_3 which are not all 0.

$\therefore$ $c_1\mathbf{v}_1=-c_2\mathbf{v}_2-c_3\mathbf{v}_3$ if $c_1\neq0$

$\therefore$ $\mathbf{v}_1=-\dfrac{c_2}{c_1}\mathbf{v}_2-\dfrac{c_3}{c_1}\mathbf{v}_3$

$\Rightarrow$ $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

$(\Leftarrow)$ If $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$

then $\mathbf{v}_1=x_2\mathbf{v}_2+x_3\mathbf{v}_3$

$\therefore$ $-\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3=\mathbf{0}$

$\therefore$ -1, x_2, x_3 is a non-trivial solution to $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$

$\Rightarrow$ $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

EXERCISE 11.5

1 $\mathbf{v}_1=\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix}3\\3\\4\end{pmatrix}$, $\mathbf{v}_3=\begin{pmatrix}1\\-1\\3\end{pmatrix}$

(1) Suppose there exists x_1, x_2, x_3 in $\mathbb{R}$ such that:

$$x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3=\mathbf{0}$$

$$\Rightarrow x_1\begin{pmatrix}1\\2\\1\end{pmatrix}+x_2\begin{pmatrix}3\\3\\4\end{pmatrix}+x_3\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1&3&1\\2&3&-1\\1&4&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

which has augmented matrix

$$\left(\begin{array}{ccc|c}1&3&1&0\\2&3&-1&0\\1&4&3&0\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)\quad\{\text{using technology}\}$$

The system has the trivial solution $x_1=x_2=x_3=0$, so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

(2) Now $\mathbf{A}=(\mathbf{v}_1\,|\,\mathbf{v}_2\,|\,\mathbf{v}_3)=\begin{pmatrix}1&3&1\\2&3&-1\\1&4&3\end{pmatrix}$

has $|\mathbf{A}|=-3\neq0$

$\therefore$ $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbb{R}^3$.

From (1) and (2), $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent, and form a basis for $\mathbb{R}^3$.

2 **a** $\mathbf{v}_1=\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix}0\\1\\1\end{pmatrix}$, $\mathbf{v}_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$

(1) Suppose there exists x_1, x_2, x_3 in $\mathbb{R}$ such that:

$$x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3=\mathbf{0}$$

$$\Rightarrow x_1\begin{pmatrix}0\\0\\1\end{pmatrix}+x_2\begin{pmatrix}0\\1\\1\end{pmatrix}+x_3\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

which has augmented matrix

$$\left(\begin{array}{ccc|c}0&0&1&0\\0&1&1&0\\1&1&1&0\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)\quad\{\text{using technology}\}$$

The system has the trivial solution $x_1=x_2=x_3=0$, so $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

(2) Now $\mathbf{A}=(\mathbf{v}_1\,|\,\mathbf{v}_2\,|\,\mathbf{v}_3)=\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}$

has $|\mathbf{A}|=-1\neq0$

$\therefore$ $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbb{R}^3$.

From (1) and (2), $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

b $\mathbf{v}_1=\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix}2\\1\\-1\end{pmatrix}$, $\mathbf{v}_3=\begin{pmatrix}3\\-1\\-4\end{pmatrix}$

$\mathbf{A}=(\mathbf{v}_1\,|\,\mathbf{v}_2\,|\,\mathbf{v}_3)=\begin{pmatrix}1&2&3\\0&1&-1\\-1&-1&-4\end{pmatrix}$

has $|\mathbf{A}|=0$

$\therefore$ $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ do not span $\mathbb{R}^3$

$\therefore$ $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ is not a basis for $\mathbb{R}^3$.

c $\mathbf{v}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}-1\\1\\4\end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix}1\\3\\10\end{pmatrix}$

(1) Suppose there exists x_1, x_2, x_3 in $\mathbb{R}$ such that:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$$

$$\Rightarrow \quad x_1\begin{pmatrix}1\\2\\3\end{pmatrix} + x_2\begin{pmatrix}-1\\1\\4\end{pmatrix} + x_3\begin{pmatrix}1\\3\\10\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}1 & -1 & 1\\2 & 1 & 3\\3 & 4 & 10\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

which has augmented matrix

$$\left(\begin{array}{ccc|c}1 & -1 & 1 & 0\\2 & 1 & 3 & 0\\3 & 4 & 10 & 0\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right) \quad \{\text{using technology}\}$$

The system has the trivial solution $x_1 = x_2 = x_3 = 0$, so $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

(2) Now $\mathbf{A} = (\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \mathbf{v}_3) = \begin{pmatrix}1 & -1 & 1\\2 & 1 & 3\\3 & 4 & 10\end{pmatrix}$

has $|\,\mathbf{A}\,| = 14 \neq 0$

$\therefore$ $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbb{R}^3$.

From (1) and (2), $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

3 **a** If $x + 2y - 3z = 0$ and we let $y = s$, $z = t$ then $x = -2s + 3t$

$$\Rightarrow \quad \begin{pmatrix}x\\y\\z\end{pmatrix} = s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}3\\0\\1\end{pmatrix} \quad \text{for all } s, t \in \mathbb{R}$$

$\therefore$ $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix}\right\}$ is a basis.

b If $x + z = 0$

then $x = -z$.

So, if $z = t$, $x = -t$, $y = s$

$$\Rightarrow \quad \begin{pmatrix}x\\y\\z\end{pmatrix} = s\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}-1\\0\\1\end{pmatrix} \quad \text{for all } s, t \in \mathbb{R}$$

$\therefore$ $\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ is a basis.

c $\begin{pmatrix}x\\y\\z\end{pmatrix} = t\begin{pmatrix}1\\-5\\2\end{pmatrix}$ for all $t \in \mathbb{R}$,

$\therefore$ $\left\{\begin{pmatrix}1\\-5\\2\end{pmatrix}\right\}$ is a basis.

d $\begin{pmatrix}a\\b\\a-b\end{pmatrix} = a\begin{pmatrix}1\\0\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\-1\end{pmatrix}$ for all $a, b \in \mathbb{R}$

$\therefore$ $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$ is a basis.

4 **a** The system has AM

$$\left(\begin{array}{ccc|c}1 & 2 & 3 & 0\\2 & -1 & 7 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}\boxed{1} & 0 & \frac{17}{5} & 0\\0 & \boxed{1} & -\frac{1}{5} & 0\end{array}\right)$$

$\{\text{using technology}\}$

The free variable is x_3.

Letting $x_3 = t$, we find $x_2 = \frac{1}{5}t$ and $x_1 = -\frac{17}{5}t$.

Thus $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = t\begin{pmatrix}-\frac{17}{5}\\\frac{1}{5}\\1\end{pmatrix}$, $t \in \mathbb{R}$

$\therefore$ $\begin{pmatrix}-\frac{17}{5}\\\frac{1}{5}\\1\end{pmatrix}$ spans the solution space

$\therefore$ $\left\{\begin{pmatrix}-\frac{17}{5}\\\frac{1}{5}\\1\end{pmatrix}\right\}$ is a basis for the solution space,

and $\dim(S) = 1$.

b The system has AM

$$\left(\begin{array}{ccc|c}1 & 1 & -1 & 0\\1 & -1 & 2 & 0\\2 & 4 & -5 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}\boxed{1} & 0 & \frac{1}{2} & 0\\0 & \boxed{1} & -\frac{3}{2} & 0\\0 & 0 & 0 & 0\end{array}\right)$$

$\{\text{using technology}\}$

The free variable is x_3.

Letting $x_3 = t$, we find $x_2 = \frac{3}{2}t$ and $x_1 = -\frac{1}{2}t$.

Thus $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = t\begin{pmatrix}-\frac{1}{2}\\\frac{3}{2}\\1\end{pmatrix}$, $t \in \mathbb{R}$

$\therefore$ $\begin{pmatrix}-\frac{1}{2}\\\frac{3}{2}\\1\end{pmatrix}$ spans the solution space

$\therefore$ $S = \left\{\begin{pmatrix}-\frac{1}{2}\\\frac{3}{2}\\1\end{pmatrix}\right\}$ is a basis for the solution space,

and $\dim(S) = 1$.

c The system has AM

$$\left(\begin{array}{cccc|c}1 & 1 & -1 & 1 & 0\\2 & -1 & 2 & -1 & 0\end{array}\right) \sim \left(\begin{array}{cccc|c}\boxed{1} & 0 & \frac{1}{3} & 0 & 0\\0 & \boxed{1} & -\frac{4}{3} & 1 & 0\end{array}\right)$$

$\{\text{using technology}\}$

The free variables are x_3 and x_4.

Letting $x_3 = s$, $x_4 = t$, we find $x_2 = \frac{4}{3}s - t$ and $x_1 = -\frac{1}{3}s$.

$$\therefore \quad \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = s\begin{pmatrix}-\frac{1}{3}\\\frac{4}{3}\\1\\0\end{pmatrix} + t\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}, \quad s, t \in \mathbb{R}$$

$\therefore$ $\begin{pmatrix}-\frac{1}{3}\\\frac{4}{3}\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$ span the solution space and are linearly independent.

$$\therefore \quad S = \left\{ \begin{pmatrix} -\frac{1}{3} \\ \frac{4}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis for the}$$

solution space, and $\dim(S) = 2$.

d The system has AM

$$\left(\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 & 3 & 0 \\ 2 & 3 & 0 & 1 & 3 & 0 \\ 2 & 3 & 2 & 2 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 & 0 \end{array}\right)$$

{using technology}

The free variable is x_5.

Letting $x_5 = t$, we find $x_4 = -3t$, $x_3 = \frac{1}{2}t$, $x_2 = 0$, and $x_1 = 0$.

$$\therefore \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -3 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\therefore \quad \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -3 \\ 1 \end{pmatrix} \text{ spans the solution space}$$

$$\therefore \quad S = \left\{ \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -3 \\ 1 \end{pmatrix} \right\} \text{ is a basis for the solution space,}$$

and $\dim(S) = 1$.

e The system has AM

$$\left(\begin{array}{ccccc|c} 1 & 1 & 2 & -1 & 3 & 0 \\ 2 & -1 & 0 & 1 & 2 & 0 \\ 3 & 0 & 2 & 0 & 5 & 0 \\ 1 & -2 & -2 & 2 & -1 & 0 \\ 4 & 1 & 1 & -1 & 3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 0 & \frac{5}{9} & 0 \\ 0 & \boxed{1} & 0 & -1 & -\frac{8}{9} & 0 \\ 0 & 0 & \boxed{1} & 0 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \text{\{using technology\}}$$

The free variables are x_4 and x_5.

Letting $x_4 = s$, $x_5 = t$ we find $x_3 = -\frac{5}{3}t$, $x_2 = s + \frac{8}{9}t$, and $x_1 = -\frac{5}{9}t$.

$$\therefore \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{5}{9} \\ \frac{8}{9} \\ -\frac{5}{3} \\ 0 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

$$\therefore \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{5}{9} \\ \frac{8}{9} \\ -\frac{5}{3} \\ 0 \\ 1 \end{pmatrix} \text{ span the solution space and}$$

are linearly independent.

$$\therefore \quad S = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{5}{9} \\ \frac{8}{9} \\ -\frac{5}{3} \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis for the}$$

solution space, and $\dim(S) = 2$.

5 **a** $$\begin{pmatrix} a \\ b \\ 0 \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$

$$\therefore \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis}$$

$\therefore$ the subspace has dimension 3.

b $$\begin{pmatrix} a \\ b \\ 2a \\ a+b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{where } a, b \in \mathbb{R}$$

$$\therefore \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis}$$

$\therefore$ the subspace has dimension 2.

6 $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of V in $\mathbb{R}^3$

$\therefore$ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent .... (1)

and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span V .... (2)

Consider $a(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + b(\mathbf{v}_1 + \mathbf{v}_2) + c(\mathbf{v}_1) = \mathbf{0}$

$\therefore$ $(a + b + c)\mathbf{v}_1 + (a + b)\mathbf{v}_2 + a\mathbf{v}_3 = \mathbf{0}$

But, using (1) $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has only the trivial solution $x_1 = x_2 = x_3 = 0$

$\therefore$ $a + b + c = 0$, $a + b = 0$, and $a = 0$

$\therefore$ $a = b = c = 0$

$\therefore$ $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1$ are linearly independent.

$$\begin{aligned} \text{Now} \quad & a(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + b(\mathbf{v}_1 + \mathbf{v}_2) + c\mathbf{v}_1 \\ &= (a + b + c)\mathbf{v}_1 + (a + b)\mathbf{v}_2 + a\mathbf{v}_3 \\ &= x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 \\ &= \text{lin}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \end{aligned}$$

$\therefore$ $\text{lin}\{\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1\} = \text{lin}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span V {from (2)}

$\therefore$ $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1$ also span V.

$\therefore$ $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1$ are linearly independent and span V

$\therefore$ $\{\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1\}$ is a basis for V.

EXERCISE 11.6

1 **a** $\begin{pmatrix}1 & 0 & 1 & -2\end{pmatrix}$, $\begin{pmatrix}2 & -1 & 1 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 3 & 2 & 1\end{pmatrix}$

b $\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\begin{pmatrix}0\\-1\\3\end{pmatrix}$, $\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\begin{pmatrix}-2\\3\\1\end{pmatrix}$

2 **a** $\mathbf{Ax}=\mathbf{0}$ has AM

$$\left(\begin{array}{cccc|c}1 & 2 & 1 & 0 & 0\\ 2 & -1 & 1 & 3 & 0\\ 3 & -4 & 1 & 1 & 0\end{array}\right) \sim \left(\begin{array}{cccc|c}\boxed{1} & 0 & \frac{3}{5} & 0 & 0\\ 0 & \boxed{1} & \frac{1}{5} & 0 & 0\\ 0 & 0 & 0 & \boxed{1} & 0\end{array}\right)$$

{using technology}

Letting $x_3 = t$, we find $x_4 = 0$, $x_2 = -\frac{1}{5}t$, and $x_1 = -\frac{3}{5}t$

$\therefore$ Null $\mathbf{A}$ is $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = t\begin{pmatrix}-\frac{3}{5}\\-\frac{1}{5}\\1\\0\end{pmatrix}$, $t \in \mathbb{R}$

which is the subspace spanned by $\left\{\begin{pmatrix}-\frac{3}{5}\\-\frac{1}{5}\\1\\0\end{pmatrix}\right\}$

$\therefore$ nullity $(\mathbf{A}) = 1$.

b $\mathbf{Ax}=\mathbf{0}$ has AM

$$\left(\begin{array}{ccccc|c}2 & 1 & 1 & 3 & 1 & 0\\ 3 & 3 & 1 & 5 & 6 & 0\\ 1 & -1 & 1 & 1 & -2 & 0\end{array}\right) \sim \left(\begin{array}{ccccc|c}\boxed{1} & 0 & \frac{2}{3} & \frac{4}{3} & 0 & 0\\ 0 & \boxed{1} & -\frac{1}{3} & \frac{1}{3} & 0 & 0\\ 0 & 0 & 0 & 0 & \boxed{1} & 0\end{array}\right)$$

{using technology}

Letting $x_3 = s$, $x_4 = t$, we find $x_5 = 0$, $x_2 = \frac{1}{3}s - \frac{1}{3}t$, and $x_1 = -\frac{2}{3}s - \frac{4}{3}t$

$\therefore$ Null $\mathbf{A}$ is $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = s\begin{pmatrix}-\frac{2}{3}\\\frac{1}{3}\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}-\frac{4}{3}\\-\frac{1}{3}\\0\\1\\0\end{pmatrix}$,

where $s, t \in \mathbb{R}$ which is the subspace spanned by

$$\left\{\begin{pmatrix}-\frac{2}{3}\\\frac{1}{3}\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-\frac{4}{3}\\-\frac{1}{3}\\0\\1\\0\end{pmatrix}\right\}$$

$\therefore$ nullity $(\mathbf{A}) = 2$.

3 **a** $\mathbf{A} = \begin{pmatrix}1 & -2\\ -3 & 6\end{pmatrix} \sim \mathbf{R} = \begin{pmatrix}\boxed{1} & -2\\ 0 & 0\end{pmatrix}$ {using technology}

i By **Theorem 3**, a basis for the row space of $\mathbf{R}$ is $\left\{\begin{pmatrix}1 & -2\end{pmatrix}\right\}$

$\therefore$ by **Theorem 1**, a basis for the row space of $\mathbf{A}$ is $\left\{\begin{pmatrix}1 & -2\end{pmatrix}\right\}$.

ii By **Theorem 3**, a basis for the column space of $\mathbf{R}$ is $\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$

$\therefore$ by **Theorem 2**, a basis for the column space of $\mathbf{A}$ is $\left\{\begin{pmatrix}1\\-3\end{pmatrix}\right\}$.

iii rank $(\mathbf{A})$ = row rank of $\mathbf{A}$ = column rank of $\mathbf{A}$ = 1.

b $\mathbf{A} = \begin{pmatrix}1 & 0 & 1 & 2\\ 1 & 2 & -1 & 3\\ 3 & 2 & 1 & 5\end{pmatrix} \sim \mathbf{R} = \begin{pmatrix}\boxed{1} & 0 & 1 & 0\\ 0 & \boxed{1} & -1 & 0\\ 0 & 0 & 0 & \boxed{1}\end{pmatrix}$

{using technology}

i By **Theorem 3**, a basis for the row space of $\mathbf{R}$ is $\left\{\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}\right\}$

$\therefore$ by **Theorem 1**, a basis for the row space of $\mathbf{A}$ is $\left\{\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}\right\}$.

ii By **Theorem 3**, a basis for the column space of $\mathbf{R}$ is

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$\therefore$ by **Theorem 2**, a basis for the column space of $\mathbf{A}$ is

$$\left\{\begin{pmatrix}1\\1\\3\end{pmatrix}, \begin{pmatrix}0\\2\\2\end{pmatrix}, \begin{pmatrix}2\\3\\5\end{pmatrix}\right\}.$$

iii rank $(\mathbf{A})$ = row rank of $\mathbf{A}$ = column rank of $\mathbf{A}$ = 3.

c $\mathbf{A} = \begin{pmatrix}1 & 1 & 1 & 2 & 4\\ 2 & 0 & -1 & 1 & 0\\ 3 & -1 & -3 & 0 & -8\\ 2 & 2 & 2 & 4 & 1\\ 5 & 1 & -1 & 4 & -7\end{pmatrix}$

$\sim \mathbf{R} = \begin{pmatrix}\boxed{1} & 0 & -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & \boxed{1} & \frac{3}{2} & \frac{3}{2} & 0\\ 0 & 0 & 0 & 0 & \boxed{1}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$ {using technology}

i By **Theorem 3**, a basis for the row space of $\mathbf{R}$ is $\left\{\begin{pmatrix}1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & \frac{3}{2} & \frac{3}{2} & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0 & 0 & 1\end{pmatrix}\right\}$

$\therefore$ by **Theorem 1**, a basis for the row space of $\mathbf{A}$ is $\left\{\begin{pmatrix}1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & \frac{3}{2} & \frac{3}{2} & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0 & 0 & 1\end{pmatrix}\right\}$.

ii By **Theorem 3**, a basis for the column space of $\mathbf{R}$ is

$$\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}\right\}$$

$\therefore$ by **Theorem 2**, a basis for the column space of $\mathbf{A}$ is

$$\left\{\begin{pmatrix}1\\2\\3\\2\\5\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\2\\1\end{pmatrix}, \begin{pmatrix}4\\0\\-8\\1\\-7\end{pmatrix}\right\}.$$

iii rank $(\mathbf{A})$ = row rank of $\mathbf{A}$ = column rank of $\mathbf{A}$ = 3.

4 **a** For $\mathbf{A} = \begin{pmatrix}1 & 3 & 2\\ 2 & 6 & 4\end{pmatrix}$,

$\mathbf{A}^T = \begin{pmatrix}1 & 2\\ 3 & 6\\ 2 & 4\end{pmatrix} \sim \mathbf{R} = \begin{pmatrix}\boxed{1} & 2\\ 0 & 0\\ 0 & 0\end{pmatrix}$ {using technology}

The first column of $\mathbf{R}$ forms a basis for the column space of $\mathbf{R}$

$\therefore$ the first column of $\mathbf{A}^T$ forms a basis for the column space of $\mathbf{A}^T$

$\therefore$ the first row of $\mathbf{A}$ forms a basis for the row space of $\mathbf{A}$

$\therefore$ $\left\{\begin{pmatrix}1 & 3 & 2\end{pmatrix}\right\}$ forms a basis for the row space of $\mathbf{A}$.

b For $\mathbf{A}=\begin{pmatrix}1&0&1&2\\1&2&-1&3\\3&2&1&7\end{pmatrix}$,

$$\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}1&1&3\\0&2&2\\1&-1&1\\2&3&7\end{pmatrix}\sim\mathbf{R}=\begin{pmatrix}\boxed{1}&0&2\\0&\boxed{1}&1\\0&0&0\\0&0&0\end{pmatrix}$$

{using technology}

The first two columns of $\mathbf{R}$ form a basis for the column space of $\mathbf{R}$

$\therefore$ the first two columns of $\mathbf{A}^{\mathrm{T}}$ form a basis for the column space of $\mathbf{A}^{\mathrm{T}}$

$\therefore$ the first two rows of $\mathbf{A}$ form a basis for the row space of $\mathbf{A}$

$\therefore$ $\left\{\begin{pmatrix}1&0&1&2\end{pmatrix},\begin{pmatrix}1&2&-1&3\end{pmatrix}\right\}$ forms a basis for the row space of $\mathbf{A}$.

c For $\mathbf{A}=\begin{pmatrix}1&1&1&2&4\\2&0&-1&1&0\\3&-1&-3&0&-8\\2&2&2&4&1\\5&1&-1&4&-7\end{pmatrix}$,

$$\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}1&2&3&2&5\\1&0&-1&2&1\\1&-1&-3&2&-1\\2&1&0&4&4\\4&0&-8&1&-7\end{pmatrix}$$

$$\sim\mathbf{R}=\begin{pmatrix}\boxed{1}&0&0&\frac{15}{4}&\frac{15}{4}\\0&\boxed{1}&0&-\frac{7}{2}&-\frac{7}{2}\\0&0&\boxed{1}&\frac{7}{4}&\frac{11}{4}\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\quad\text{\{using technology\}}$$

The first three columns of $\mathbf{R}$ form a basis for the column space of $\mathbf{R}$

$\therefore$ the first three columns of $\mathbf{A}^{\mathrm{T}}$ form a basis for the column space of $\mathbf{A}^{\mathrm{T}}$

$\therefore$ the first three rows of $\mathbf{A}$ form a basis for the row space of $\mathbf{A}$

$\therefore$ $\left\{\begin{pmatrix}1&1&1&2&4\end{pmatrix},\begin{pmatrix}2&0&-1&1&0\end{pmatrix},\begin{pmatrix}3&-1&-3&0&-8\end{pmatrix}\right\}$ forms a basis for the row space of $\mathbf{A}$.

5 The system has the form $\mathbf{A}\mathbf{x}=\mathbf{b}$

where $\mathbf{A}=\begin{pmatrix}1&1&-1\\1&-2&1\\2&-1&0\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}4\\6\\11\end{pmatrix}$

$$\mathbf{A}\sim\mathbf{R}=\begin{pmatrix}\boxed{1}&0&-\frac{1}{3}\\0&\boxed{1}&-\frac{2}{3}\\0&0&0\end{pmatrix}\quad\text{\{using technology\}}$$

$\therefore$ a basis for the column space of $\mathbf{R}$ is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$.

$\therefore$ a basis for the column space of $\mathbf{A}$ is

$$S=\left\{\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}1\\-2\\-1\end{pmatrix}\right\}.$$

Now if $\begin{pmatrix}4\\6\\11\end{pmatrix}=a\begin{pmatrix}1\\1\\2\end{pmatrix}+b\begin{pmatrix}1\\-2\\-1\end{pmatrix}$

then $\begin{cases}a+b=4\\a-2b=6\\2a-b=11\end{cases}$ which has no solution.

$\therefore$ $\mathbf{b}=\begin{pmatrix}4\\6\\11\end{pmatrix}$ is not in the column space of $\mathbf{A}$

$\therefore$ $\mathbf{A}\mathbf{x}=\mathbf{b}$ is inconsistent.

6 For $\mathbf{A}\mathbf{x}=\mathbf{0}$, the AM is

$$\left(\begin{array}{ccccc|c}1&3&1&-2&1&0\\2&6&4&-8&3&0\\-1&-3&1&-2&5&0\end{array}\right)\sim\left(\begin{array}{ccccc|c}\boxed{1}&3&0&0&0&0\\0&0&\boxed{1}&-2&0&0\\0&0&0&0&\boxed{1}&0\end{array}\right)$$

Letting $x_2=s$, $x_4=t$, we find $x_5=0$, $x_3=2t$, and $x_1=-3s$

$\therefore$ Null $\mathbf{A}$ is $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=s\begin{pmatrix}-3\\1\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}0\\0\\2\\1\\0\end{pmatrix}$,

where $s,t\in\mathbb{R}$

which is the subspace spanned by $\left\{\begin{pmatrix}-3\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\2\\1\\0\end{pmatrix}\right\}$.

$\therefore$ nullity $(\mathbf{A})=2$.

We will use the transpose method to find a basis for the row space of $\mathbf{A}$.

$$\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}1&2&-1\\3&6&-3\\1&4&1\\-2&-8&-2\\1&3&5\end{pmatrix}\sim\mathbf{R}=\begin{pmatrix}\boxed{1}&0&0\\0&\boxed{1}&0\\0&0&\boxed{1}\\0&0&0\\0&0&0\end{pmatrix}$$

$\therefore$ the three rows of $\mathbf{A}$ form a basis for the row space of $\mathbf{A}$.

$\therefore$ $\left\{\begin{pmatrix}1&3&1&-2&1\end{pmatrix},\begin{pmatrix}2&6&4&-8&3\end{pmatrix},\begin{pmatrix}-1&-3&1&-2&5\end{pmatrix}\right\}$ forms a basis for the row space of $\mathbf{A}$.

$\therefore$ rank $(\mathbf{A})=3$

$\therefore$ rank $(\mathbf{A})$ + nullity $(\mathbf{A})=3+2=5$, which is the number of columns of $\mathbf{A}$ ✓

7 **a** $\begin{pmatrix}1\\2\\0\end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$\therefore$ these vectors are linearly independent

$\therefore$ a basis for the subspace is $\left\{\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$

b The subspace is the column space of $\mathbf{A}$, where

$$\mathbf{A}=\begin{pmatrix}1&0&1\\0&1&2\\1&1&3\end{pmatrix}\sim\mathbf{R}=\begin{pmatrix}\boxed{1}&0&1\\0&\boxed{1}&2\\0&0&0\end{pmatrix}$$

{using technology}

$\therefore$ a basis for the column space of $\mathbf{A}$ is

$$\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$$

$\therefore$ a basis for the subspace is $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$

$\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\right.$ is a linear combination of $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$ so it is not needed in the basis$\left.\right)$.

8 **a** The subspace is the column space of $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix}0&0&-1\\1&0&1\\0&1&0\\1&1&1\end{pmatrix} \sim \mathbf{R} = \begin{pmatrix}\boxed{1}&0&0\\0&\boxed{1}&0\\0&0&\boxed{1}\\0&0&0\end{pmatrix}$$

{using technology}

$\therefore$ a basis for the column space of $\mathbf{A}$ is

$$\left\{\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\1\end{pmatrix}\right\}$$

$\therefore$ a basis for the subspace is

$$\left\{\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\1\end{pmatrix}\right\}.$$

b The subspace is the column space of $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix}0&1&-1&2\\1&0&0&1.5\\1&0&1&0\\0&1&1&-1\end{pmatrix} \sim \mathbf{R} = \begin{pmatrix}\boxed{1}&0&0&\frac{3}{2}\\0&\boxed{1}&0&\frac{1}{2}\\0&0&\boxed{1}&-\frac{3}{2}\\0&0&0&0\end{pmatrix}$$

$\therefore$ a basis for the column space of $\mathbf{A}$ is

$$\left\{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\1\end{pmatrix}\right\}$$

$\therefore$ a basis for the subspace is

$$\left\{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\1\end{pmatrix}\right\}$$

$\left(\begin{pmatrix}2\\1.5\\0\\-1\end{pmatrix}\right.$ is a linear combination of the other three vectors.$\left.\right)$

EXERCISE 1J.1

1 Let $\mathbf{u} = \begin{pmatrix}x_1\\y_1\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}x_2\\y_2\end{pmatrix}$

(1) $$\mathrm{T}(\mathbf{u}+\mathbf{v}) = \mathrm{T}\left(\begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix}\right) = \begin{pmatrix}x_1+x_2\\y_1+y_2\\x_1+x_2+2(y_1+y_2)\end{pmatrix} = \begin{pmatrix}x_1\\y_1\\x_1+2y_1\end{pmatrix} + \begin{pmatrix}x_2\\y_2\\x_2+2y_2\end{pmatrix} = \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})$$

(2) For all $k \in \mathbb{R}$, $$\mathrm{T}(k\mathbf{u}) = \mathrm{T}\left(\begin{pmatrix}kx_1\\ky_1\end{pmatrix}\right) = \begin{pmatrix}kx_1\\ky_1\\kx_1+2ky_1\end{pmatrix} = k\begin{pmatrix}x_1\\y_1\\x_1+2y_1\end{pmatrix} = k\mathrm{T}(\mathbf{u})$$

Since the addition and scalar multiplication properties are satisfied, T is a linear transformation.

2 Let $\mathbf{u} = \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$

(1) $$\mathrm{T}(\mathbf{u}+\mathbf{v}) = \mathrm{T}\left(\begin{pmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2\end{pmatrix}\right) = \begin{pmatrix}x_1+x_2+y_1+y_2\\z_1+z_2-(y_1+y_2)\end{pmatrix} = \begin{pmatrix}x_1+y_1\\z_1-y_1\end{pmatrix} + \begin{pmatrix}x_2+y_2\\z_2-y_2\end{pmatrix} = \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})$$

(2) For all $k \in \mathbb{R}$, $$\mathrm{T}(k\mathbf{u}) = \mathrm{T}\left(\begin{pmatrix}kx_1\\ky_1\\kz_1\end{pmatrix}\right) = \begin{pmatrix}kx_1+ky_1\\kz_1-ky_1\end{pmatrix} = k\begin{pmatrix}x_1+y_1\\z_1-y_1\end{pmatrix} = k\mathrm{T}(\mathbf{u})$$

Since the addition and scalar multiplication properties are satisfied, T is a linear transformation.

3 Let $\mathbf{u} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}2\\1\\0\end{pmatrix}$.

Now $\mathrm{T}(\mathbf{u}+\mathbf{v}) = \mathrm{T}\left(\begin{pmatrix}3\\3\\3\end{pmatrix}\right) = \begin{pmatrix}9\\9\end{pmatrix}$

and $$\mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v}) = \mathrm{T}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\right) + \mathrm{T}\left(\begin{pmatrix}2\\1\\0\end{pmatrix}\right) = \begin{pmatrix}3\\6\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}3\\6\end{pmatrix}$$

For this $\mathbf{u}$ and $\mathbf{v}$, $\mathrm{T}(\mathbf{u}+\mathbf{v}) \neq \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})$

$\therefore$ T is not a linear transformation.

4 **Proof:** (By the Principle of Mathematical Induction)

P_r is that $\mathrm{T}(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + ... + k_r\mathbf{u}_r) = k_1\mathrm{T}(\mathbf{u}_1) + k_2\mathrm{T}(\mathbf{u}_2) + ... + k_r\mathrm{T}(\mathbf{u}_r)$ for $r \in \mathbb{Z}^+$

(1) If $r = 1$, $\mathrm{T}(k_1\mathbf{u}_1) = k_1\mathrm{T}(\mathbf{u}_1)$ is true {scalar multiplication property}

$\therefore$ P_1 is true.

(2) If P_j is true, then

$$\mathrm{T}(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + ... + k_j\mathbf{u}_j) = k_1\mathrm{T}(\mathbf{u}_1) + k_2\mathrm{T}(\mathbf{u}_2) + ... + k_j\mathrm{T}(\mathbf{u}_j), \quad j \in \mathbb{Z}^+ \quad .... \ (*)$$

$$\begin{aligned}\text{Now}\quad &\mathrm{T}(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + ... + k_j\mathbf{u}_j + k_{j+1}\mathbf{u}_{j+1})\\ &= \mathrm{T}(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + ... + k_j\mathbf{u}_j) + \mathrm{T}(k_{j+1}\mathbf{u}_{j+1})\\ &\quad \{\text{addition property}\}\\ &= k_1\mathrm{T}(\mathbf{u}_1) + k_2\mathrm{T}(\mathbf{u}_2) + ... + k_j\mathrm{T}(\mathbf{u}_j) + k_{j+1}\mathrm{T}(\mathbf{u}_{j+1})\\ &\quad \{\text{using } (*) \text{ and scalar multiplication property}\}\end{aligned}$$

Thus P_1 is true, and P_{j+1} is true whenever P_j is true.

$\therefore$ P_r is true for $r \in \mathbb{Z}^+$.

5 $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is the standard basis for $\mathbb{R}^2$.

$$\begin{aligned}\therefore\ \mathrm{T}\left(\begin{pmatrix}5\\-3\end{pmatrix}\right) &= \mathrm{T}\left(5\begin{pmatrix}1\\0\end{pmatrix} - 3\begin{pmatrix}0\\1\end{pmatrix}\right)\\ &= 5\mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right) - 3\mathrm{T}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)\\ &= 5\begin{pmatrix}1\\3\end{pmatrix} - 3\begin{pmatrix}1\\3\end{pmatrix}\\ &= \begin{pmatrix}2\\6\end{pmatrix}\end{aligned}$$

6 Let $\mathbf{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix}1\\2\end{pmatrix}$.

Suppose there exists $x_1, x_2 \in \mathbb{R}$ such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}$$

$$\therefore\ \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

which has augmented matrix

$$\left(\begin{array}{cc|c}1 & 1 & 0\\0 & 2 & 0\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\end{array}\right) \quad \{\text{using technology}\}$$

$\therefore$ the system has only the trivial solution $x_1 = x_2 = 0$.

So, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Now $\mathbf{A} = (\mathbf{v}_1 \,|\, \mathbf{v}_2) = \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}$ has $|\,\mathbf{A}\,| = 2 \neq 0$

$\therefore$ $\mathbf{v}_1$ and $\mathbf{v}_2$ span $\mathbb{R}^2$.

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and span $\mathbb{R}^2$, they form a basis for $\mathbb{R}^2$.

$$\begin{aligned}\therefore\ \mathrm{T}\left(\begin{pmatrix}4\\6\end{pmatrix}\right) &= \mathrm{T}\left(1\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}1\\2\end{pmatrix}\right)\\ &= \mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right) + 3\mathrm{T}\left(\begin{pmatrix}1\\2\end{pmatrix}\right)\\ &= \begin{pmatrix}1\\3\end{pmatrix} + 3\begin{pmatrix}-1\\-3\end{pmatrix}\\ &= \begin{pmatrix}-2\\-6\end{pmatrix}\end{aligned}$$

7 $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is the standard basis for $\mathbb{R}^2$.

$$\begin{aligned}\therefore\ \mathrm{T}\left(\begin{pmatrix}3\\7\end{pmatrix}\right) &= \mathrm{T}\left(3\begin{pmatrix}1\\0\end{pmatrix} + 7\begin{pmatrix}0\\1\end{pmatrix}\right)\\ &= 3\mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right) + 7\mathrm{T}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)\\ &= 3\begin{pmatrix}2\\1\\7\end{pmatrix} + 7\begin{pmatrix}-3\\0\\-4\end{pmatrix}\\ &= \begin{pmatrix}-15\\3\\-7\end{pmatrix}\end{aligned}$$

8 $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is the standard basis for $\mathbb{R}^3$.

a

$$\begin{aligned}\mathrm{T}\left(\begin{pmatrix}1\\3\\0\end{pmatrix}\right) &= \mathrm{T}\left(1\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\\ &= \mathrm{T}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) + 3\mathrm{T}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\\ &= \begin{pmatrix}1\\-1\\2\end{pmatrix} + 3\begin{pmatrix}2\\1\\3\end{pmatrix}\\ &= \begin{pmatrix}7\\2\\11\end{pmatrix}\end{aligned}$$

b

$$\begin{aligned}&\mathrm{T}\left(\begin{pmatrix}a\\b\\c\end{pmatrix}\right)\\ &= \mathrm{T}\left(a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\\ &= a\mathrm{T}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) + b\mathrm{T}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) + c\mathrm{T}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\\ &= a\begin{pmatrix}1\\-1\\2\end{pmatrix} + b\begin{pmatrix}2\\1\\3\end{pmatrix} + c\begin{pmatrix}5\\1\\4\end{pmatrix}\\ &= \begin{pmatrix}a + 2b + 5c\\-a + b + c\\2a + 3b + 4c\end{pmatrix}\end{aligned}$$

9 **a** Consider $\mathbf{u} = \begin{pmatrix}1\\2\end{pmatrix}$.

$$\mathrm{T}(\mathbf{u}) = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad\text{and}\quad \mathrm{T}(-\mathbf{u}) = \mathrm{T}\left(\begin{pmatrix}-1\\-2\end{pmatrix}\right) = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

Thus $\mathrm{T}(-\mathbf{u}) \neq -\mathrm{T}(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^2$.

$\therefore$ T is not a linear transformation as Property 2 is violated.

b $\mathrm{T}(\mathbf{0}) = \mathrm{T}\left(\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}0\\2\end{pmatrix} \neq \mathbf{0}$

$\therefore$ T is not a linear transformation as Property 1 is violated.

c Let $\mathbf{u} = \begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$, $\mathbf{v} = \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$

$$\begin{aligned}(1)\quad \mathrm{T}(\mathbf{u} + \mathbf{v}) &= \mathrm{T}\left(\begin{pmatrix}x_1 + x_2\\y_1 + y_2\\z_1 + z_2\end{pmatrix}\right)\\ &= (x_1 + x_2 + y_1 + y_2 - 2(z_1 + z_2))\\ &= (x_1 + y_1 - 2z_1) + (x_2 + y_2 - 2z_2)\\ &= \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v})\end{aligned}$$

$$\begin{aligned}(2)\quad \mathrm{T}(k\mathbf{u}) &= \mathrm{T}\left(\begin{pmatrix}kx_1\\ky_1\\kz_1\end{pmatrix}\right)\\ &= (kx_1 + ky_1 - 2kz_1)\\ &= k(x_1 + y_1 - 2z_1)\\ &= k\mathrm{T}(\mathbf{u})\end{aligned}$$

Since the addition and scalar multiplication properties are satisfied, T is a linear transformation.

d Let $\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

(1) $\mathrm{T}(\mathbf{u}+\mathbf{v})$

$$\begin{aligned} &= \mathrm{T}\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} x_1+x_2+y_1+y_2 \\ x_1+x_2-4(z_1+z_2) \end{pmatrix} \\ &= \begin{pmatrix} x_1+y_1 \\ x_1-4z_1 \end{pmatrix} + \begin{pmatrix} x_2+y_2 \\ x_2-4z_2 \end{pmatrix} \\ &= \mathrm{T}(\mathbf{u}) + \mathrm{T}(\mathbf{v}) \end{aligned}$$

(2) $\mathrm{T}(k\mathbf{u})$

$$\begin{aligned} &= \mathrm{T}\begin{pmatrix} kx_1 \\ ky_1 \\ kz_1 \end{pmatrix} \\ &= \begin{pmatrix} kx_1+ky_1 \\ kx_1-4kz_1 \end{pmatrix} \\ &= k\begin{pmatrix} x_1+y_1 \\ x_1-4z_1 \end{pmatrix} \\ &= k\mathrm{T}(\mathbf{u}) \end{aligned}$$

Since the addition and scalar multiplication properties are satisfied, T is a linear transformation.

EXERCISE 1J.2

1 **a** Let $\mathrm{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore\ y = z = 0$$

$$\therefore\ \ker(\mathrm{T}) = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \ker(\mathrm{T}) = \operatorname{lin}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

b Consider $\mathbf{w} = \mathrm{T}(\mathbf{v})$

$$= \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix} = z\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - y\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore\ \mathscr{R}(\mathrm{T}) = \operatorname{lin}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

c nullity (T) = dimension of $\ker(\mathrm{T}) = 1$

d rank (T) = dimension of $\mathscr{R}(\mathrm{T}) = 2$

2 $\mathrm{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y \\ z-x \\ 2x \\ x+y+z \end{pmatrix}$

a Let $\mathrm{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} y \\ z-x \\ 2x \\ x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore\ x = y = z = 0$$

Thus $\ker(\mathrm{T}) = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$.

b Consider $\mathbf{w} = \mathrm{T}(\mathbf{v}) = \begin{pmatrix} y \\ z-x \\ 2x \\ x+y+z \end{pmatrix}$

$$= x\begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore\ \mathscr{R}(\mathrm{T}) = \operatorname{lin}\left\{\begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

c nullity (T) = dimension of $\ker(\mathrm{T}) = 0$

d rank (T) = dimension of $\mathscr{R}(\mathrm{T}) = 3$

3 **a** $\mathrm{T}\left(\begin{pmatrix} 9 \\ -3 \end{pmatrix}\right) = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix}\begin{pmatrix} 9 \\ -3 \end{pmatrix}$

$$\begin{aligned} &= \begin{pmatrix} 9-9 \\ -18+18 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\therefore\ \begin{pmatrix} 9 \\ -3 \end{pmatrix} \in \ker(T)$$

$$\begin{aligned} \mathrm{T}\begin{pmatrix} -3 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix}\begin{pmatrix} -3 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -3+0 \\ 6+0 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 6 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\therefore\ \begin{pmatrix} -3 \\ 0 \end{pmatrix} \notin \ker(\mathrm{T})$$

b $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix}$

$\therefore\ \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$ {using technology}

$\therefore$ a basis for the row space of $\mathbf{A}^{\mathrm{T}}$ is $\left\{\begin{pmatrix} 1 & -2 \end{pmatrix}\right\}$.

$\therefore$ a basis for the column space of $\mathbf{A}$ is $\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$.

$$\therefore\ \mathscr{R}(\mathrm{T}) = \operatorname{lin}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$$

i $\begin{pmatrix} 2 \\ 2 \end{pmatrix} \notin \mathscr{R}(\mathrm{T})$ **ii** $\begin{pmatrix} -4 \\ 8 \end{pmatrix} \in \mathscr{R}(\mathrm{T})$

4 **a** $\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -y \\ x \end{pmatrix} = x\begin{pmatrix} 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So, $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

or $\mathrm{T}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and $\mathrm{T}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

So, $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b $$\mathrm{T}\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=\begin{pmatrix}x+z\\y-z\end{pmatrix}=x\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix}+z\begin{pmatrix}1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}1&0&1\\0&1&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\text{So,}\quad \mathbf{A}=\begin{pmatrix}1&0&1\\0&1&-1\end{pmatrix}$$

or $$\mathrm{T}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)=\begin{pmatrix}1\\0\end{pmatrix},\quad \mathrm{T}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)=\begin{pmatrix}0\\1\end{pmatrix},$$

and $$\mathrm{T}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)=\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\text{So,}\quad \mathbf{A}=\begin{pmatrix}1&0&1\\0&1&-1\end{pmatrix}$$

c $$\mathrm{T}\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=\begin{pmatrix}y\\-z\\x+y\end{pmatrix}$$

$$=x\begin{pmatrix}0\\0\\1\end{pmatrix}+y\begin{pmatrix}1\\0\\1\end{pmatrix}+z\begin{pmatrix}0\\-1\\0\end{pmatrix}$$

$$=\begin{pmatrix}0&1&0\\0&0&-1\\1&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\text{So,}\quad \mathbf{A}=\begin{pmatrix}0&1&0\\0&0&-1\\1&1&0\end{pmatrix}$$

or $$\mathrm{T}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)=\begin{pmatrix}0\\0\\1\end{pmatrix},\quad \mathrm{T}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)=\begin{pmatrix}1\\0\\1\end{pmatrix},$$

and $$\mathrm{T}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)=\begin{pmatrix}0\\-1\\0\end{pmatrix}$$

$$\text{So,}\quad \mathbf{A}=\begin{pmatrix}0&1&0\\0&0&-1\\1&1&0\end{pmatrix}$$

d $$\mathrm{T}\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}-x\\x+y\\y\\y-x\end{pmatrix}$$

$$=x\begin{pmatrix}-1\\1\\0\\-1\end{pmatrix}+y\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

$$=\begin{pmatrix}-1&0\\1&1\\0&1\\-1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\text{So,}\quad \mathbf{A}=\begin{pmatrix}-1&0\\1&1\\0&1\\-1&1\end{pmatrix}$$

or $$\mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)=\begin{pmatrix}-1\\1\\0\\-1\end{pmatrix}\quad\text{and}\quad \mathrm{T}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)=\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

$$\text{So,}\quad \mathbf{A}=\begin{pmatrix}-1&0\\1&1\\0&1\\-1&1\end{pmatrix}$$

5 **a** $$\mathbf{A}=\begin{pmatrix}3&1&2&-1\\1&2&0&4\end{pmatrix}$$

$\mathbf{Av}=\mathbf{0}$ has augmented matrix

$$\left(\begin{array}{cccc|c}3&1&2&-1&0\\1&2&0&4&0\end{array}\right)\sim\left(\begin{array}{cccc|c}1&0&\frac{4}{5}&-\frac{6}{5}&0\\0&1&-\frac{2}{5}&\frac{13}{5}&0\end{array}\right)$$

{using technology}

Letting $x_3=s$, $x_4=t$, where $s, t\in\mathbb{R}$, we find that $x_2=\frac{2}{5}s-\frac{13}{5}t$ and $x_1=-\frac{4}{5}s+\frac{6}{5}t$.

$$\therefore\quad \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=s\begin{pmatrix}-\frac{4}{5}\\\frac{2}{5}\\1\\0\end{pmatrix}+t\begin{pmatrix}\frac{6}{5}\\-\frac{13}{5}\\0\\1\end{pmatrix}\quad\text{where } s, t\in\mathbb{R}$$

$$\therefore\quad \ker(\mathrm{T})=\operatorname{lin}\left\{\begin{pmatrix}-\frac{4}{5}\\\frac{2}{5}\\1\\0\end{pmatrix},\begin{pmatrix}\frac{6}{5}\\-\frac{13}{5}\\0\\1\end{pmatrix}\right\}$$

b $$\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}3&1\\1&2\\2&0\\-1&4\end{pmatrix}\sim\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}\quad\{\text{using technology}\}$$

$\therefore$ a basis for the row space of $\mathbf{A}^{\mathrm{T}}$ is $\left\{\begin{pmatrix}1&0\end{pmatrix},\begin{pmatrix}0&1\end{pmatrix}\right\}$.

$\therefore$ a basis for the column space of $\mathbf{A}$ is $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

$$\therefore\quad \mathcal{R}(\mathrm{T})=\operatorname{lin}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}=\mathbb{R}^2$$

c $$\begin{aligned}\operatorname{nullity}(\mathrm{T})+\operatorname{rank}(\mathrm{T})&=2+2\\&=4\\&=\text{dimension of the domain}\end{aligned}$$

6 **a** $$\mathbf{A}=\begin{pmatrix}1&0&-1&2\\1&2&9&-4\\-2&-1&-3&-1\end{pmatrix}$$

$\mathbf{Av}=\mathbf{0}$ has augmented matrix

$$\left(\begin{array}{cccc|c}1&0&-1&2&0\\1&2&9&-4&0\\-2&-1&-3&-1&0\end{array}\right)\sim\left(\begin{array}{cccc|c}1&0&-1&2&0\\0&1&5&-3&0\\0&0&0&0&0\end{array}\right)$$

{using technology}

Letting $x_3=s$, $x_4=t$, where $s, t\in\mathbb{R}$, we find that $x_2=-5s+3t$ and $x_1=s-2t$.

$$\therefore\quad \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=s\begin{pmatrix}1\\-5\\1\\0\end{pmatrix}+t\begin{pmatrix}-2\\3\\0\\1\end{pmatrix}\quad\text{where } s, t\in\mathbb{R}$$

$$\therefore\quad \ker(\mathrm{T})=\operatorname{lin}\left\{\begin{pmatrix}1\\-5\\1\\0\end{pmatrix},\begin{pmatrix}-2\\3\\0\\1\end{pmatrix}\right\}$$

b $\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 2 & -1 \\ -1 & 9 & -3 \\ 2 & -4 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

{using technology}

$\therefore$ a basis for the row space of $\mathbf{A}^{\mathrm{T}}$ is

$$\left\{ \begin{pmatrix} 1 & 0 & -\frac{3}{2} \end{pmatrix}, \begin{pmatrix} 0 & 1 & -\frac{1}{2} \end{pmatrix} \right\}.$$

$\therefore$ a basis for the column space of $\mathbf{A}$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -\frac{3}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \right\}.$$

$$\therefore \quad \mathcal{R}(\mathrm{T}) = \mathrm{lin} \left\{ \begin{pmatrix} 1 \\ 0 \\ -\frac{3}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \right\}$$

c $\begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 9 & -4 \\ -2 & -1 & -3 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -7 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+2 \\ -14+18-4 \\ 7-6-1 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 0 \\ -7 \\ 2 \\ 1 \end{pmatrix} \in \ker(\mathrm{T})$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 9 & -4 \\ -2 & -1 & -3 & -1 \end{pmatrix} \begin{pmatrix} 7 \\ -21 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 7-3-4 \\ 7-42+27+8 \\ -14+21-9+2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 7 \\ -21 \\ 3 \\ -2 \end{pmatrix} \in \ker(\mathrm{T})$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & 9 & -4 \\ -2 & -1 & -3 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3-1+4 \\ -3+6+9-8 \\ 6-3-3-2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} -3 \\ 3 \\ 1 \\ 2 \end{pmatrix} \notin \ker(\mathrm{T})$$

7 **a** Let $\mathrm{T}\left(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \quad \begin{pmatrix} x+y-z \\ y+w-x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore \quad z = x+y$ and $w = x-y$

$$\therefore \quad \ker(\mathrm{T}) = \begin{pmatrix} x \\ y \\ x+y \\ x-y \end{pmatrix}, \quad x, y \in \mathbb{R}$$

$$= x \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad x, y \in \mathbb{R}$$

$$= \mathrm{lin} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

b Consider $\mathbf{w} = \mathrm{T}(\mathbf{v})$

$$= \begin{pmatrix} x+y-z \\ y+w-x \end{pmatrix}$$

$$= (x+y-z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (y+w-x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

{for any x and y we can choose z and w to make any combination of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$}

$$\therefore \quad \mathcal{R}(\mathrm{T}) = \mathrm{lin} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$= \mathbb{R}^2$$

c $\mathrm{nullity}(\mathrm{T}) = 2$

d $\mathrm{rank}(\mathrm{T}) = 2$

8 Let $\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \quad \begin{pmatrix} -y \\ -x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad x = y = 0$$

$$\therefore \quad \ker(\mathrm{T}) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

Consider $\mathbf{w} = \mathrm{T}(\mathbf{v})$

$$= \begin{pmatrix} -y \\ -x \end{pmatrix}$$

$$= -y \begin{pmatrix} 1 \\ 0 \end{pmatrix} - x \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore \quad \mathcal{R}(\mathrm{T}) = \mathrm{lin} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$= \mathbb{R}^2$$

9 **a** $\mathrm{T}\left(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \right) = \begin{pmatrix} x-y \\ w+z \end{pmatrix}$

$$= x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \end{pmatrix} + w \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

So, $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

or $\mathrm{T}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right)=\begin{pmatrix}1\\0\end{pmatrix}$, $\mathrm{T}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right)=\begin{pmatrix}-1\\0\end{pmatrix}$,

$\mathrm{T}\left(\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right)=\begin{pmatrix}0\\1\end{pmatrix}$, and $\mathrm{T}\left(\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)=\begin{pmatrix}0\\1\end{pmatrix}$

So, $\mathbf{A}=\begin{pmatrix}1&-1&0&0\\0&0&1&1\end{pmatrix}$

b $\mathbf{Av}=\mathbf{0}$ has augmented matrix

$\left(\begin{array}{cccc|c}1&-1&0&0&0\\0&0&1&1&0\end{array}\right)$ which is already in reduced row echelon form.

Letting $y=s$, $w=t$ where $s,t\in\mathbb{R}$, $z=-t$ and $x=s$.

$$\therefore\ \begin{pmatrix}x\\y\\z\\w\end{pmatrix}=s\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}0\\0\\-1\\1\end{pmatrix},\quad\text{where } s,t\in\mathbb{R}$$

$$\therefore\ \ker(\mathrm{T})=\operatorname{lin}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}\right\}$$

c $\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}1&0\\-1&0\\0&1\\0&1\end{pmatrix}\sim\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}$ {using technology}

$\therefore$ a basis for the row space of $\mathbf{A}^{\mathrm{T}}$ is $\left\{\begin{pmatrix}1&0\end{pmatrix},\begin{pmatrix}0&1\end{pmatrix}\right\}$.

$\therefore$ a basis for the column space of $\mathbf{A}$ is $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

$$\begin{aligned}\therefore\ \mathcal{R}(\mathrm{T})&=\operatorname{lin}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}\\&=\mathbb{R}^2\end{aligned}$$

d
$$\begin{aligned}\operatorname{nullity}(\mathrm{T})+\operatorname{rank}(\mathrm{T})&=2+2\\&=4\\&=\text{dimension of the domain}\end{aligned}$$

EXERCISE 1J.3

1 **a**
$$\begin{aligned}(\mathrm{T}\circ\mathrm{S})(\mathbf{v})&=\mathrm{T}(\mathrm{S}(\mathbf{v}))\\&=\mathrm{T}\left(\begin{pmatrix}2x\\z+x-y\end{pmatrix}\right)\\&=\begin{pmatrix}2x\\z+x-y-2x\\z+x-y+2x\end{pmatrix}\\&=\begin{pmatrix}2x\\-x-y+z\\3x-y+z\end{pmatrix}\end{aligned}$$

b Let T have standard matrix **A** and S have standard matrix **B**.

Now $\mathbf{A}=\left(\mathrm{T}\left(\begin{pmatrix}1\\0\end{pmatrix}\right)\middle|\,\mathrm{T}\left(\begin{pmatrix}0\\1\end{pmatrix}\right)\right)=\begin{pmatrix}1&0\\-1&1\\1&1\end{pmatrix}$

and $\mathbf{B}=\left(\mathrm{S}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\middle|\,\mathrm{S}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\middle|\,\mathrm{S}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right)$
$=\begin{pmatrix}2&0&0\\1&-1&1\end{pmatrix}$

$\therefore$ $\mathrm{T}\circ\mathrm{S}$ has standard matrix

$$\begin{aligned}\mathbf{AB}&=\begin{pmatrix}1&0\\-1&1\\1&1\end{pmatrix}\begin{pmatrix}2&0&0\\1&-1&1\end{pmatrix}\\&=\begin{pmatrix}2&0&0\\-1&-1&1\\3&-1&1\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore\ (\mathrm{T}\circ\mathrm{S})\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)&=\begin{pmatrix}2&0&0\\-1&-1&1\\3&-1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\\&=\begin{pmatrix}2x\\-x-y+z\\3x-y+z\end{pmatrix}\end{aligned}$$

2 **a**
$$\begin{aligned}(\mathrm{S}\circ\mathrm{T})(\mathbf{v})&=\mathrm{S}(\mathrm{T}(\mathbf{v}))\\&=\mathrm{S}\left(\begin{pmatrix}2x\\-y\\z-x\end{pmatrix}\right)\\&=\begin{pmatrix}-y\\-2x\\-y+z-x\end{pmatrix}\\&=\begin{pmatrix}-y\\-2x\\-x-y+z\end{pmatrix}\end{aligned}$$

b Let S have standard matrix **A** and T have standard matrix **B**.

Now $\mathbf{A}=\left(\mathrm{S}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\middle|\,\mathrm{S}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\middle|\,\mathrm{S}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right)$
$=\begin{pmatrix}0&1&0\\-1&0&0\\0&1&1\end{pmatrix}$

and $\mathbf{B}=\left(\mathrm{T}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\middle|\,\mathrm{T}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\middle|\,\mathrm{T}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right)$
$=\begin{pmatrix}2&0&0\\0&-1&0\\-1&0&1\end{pmatrix}$

$\therefore$ $\mathrm{S}\circ\mathrm{T}$ has standard matrix

$$\begin{aligned}\mathbf{AB}&=\begin{pmatrix}0&1&0\\-1&0&0\\0&1&1\end{pmatrix}\begin{pmatrix}2&0&0\\0&-1&0\\-1&0&1\end{pmatrix}\\&=\begin{pmatrix}0&-1&0\\-2&0&0\\-1&-1&1\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore\ (\mathrm{S}\circ\mathrm{T})\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)&=\begin{pmatrix}0&-1&0\\-2&0&0\\-1&-1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\\&=\begin{pmatrix}-y\\-2x\\-x-y+z\end{pmatrix}\end{aligned}$$

3 $\mathrm{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore$ T has standard matrix $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$

Now $\mathbf{A}^{-1} = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$

$$\therefore\ \mathrm{S}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore\ \mathrm{S}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -3x + 2y \\ -2x + y \end{pmatrix}$$

4 $\mathrm{T}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\therefore$ T has standard matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

Now $\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ -1 & 1 & 1 \end{pmatrix}$

$$\therefore\ \mathrm{S}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore\ \mathrm{S}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}x - \frac{1}{2}z \\ -\frac{1}{2}x + y + \frac{3}{2}z \\ -x + y + z \end{pmatrix}$$

5 **a** $(\mathrm{T} \circ \mathrm{S})(\mathbf{v})$ and $(\mathrm{S} \circ \mathrm{T})(\mathbf{v})$

$$= \mathrm{T}(\mathrm{S}(\mathbf{v})) \qquad\qquad = \mathrm{S}(\mathrm{T}(\mathbf{v}))$$

$$= \mathrm{T}\left(\begin{pmatrix} x \\ 0 \\ -y \end{pmatrix}\right) \qquad\qquad = \mathrm{S}\left(\begin{pmatrix} x \\ z \\ 2y \end{pmatrix}\right)$$

$$= \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \qquad\qquad = \begin{pmatrix} x \\ 0 \\ -z \end{pmatrix}$$

$\therefore$ $\mathrm{T} \circ \mathrm{S} \neq \mathrm{S} \circ \mathrm{T}$

b **i** $$\mathbf{A} = \left(\mathrm{T}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \middle| \mathrm{T}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \middle| \mathrm{T}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

and $$\mathbf{B} = \left(\mathrm{S}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \middle| \mathrm{S}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \middle| \mathrm{S}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

ii $$\mathbf{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

iii **AB** is the standard matrix for $\mathrm{T} \circ \mathrm{S}$ and
BA is the standard matrix for $\mathrm{S} \circ \mathrm{T}$.

$$\mathbf{AB} \neq \mathbf{BA}$$

$\therefore$ $\mathrm{T} \circ \mathrm{S} \neq \mathrm{S} \circ \mathrm{T}$, which agrees with part **a**.

6 Consider the linear transformations
$\mathrm{T} : \mathbb{R}^k \to \mathbb{R}^m$ and $\mathrm{S} : \mathbb{R}^n \to \mathbb{R}^k$

$$\begin{aligned}
\text{Now}\quad & (\mathrm{T} \circ \mathrm{S})(\mathbf{u} + \mathbf{v}) \\
&= \mathrm{T}(\mathrm{S}(\mathbf{u} + \mathbf{v})) \\
&= \mathrm{T}(\mathrm{S}(\mathbf{u}) + \mathrm{S}(\mathbf{v})) && \{\text{addition property}\} \\
&= \mathrm{T}(\mathrm{S}(\mathbf{u})) + \mathrm{T}(\mathrm{S}(\mathbf{v})) && \{\text{addition property}\} \\
&= (\mathrm{T} \circ \mathrm{S})(\mathbf{u}) + (\mathrm{T} \circ \mathrm{S})(\mathbf{v})
\end{aligned}$$

$$\begin{aligned}
\text{and}\quad & (\mathrm{T} \circ \mathrm{S})(c\mathbf{v}) \\
&= \mathrm{T}(\mathrm{S}(c\mathbf{v})) \\
&= \mathrm{T}(c\mathrm{S}(\mathbf{v})) && \{\text{scalar multiplication property}\} \\
&= c\mathrm{T}(\mathrm{S}(\mathbf{v})) && \{\text{scalar multiplication property}\} \\
&= c(\mathrm{T} \circ \mathrm{S})(\mathbf{v})
\end{aligned}$$

Since the addition and scalar multiplication properties are satisfied, $\mathrm{T} \circ \mathrm{S}$ is a linear transformation.

EXERCISE 1J.4

1 **a** $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 2 \\ 0 & 0 \end{pmatrix}$

A basis of the row space is $\left\{\begin{pmatrix} 1 & 2 \end{pmatrix}\right\}$,
so row rank = 1.

A basis of the column space is $\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}$,
so column rank = 1.

But $\mathscr{R}(\mathrm{T})$ is the column space of **A**,
so rank (T) = column rank = 1. ✓

b $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \boxed{1} & \frac{3}{4} & \frac{5}{4} \end{pmatrix}$

A basis of the row space is
$\left\{\begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 1 & \frac{3}{4} & \frac{5}{4} \end{pmatrix}\right\}$,
so row rank = 2.

A basis of the column space is $\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right\}$,
so column rank = 2.

But $\mathscr{R}(\mathrm{T})$ is the column space of **A**,
so rank (T) = column rank = 2. ✓

c $\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 & 3 & 1 \\ 1 & -1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 0 & -1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$

A basis of the row space is
$\left\{\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \end{pmatrix},\right.$
$\left.\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$,
so row rank = 4.

A basis of the column space is

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}\right\},$$

so column rank = 4.

But $\mathscr{R}(\mathrm{T})$ is the column space of **A**,
so rank (T) = column rank = 4. ✓

2 **a** The system has the form $\mathbf{Ax}=\mathbf{b}$

where $\mathbf{A}=\begin{pmatrix}1&-1&1\\2&1&1\\3&3&1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}4\\6\\8\end{pmatrix}$.

$$\text{Now}\quad \mathbf{A}=\begin{pmatrix}1&-1&1\\2&1&1\\3&3&1\end{pmatrix}\sim\begin{pmatrix}\boxed{1}&0&\frac{2}{3}\\0&\boxed{1}&-\frac{1}{3}\\0&0&0\end{pmatrix}$$

so rank$(\mathbf{A})=2$.

$$(\mathbf{A}\mid\mathbf{b})=\left(\begin{array}{ccc|c}1&-1&1&4\\2&1&1&6\\3&3&1&8\end{array}\right)\sim\left(\begin{array}{ccc|c}\boxed{1}&0&\frac{2}{3}&\frac{10}{3}\\0&\boxed{1}&-\frac{1}{3}&-\frac{2}{3}\\0&0&0&0\end{array}\right)$$

so rank$(\mathbf{A}\mid\mathbf{b})=2$.

Since rank$(\mathbf{A})=$ rank$(\mathbf{A}\mid\mathbf{b})$, $\mathbf{Ax}=\mathbf{b}$ is consistent and thus has a solution.

b The system has the form $\mathbf{Ax}=\mathbf{b}$

where $\mathbf{A}=\begin{pmatrix}1&2&1\\1&1&1\\3&4&3\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\-1\\10\end{pmatrix}$.

$$\text{Now}\quad \mathbf{A}=\begin{pmatrix}1&2&1\\1&1&1\\3&4&3\end{pmatrix}\sim\begin{pmatrix}\boxed{1}&0&1\\0&\boxed{1}&0\\0&0&0\end{pmatrix}$$

so rank$(\mathbf{A})=2$.

$$(\mathbf{A}\mid\mathbf{b})=\left(\begin{array}{ccc|c}1&2&1&2\\1&1&1&-1\\3&4&3&10\end{array}\right)\sim\left(\begin{array}{ccc|c}\boxed{1}&0&1&0\\0&\boxed{1}&0&0\\0&0&0&\boxed{1}\end{array}\right)$$

so rank$(\mathbf{A}\mid\mathbf{b})=3$.

Since rank$(\mathbf{A})\neq$ rank$(\mathbf{A}\mid\mathbf{b})$, there are no solutions to $\mathbf{Ax}=\mathbf{b}$.

3 **a** By inspection, $x_1=3$, $x_2=0$, $x_3=0$, $x_4=0$ is a particular solution.

The corresponding homogeneous system of equations has augmented matrix

$$\left(\begin{array}{cccc|c}1&1&-1&1&0\\1&-1&1&2&0\end{array}\right)\sim\left(\begin{array}{cccc|c}\boxed{1}&0&0&\frac{3}{2}&0\\0&\boxed{1}&-1&-\frac{1}{2}&0\end{array}\right)$$

Letting $x_3=s$ and $x_4=t$ we find $x_2=s+\frac{1}{2}t$ and $x_1=-\frac{3}{2}t$.

$$\therefore\quad \mathbf{x}=\begin{pmatrix}3\\0\\0\\0\end{pmatrix}+s\begin{pmatrix}0\\1\\1\\0\end{pmatrix}+t\begin{pmatrix}-\frac{3}{2}\\\frac{1}{2}\\0\\1\end{pmatrix}\quad\text{where } s,t\in\mathbb{R}$$

b By inspection, $x_3=1$, $x_1=0$, $x_2=0$, $x_4=0$, $x_5=0$ is a particular solution.

The corresponding homogeneous system of equations has augmented matrix

$$\left(\begin{array}{ccccc|c}1&-1&1&-1&1&0\\1&1&-1&-1&-1&0\\2&1&3&-2&1&0\end{array}\right)\sim\left(\begin{array}{ccccc|c}\boxed{1}&0&0&-1&0&0\\0&\boxed{1}&0&0&-\frac{1}{2}&0\\0&0&\boxed{1}&0&\frac{1}{2}&0\end{array}\right)$$

Letting $x_4=s$ and $x_5=t$, we find $x_3=-\frac{1}{2}t$, $x_2=\frac{1}{2}t$, and $x_1=s$.

$$\therefore\quad \mathbf{x}=\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+s\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}+t\begin{pmatrix}0\\\frac{1}{2}\\-\frac{1}{2}\\0\\1\end{pmatrix}\quad\text{where } s,t\in\mathbb{R}$$

EXERCISE 1K.1

1 **a** For $(0,1)$, $x'=2x+y=2(0)+(1)=1$

and $y'=x-y=0-1=-1$

$\therefore\ (0,1)\xrightarrow{T}(1,-1)$

b For $(-1,-3)$, $x'=2x+y=2(-1)+(-3)=-5$

and $y'=x-y=-1-(-3)=2$

$\therefore\ (-1,-3)\xrightarrow{T}(-5,2)$

c
$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}2&1\\1&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}2&1\\1&-1\end{pmatrix}^{-1}\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$$

$$\therefore\ \begin{pmatrix}x\\y\end{pmatrix}=-\frac{1}{3}\begin{pmatrix}-1&-1\\-1&2\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}\frac{1}{3}x'+\frac{1}{3}y'\\\frac{1}{3}x'-\frac{2}{3}y'\end{pmatrix}$$

$$\therefore\ y=3x+2\ \text{ becomes }\ \frac{x'-2y'}{3}=3\left(\frac{x'+y'}{3}\right)+2$$

$$\therefore\ x'-2y'=3x'+3y'+6$$

$$\therefore\ 2x'+5y'=-6$$

Hence $y=3x+2\xrightarrow{T}2x+5y=-6$

d From **c**, $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{1}{3}x'+\frac{1}{3}y'\\\frac{1}{3}x'-\frac{2}{3}y'\end{pmatrix}$

$\therefore\ x^2+y^2=1$ becomes

$$\left(\frac{x'+y'}{3}\right)^2+\left(\frac{x'-2y'}{3}\right)^2=1$$

$$\therefore\ x'^{\,2}+2x'y'+y'^{\,2}+x'^{\,2}-4x'y'+4y'^{\,2}=9$$

$$\therefore\ 2x'^{\,2}-2x'y'+5y'^{\,2}=9$$

$$\therefore\ x^2+y^2=1\xrightarrow{T}2x^2-2xy+5y^2=9$$

e From **c**, $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{1}{3}x'+\frac{1}{3}y'\\\frac{1}{3}x'-\frac{2}{3}y'\end{pmatrix}$

$\therefore\ y=x^2+1$ becomes

$$\frac{x'-2y'}{3}=\left(\frac{x'+y'}{3}\right)^2+1$$

$$\therefore\ 3(x'-2y')=x'^{\,2}+2x'y'+y'^{\,2}+9$$

$$\therefore\ 3x'-6y'=x'^{\,2}+2x'y'+y'^{\,2}+9$$

$$\therefore\ x'^{\,2}+2x'y'+y'^{\,2}-3x'+6y'+9=0$$

$$\therefore\ y=x^2+1\xrightarrow{T}x^2+2xy+y^2-3x+6y+9=0$$

2 **a** Let S have equations $\begin{cases}x'=ax+by\\y'=cx+dy\end{cases}$

We see that the y-coordinate of each point does not change.

$\therefore\ c=0$ and $d=1$ and so $y'=y$.

We know $(2,1)$ maps on to $(4,1)$

and $(-1,3)$ maps on to $(-7,3)$, so

$4=a(2)+b(1)$ $\qquad -7=a(-1)+b(3)$

$\therefore\ 2a+b=4$ .... (1) and $-a+3b=-7$ .... (2)

Solving simultaneously, we get $a=\frac{19}{7}$ and $b=-\frac{10}{7}$.

Thus, S has equations $\begin{cases}x'=\frac{19}{7}x-\frac{10}{7}y\\y'=y\end{cases}$.

b Let the object point be $(a,\ b)$.

We know $(a,\ b) \xrightarrow{S} (3,\ -1)$, so

$$3 = \tfrac{19}{7}a - \tfrac{10}{7}b \quad \text{and} \quad -1 = b$$

$$\therefore\ \tfrac{19}{7}a + \tfrac{10}{7} = 3 \qquad \therefore\ b = -1$$

$$\therefore\ 19a + 10 = 21$$

$$\therefore\ 19a = 11$$

$$\therefore\ a = \tfrac{11}{19}$$

So, the object point is $(\frac{11}{19},\ -1)$.

3 Suppose $\mathbf{v}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{v}$

then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\mathbf{v}' = \mathbf{v}$

which is only possible if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible; that is, if $|\mathbf{A}| \neq 0$.

So, $\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{|\mathbf{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{|\mathbf{A}|}\begin{pmatrix} dx' - by' \\ -cx' + ay' \end{pmatrix}$$

$\therefore\ y = mx + k$ becomes

$$\frac{1}{|\mathbf{A}|}(-cx' + ay') = m\frac{1}{|\mathbf{A}|}(dx' - by') + k$$

$$\therefore\ -cx' + ay' = mdx' - mby' + k|\mathbf{A}|$$

$$\therefore\ -cx' - mdx' + ay' + mby' = k|\mathbf{A}|$$

$$\therefore\ -(c + md)x' + (a + mb)y' = k|\mathbf{A}|$$

which has the form $Ax' + By' = C$ where A, B, and C are constants.

So, under the linear transformation $\mathbf{v}' = \mathbf{Av}$, the line with equation $y = mx + k$ maps on to another line provided $|\mathbf{A}| \neq 0$.

4 From **3**, $\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{|\mathbf{A}|}\begin{pmatrix} dx' - by' \\ -cx' + ay' \end{pmatrix}$

$\therefore\ x^2 + y^2 = 1$ becomes

$$\left(\frac{dx' - by'}{|\mathbf{A}|}\right)^2 + \left(\frac{-cx' + ay'}{|\mathbf{A}|}\right)^2 = 1$$

$$\therefore\ d^2x'^{\,2} - 2bdx'y' + b^2y'^{\,2} + c^2x'^{\,2} - 2acx'y' + a^2y'^{\,2} = |\mathbf{A}|^2$$

$$\therefore\ (c^2 + d^2)x'^{\,2} - 2(ac + bd)x'y' + (a^2 + b^2)y^2 = |\mathbf{A}|^2$$

a An ellipse has the form $Ax^2 + By^2 = C$

So, we require $-2(ac + bd) = 0$

b A circle has the form $Ax^2 + By^2 = C$ where $A = B$.

So, we require $-2(ac + bd) = 0$

and $(c^2 + d^2) = (a^2 + b^2)$

EXERCISE 1K.2

1 **a** For a reflection in the y-axis, $x' = -x$ and $y' = y$

$$\therefore\ \mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

b $\mathbf{A} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

c $\mathbf{A} = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

d $m = \tan\alpha = -1$

$$\text{Now}\quad \cos 2\alpha = \frac{1 - m^2}{1 + m^2} = \frac{0}{2} = 0 \quad \text{and} \quad \sin 2\alpha = \frac{2m}{1 + m^2} = \frac{-2}{2} = -1$$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

e $\cos\theta = \cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\sin\theta = \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

f $m = \tan\alpha = 5$

$$\text{Now}\quad \cos 2\alpha = \frac{1 - m^2}{1 + m^2} = \frac{-24}{26} = -\tfrac{12}{13} \quad \text{and} \quad \sin 2\alpha = \frac{2m}{1 + m^2} = \frac{10}{26} = \tfrac{5}{13}$$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$$

g $\cos\theta = \cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, $\sin\theta = \sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

h $m = \tan\alpha = \sqrt{3}$

$$\text{Now}\quad \cos 2\alpha = \frac{1 - m^2}{1 + m^2} = \frac{-2}{4} = -\tfrac{1}{2} \quad \text{and} \quad \sin 2\alpha = \frac{2m}{1 + m^2} = \frac{2\sqrt{3}}{4} = \tfrac{\sqrt{3}}{2}$$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

i $\cos\theta = \cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, $\sin\theta = \sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

2 **a** $\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ where $|\mathbf{A}| = 1$

Since $\mathbf{A}$ has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $\mathbf{A}$ is a rotation matrix.

If the angle of rotation is θ, $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$.

$$\therefore\ \tan\theta = -1$$

$$\therefore\ \theta = -\tfrac{\pi}{4}$$

∴ the transformation is a clockwise rotation about O through $\frac{\pi}{4}$.

b $\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ where $|\mathbf{A}| = -1$

$\mathbf{A}$ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ so $\mathbf{A}$ is a reflection matrix where $\cos 2\alpha = \frac{1}{\sqrt{2}}$ and $\sin 2\alpha = \frac{1}{\sqrt{2}}$.

$\therefore \ \tan 2\alpha = 1$ and $0 < 2\alpha < \pi$

If $m = \tan\alpha$ then

$$\frac{2m}{1-m^2} = 1 \quad \text{which simplifies to}$$

$$2m = 1 - m^2$$

$$\therefore \ m^2 + 2m - 1 = 0$$

$$\therefore \ m = \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2}$$

$$\therefore \ m = -1 \pm \sqrt{2}$$

But $0 < \alpha < \frac{\pi}{2}$ so $m > 0$

$$\therefore \ \tan\alpha = -1 + \sqrt{2}$$

$\therefore$ the transformation is a reflection in the line $y = (-1 + \sqrt{2})x$.

Alternatively: $2\alpha = \frac{\pi}{4}$

$$\therefore \ \alpha = \frac{\pi}{8}$$

Line of reflection is $y = \left(\tan\frac{\pi}{8}\right)x$.

c $\mathbf{A} = \begin{pmatrix} -\frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & -\frac{5}{13} \end{pmatrix}$ where $|\mathbf{A}| = 1$

Since $\mathbf{A}$ has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $\mathbf{A}$ is a rotation matrix.

If the angle of rotation is θ, $\cos\theta = -\frac{5}{13}$ and $\sin\theta = -\frac{12}{13}$.

$\therefore \ \tan\theta = \frac{12}{5}$ and $-\pi < \theta < -\frac{\pi}{2}$

$\therefore \ \theta = \arctan\left(\frac{12}{5}\right)$

$\therefore$ the transformation is a clockwise rotation about O through $\pi - \arctan\left(\frac{12}{5}\right)$ (since $\cos\theta, \sin\theta < 0$).

d $\mathbf{A} = \begin{pmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{pmatrix}$ where $|\mathbf{A}| = -1$

$\mathbf{A}$ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ so $\mathbf{A}$ is a reflection matrix where $\cos 2\alpha = -\frac{15}{17}$ and $\sin 2\alpha = \frac{8}{17}$.

$\therefore \ \tan 2\alpha = -\frac{8}{15}$ and $\frac{\pi}{2} < 2\alpha < \pi$

If $m = \tan\alpha$ then $\dfrac{\cancel{2}m}{1-m^2} = -\dfrac{\cancel{8}^{\,4}}{15}$

which simplifies to $4m^2 - 15m - 4 = 0$

$$\therefore \ (4m+1)(m-4) = 0$$

$$\therefore \ m = -\tfrac{1}{4} \text{ or } 4$$

But $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$ so $m > 0$

$$\therefore \ \tan\alpha = 4$$

$\therefore$ the transformation is a reflection in the line $y = 4x$.

3 **a** Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\therefore \ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\therefore \ x' = ax + by$ and $y' = cx + dy$

If $(\sqrt{2}, -\sqrt{2})$ maps on to $(0, 2)$ we get

$0 = \sqrt{2}a - \sqrt{2}b$ and $2 = \sqrt{2}c - \sqrt{2}d$

$\therefore \ a = b$ and $c - d = \sqrt{2}$

If we let $a = b = s$ and $d = t$, then $c = t + \sqrt{2}$

and $\mathbf{A} = \begin{pmatrix} s & s \\ t+\sqrt{2} & t \end{pmatrix}$, where $s, t \in \mathbb{R}$.

b **i** If the transformation is a rotation about O then $|\mathbf{A}| = 1$.

$$\therefore \ \begin{vmatrix} s & s \\ t+\sqrt{2} & t \end{vmatrix} = st - s(t+\sqrt{2}) = 1$$

$$\therefore \ \cancel{st} - \cancel{st} - \sqrt{2}s = 1$$

$$\therefore \ s = -\frac{1}{\sqrt{2}}$$

In $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$,

$\cos\theta = -\frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$

$\therefore \ t = -\frac{1}{\sqrt{2}}$ and $\theta = \frac{3\pi}{4}$

So, the linear transformation is an anticlockwise rotation about O through $\frac{3\pi}{4}$.

ii If the transformation is a reflection in $y = (\tan\alpha)x$ then $|\mathbf{A}| = -1$.

$$\therefore \ \begin{vmatrix} s & s \\ t+\sqrt{2} & t \end{vmatrix} = st - s(t+\sqrt{2}) = -1$$

$$\therefore \ \cancel{st} - \cancel{st} - \sqrt{2}s = -1$$

$$\therefore \ s = \frac{1}{\sqrt{2}}$$

In $\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$,

$\cos 2\alpha = \frac{1}{\sqrt{2}}$ and $\sin 2\alpha = \frac{1}{\sqrt{2}}$

$\therefore \ t = -\frac{1}{\sqrt{2}}$ and $2\alpha = \frac{\pi}{4}$

$$\therefore \ \alpha = \frac{\pi}{8}$$

So, the linear transformation is a reflection in $y = (\tan\frac{\pi}{8})x$.

4 $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $|\mathbf{A}| = 1$

$$\therefore \ \mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\Rightarrow \ \mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$$

$\mathbf{A}^{\mathrm{T}}$ represents a clockwise rotation about O through angle θ. This is the reverse of rotating the object anticlockwise about O through angle θ.

5 For a reflection in the line $y = (\tan\alpha)x$,

$\mathbf{A} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ where $|\mathbf{A}| = -1$

$$\therefore \ \mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$= \mathbf{A}$$

$\mathbf{A}$ is its own inverse. If an object is reflected in $y = (\tan\alpha)x$, then the resulting image is reflected in $y = (\tan\alpha)x$, then we obtain the original object.

6 If T is an anticlockwise rotation about O through $-\frac{2\pi}{3}$, then

$$\mathbf{A} = \begin{pmatrix} \cos\left(-\frac{2\pi}{3}\right) & -\sin\left(-\frac{2\pi}{3}\right) \\ \sin\left(-\frac{2\pi}{3}\right) & \cos\left(-\frac{2\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$\therefore$ T has equations $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore \quad x' = \dfrac{-x + y\sqrt{3}}{2}, \quad y' = \dfrac{-x\sqrt{3} - y}{2}$

a $(5,\ -1) \xrightarrow{\text{T}} \left(\dfrac{-5 - \sqrt{3}}{2},\ \dfrac{-5\sqrt{3} + 1}{2}\right)$

b From **a**, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$

$\therefore \quad x = \dfrac{-x' - \sqrt{3}y'}{2}$ and $y = \dfrac{\sqrt{3}x' - y'}{2}$

$\therefore \quad y = 3x - 1$ becomes

$\dfrac{\sqrt{3}x' - y'}{2} = \dfrac{-3x' - 3\sqrt{3}y'}{2} - 1$

$\therefore \quad \sqrt{3}x' - y' = -3x' - 3\sqrt{3}y' - 2$

$\therefore \quad (3 + \sqrt{3})x' + (3\sqrt{3} - 1)y' = -2$

$\therefore$ the image of $y = 3x - 1$ is
$(3 + \sqrt{3})x + (3\sqrt{3} - 1)y = -2$.

c $y = \dfrac{1}{x}$ becomes $\dfrac{\sqrt{3}x' - y'}{2} = \dfrac{2}{-x' - \sqrt{3}y'}$

$\therefore \quad -\sqrt{3}x'^{\,2} + x'y' - 3x'y' + \sqrt{3}y'^{\,2} = 4$

$\therefore \quad \sqrt{3}x'^{\,2} + 2x'y' - \sqrt{3}y'^{\,2} = -4$

$\therefore$ the image of $y = \dfrac{1}{x}$ is $\sqrt{3}x^2 + 2xy - \sqrt{3}y^2 = -4$.

7 $m = \tan\alpha = -2$

Now $\cos 2\alpha = \dfrac{1 - m^2}{1 + m^2}$ and $\sin 2\alpha = \dfrac{2m}{1 + m^2}$

$\therefore \quad \cos 2\alpha = -\frac{3}{5}$ and $\sin 2\alpha = -\frac{4}{5}$

S has matrix $\mathbf{A} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$

a S has equations $x' = \dfrac{-3x - 4y}{5}$

$y' = \dfrac{-4x + 3y}{5}$

Thus $(-4,\ 2) \xrightarrow{\text{S}} \left(\dfrac{-3(-4) - 4(2)}{5},\ \dfrac{-4(-4) + 3(2)}{5}\right)$

So, $(-4,\ 2) \xrightarrow{\text{S}} \left(\frac{4}{5},\ \frac{22}{5}\right)$

b Now $\mathbf{A}^{-1} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$

$\{\mathbf{A}^{-1} = \mathbf{A}$ for a reflection$\}$

$\therefore$ S has equations $x = \dfrac{-3x' - 4y'}{5}, \quad y = \dfrac{-4x' + 3y'}{5}$

$y = 2 - x \xrightarrow{\text{S}} \dfrac{-4x' + 3y'}{5} = 2 - \left(\dfrac{-3x' - 4y'}{5}\right)$

$\therefore \quad -4x' + 3y' = 10 + 3x' + 4y'$

$\therefore \quad 7x' + y' = -10$

$\therefore$ the image of $y = 2 - x$ under S is $7x + y = -10$.

c $y = x^2 \xrightarrow{\text{S}} \dfrac{-4x' + 3y'}{5} = \dfrac{9x'^{\,2} + 24x'y' + 16y'^{\,2}}{25}$

$\therefore \quad -20x' + 15y' = 9x'^{\,2} + 24x'y' + 16y'^{\,2}$

$\therefore \quad 9x'^{\,2} + 24x'y' + 16y'^{\,2} + 20x' - 15y' = 0$

$\therefore$ the image of $y = x^2$ under S is
$9x^2 + 24xy + 16y^2 + 20x - 15y = 0$.

8 For a reflection in $y = \frac{2}{3}x$, $\tan\alpha = \frac{2}{3}$.

$$\cos 2\alpha = \frac{1 - \left(\frac{2}{3}\right)^2}{1 + \left(\frac{2}{3}\right)^2} = \frac{\frac{5}{9}}{\frac{13}{9}} = \frac{5}{13}$$

and $\sin 2\alpha = \dfrac{2\left(\frac{2}{3}\right)}{\frac{13}{9}} = \dfrac{12}{13}$

$\therefore \quad \mathbf{A} = \begin{pmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{pmatrix} = \mathbf{A}^{-1}$

Thus $x = \dfrac{5x' + 12y'}{13}, \quad y = \dfrac{12x' - 5y'}{13}$

$\therefore \quad y = mx + c \ \to\ \dfrac{12x' - 5y'}{13} = m\left(\dfrac{5x' + 12y'}{13}\right) + c$

$\therefore \quad 12x' - 5y' = 5mx' + 12my' + 13c$

$\therefore \quad (12 - 5m)x' - (5 + 12m)y' = 13c$

Hence, $12 - 5m = 32, \quad -5 - 12m = 43, \quad c = 1$

$\therefore \quad 5m = -20, \quad 12m = -48, \quad c = 1$

$\therefore \quad m = -4, \ c = 1$

$\therefore$ the object line has equation $y = -4x + 1$.

9 For a rotation about O through $\frac{3\pi}{4}$,

$\cos\theta = -\frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$.

$\therefore \quad \mathbf{A} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

and $\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (= \mathbf{A}^{\text{T}})$

Thus $x = \dfrac{-x' + y'}{\sqrt{2}}, \quad y = \dfrac{-x' - y'}{\sqrt{2}}$

$\therefore \quad y = mx + c \ \to\ \dfrac{-x' - y'}{\sqrt{2}} = m\left(\dfrac{-x' + y'}{\sqrt{2}}\right) + c$

$\therefore \quad -x' - y' = -mx' + my' + c\sqrt{2}$

$\therefore \quad (m - 1)x' - (1 + m)y' = c\sqrt{2}$

Hence $m - 1 = 1, \quad 1 + m = 3, \quad c = -1$

$\Rightarrow \quad m = 2, \ c = -1$

$\therefore$ the object line is $y = 2x - 1$.

EXERCISE 1K.3

1 The stretch has matrix $\begin{pmatrix}1 & 0\\ 0 & \frac{5}{2}\end{pmatrix}$

$\therefore\ x' = x$

$y' = \frac{5}{2}y$

a Hence $\mathrm{P}(3,\,1) \to \mathrm{P}'(3,\,\frac{5}{2})$

b As $x = x'$ and $y = \frac{2}{5}y'$,

$y = 1 - 4x \ \to\ \frac{2}{5}y' = 1 - 4x'$

$\therefore\ 2y' = 5 - 20x'$

$\therefore\ 20x' + 2y' = 5$

$\therefore\ y = 1 - 4x \ \to\ 20x + 2y = 5$

2 The shear has matrix $\begin{pmatrix}1 & \frac{3}{2}\\ 0 & 1\end{pmatrix}$

$\therefore\ x' = x + \frac{3}{2}y$

$y' = y$

a Hence $\mathrm{Q}(-2,\,6) \to \mathrm{Q}'(-2 + \frac{3}{2}(6),\,6)$

That is, $\mathrm{Q}(-2,\,6) \to \mathrm{Q}'(7,\,6)$

b Now $x = x' - \frac{3}{2}y'$ and $y = y'$

$\therefore\ x^2 + y^2 = 10 \ \to\ (x' - \frac{3}{2}y')^2 + y'^{\,2} = 10$

$\therefore\ x'^{\,2} - 3x'y' + \frac{9}{4}y'^{\,2} + y'^{\,2} = 10$

$\therefore\ 4x'^{\,2} - 12x'y' + 13y'^{\,2} = 40$

$\therefore\ x^2 + y^2 = 10 \ \to\ 4x^2 - 12xy + 13y^2 = 40$

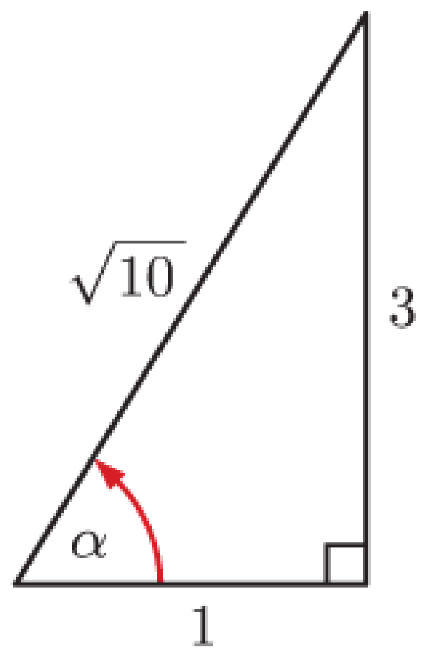
3 $\tan\alpha = -3 \quad \therefore\ \cos\alpha = \pm\frac{1}{\sqrt{10}}$

and $\sin\alpha = \mp\frac{3}{\sqrt{10}}$

Hence $\cos^2\alpha = \frac{1}{10}$,

$\sin^2\alpha = \frac{9}{10}$

and $\sin\alpha\cos\alpha = -\frac{3}{10}$



$\therefore\ \mathbf{A} = \begin{pmatrix}\frac{1}{10} & -\frac{3}{10}\\ -\frac{3}{10} & \frac{9}{10}\end{pmatrix}$

$\therefore\ x' = \dfrac{x - 3y}{10}, \quad y' = \dfrac{-3x + 9y}{10}$

$\therefore$ the projection of $(4,\,-1)$ onto $3x + y = 0$ is

$\left(\dfrac{4 - 3(-1)}{10},\ \dfrac{-3(4) + 9(-1)}{10}\right)$, that is, $(\frac{7}{10},\,-\frac{21}{10})$.

4 The shear has matrix $\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}$

$\therefore\ x' = x$ and $y' = 4x + y$

a $\mathrm{S}(-1,\,-3) \to (-1,\,4(-1) + (-3))$

$\therefore\ \mathrm{S}(-1,\,-3) \to \mathrm{S}'(-1,\,-7)$

b $x = x'$ and $y = -4x' + y'$

$\therefore\ y = -2x^2 \ \to\ -4x' + y' = -2x'^{\,2}$

$\therefore\ y = -2x^2 \ \to\ y = -2x^2 + 4x$

5 The stretch has matrix $\begin{pmatrix}\frac{7}{2} & 0\\ 0 & 1\end{pmatrix}$

$\therefore\ x' = \frac{7}{2}x$ and $y' = y$

a $\mathrm{T}(-2,\,4) \to \mathrm{T}'(-7,\,4)$

b $x = \frac{2}{7}x'$ and $y = y'$

$\therefore\ 3x - 4y = 6 \ \to\ 3(\frac{2}{7}x') - 4y' = 6$

$\Rightarrow\ 3x' - 14y' = 21$

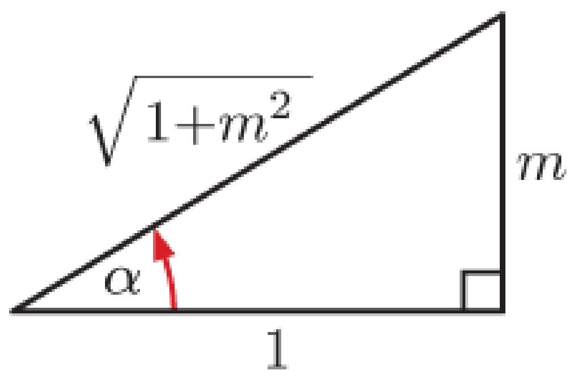
$\therefore\ 3x - 4y = 6 \ \to\ 3x - 14y = 21$

6 $m = \tan\alpha \quad \therefore\ \cos\alpha = \dfrac{1}{\sqrt{1+m^2}}, \quad \sin\alpha = \dfrac{m}{\sqrt{1+m^2}}$

$\therefore\ \cos^2\alpha = \dfrac{1}{1+m^2}, \quad \sin^2\alpha = \dfrac{m^2}{1+m^2}$

and $\sin\alpha\cos\alpha = \dfrac{m}{1+m^2}$

$\therefore\ \mathbf{A} = \begin{pmatrix}\dfrac{1}{1+m^2} & \dfrac{m}{1+m^2}\\[2mm] \dfrac{m}{1+m^2} & \dfrac{m^2}{1+m^2}\end{pmatrix}$



$\therefore\ x' = \dfrac{x + my}{1+m^2}, \quad y' = \dfrac{mx + m^2y}{1+m^2}$

But $(4,\,-\frac{1}{2}) \to (1,\,1\frac{1}{2})$

$\therefore\ 1 = \dfrac{4 - \frac{1}{2}m}{1+m^2}$ and $1\frac{1}{2} = \dfrac{4m + m^2(-\frac{1}{2})}{1+m^2}$

$\therefore\ 1 + m^2 = 4 - \frac{1}{2}m$ and $\frac{3}{2}(1 + m^2) = 4m - \frac{1}{2}m^2$

$\therefore\ 2m^2 + m - 6 = 0$ and $4m^2 - 8m + 3 = 0$

$\therefore\ (2m-3)(m+2) = 0$ and $(2m-1)(2m-3) = 0$

$\therefore\ m = \frac{3}{2}$ {the common solution}

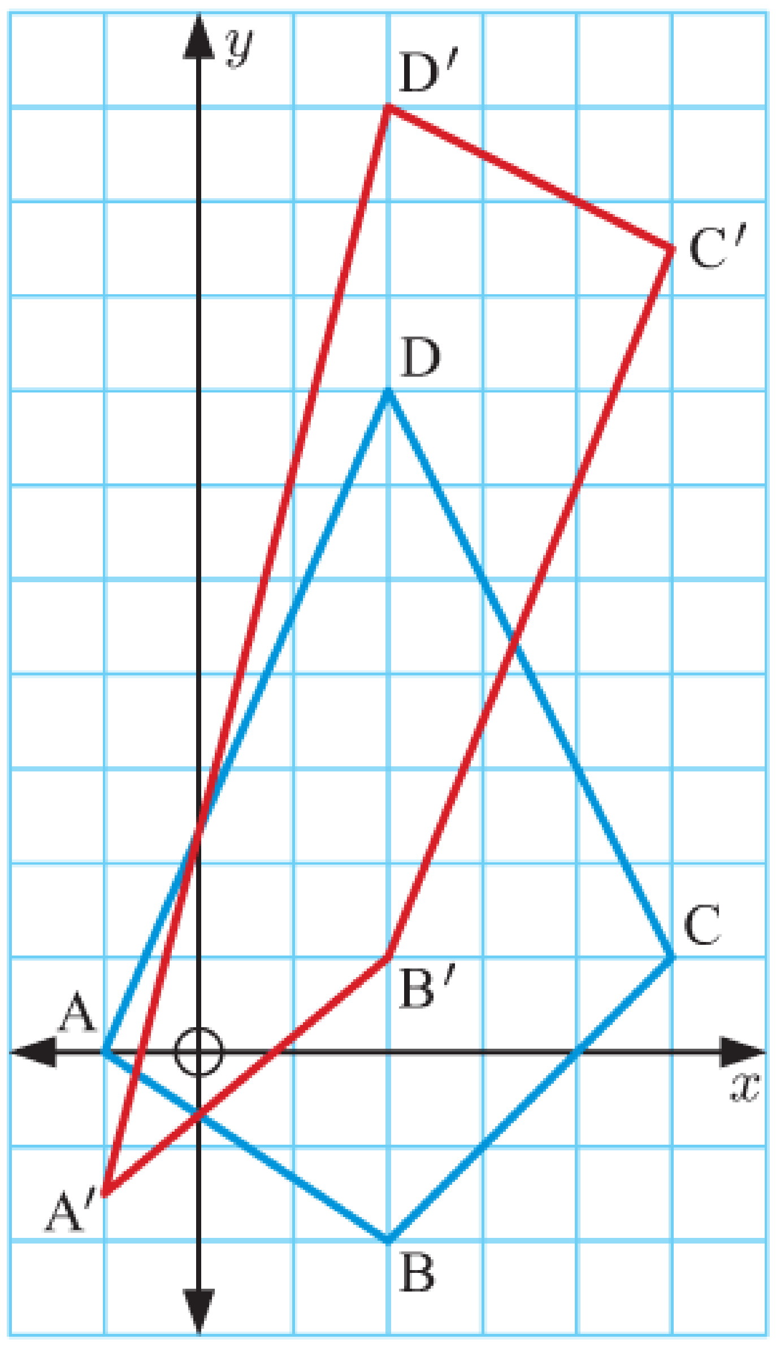
7 $\mathbf{A} = \begin{pmatrix}1 & 0\\ \frac{3}{2} & 1\end{pmatrix} \quad \therefore\ x' = x, \quad y' = \frac{3}{2}x + y$

a $\mathrm{A}(-1,\,0) \ \to\ \mathrm{A}'(-1,\,-\frac{3}{2})$

$\mathrm{B}(2,\,-2) \ \to\ \mathrm{B}'(2,\,1)$

$\mathrm{C}(5,\,1) \ \to\ \mathrm{C}'(5,\,\frac{17}{2})$

$\mathrm{D}(2,\,7) \ \to\ \mathrm{D}'(2,\,10)$



b Area ABCD = area △ABD + area △BCD

$= \frac{1}{2} \times 9 \times 3 + \frac{1}{2} \times 9 \times 3$

$= 27$ units^2

Since $|\,|\mathbf{A}|\,| = 1$, area A′B′C′D′ = 27 units^2 also.

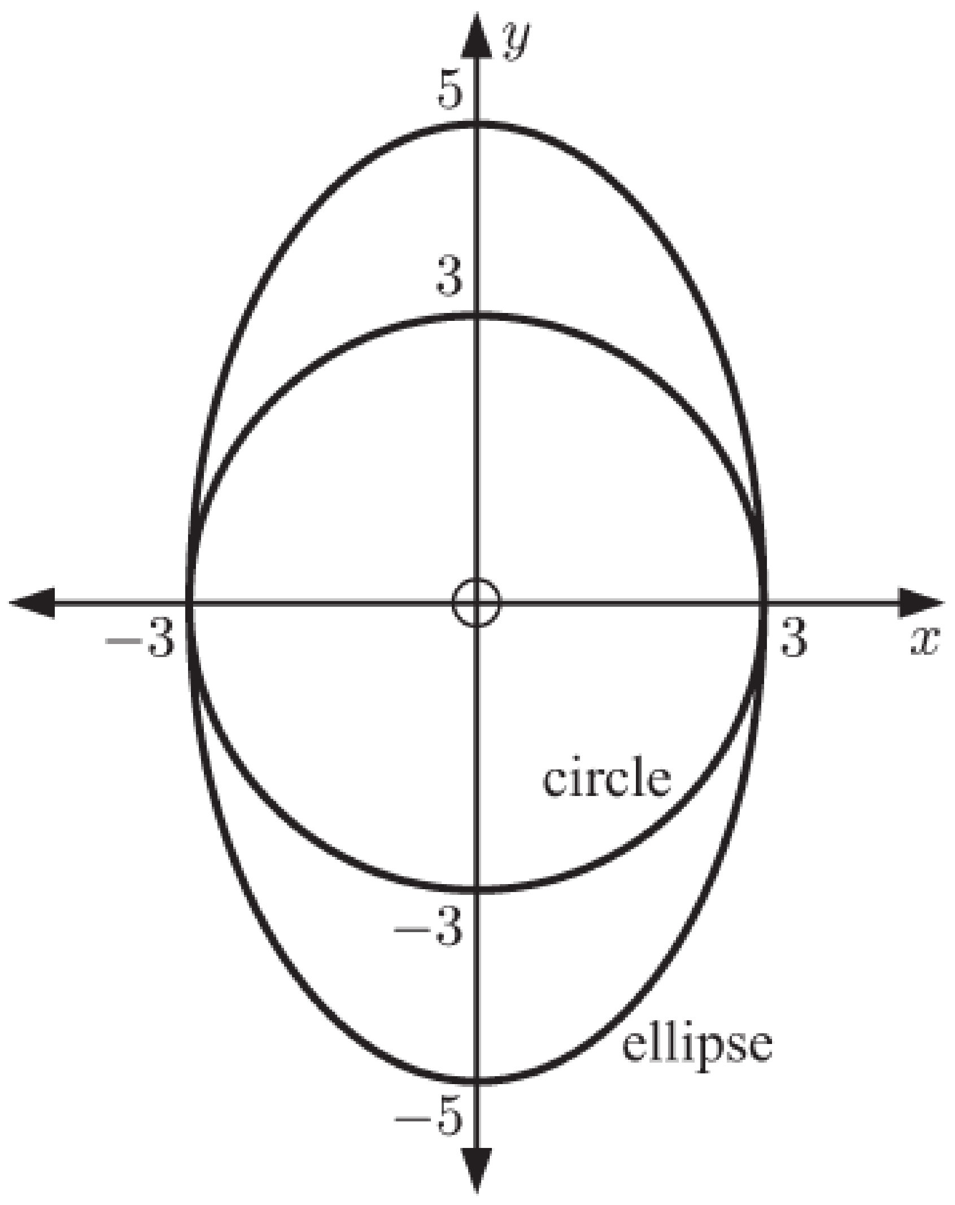
8 **a** For a vertical stretch with $k = \frac{5}{3}$,

$\mathbf{A} = \begin{pmatrix}1 & 0\\ 0 & \frac{5}{3}\end{pmatrix} \quad \therefore\ x' = x, \quad y' = \frac{5}{3}y$

The circle has the following axes intercepts, which we transform to obtain the axes intercepts of the image:

$$(3,\ 0) \to (3,\ 0)$$
$$(0,\ 3) \to (0,\ 5)$$
$$(-3,\ 0) \to (-3,\ 0)$$
$$(0,\ -3) \to (0,\ -5)$$

$\therefore$ the image is an ellipse.



b If the object has equation $x^2 + y^2 = 9$, and $x = x'$, $y = \frac{3}{5}y'$,

then the image has equation $x^2 + (\frac{3}{5}y)^2 = 9$

$$\therefore\ x^2 + \tfrac{9}{25}y^2 = 9$$

Now, $a = 3$ and $b = 5$

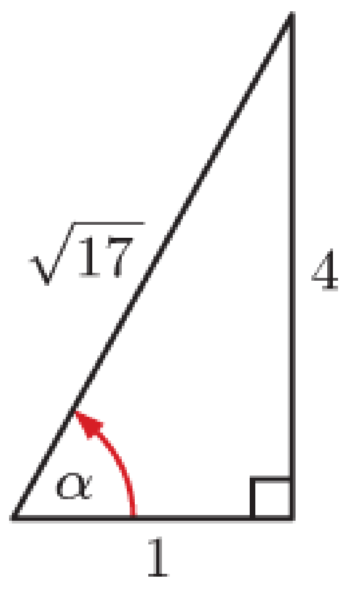
$\therefore$ area $= \pi ab = 15\pi$ units^2

Alternatively, $||\,\mathbf{A}\,|| = \frac{5}{3}$

$\therefore$ area of image $= \frac{5}{3} \times (\pi \times 3^2)$

$= 15\pi$ units^2

9 **a** For $y = 4x$, $\tan\alpha = 4$



$$\therefore\ \cos\alpha = \tfrac{1}{\sqrt{17}},\quad \sin\alpha = \tfrac{4}{\sqrt{17}}$$

$$\therefore\ \cos^2\alpha = \tfrac{1}{17},\quad \sin^2\alpha = \tfrac{16}{17},$$

$$\sin\alpha\cos\alpha = \tfrac{4}{17}$$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \frac{1}{17} & \frac{4}{17} \\ \frac{4}{17} & \frac{16}{17} \end{pmatrix}$$

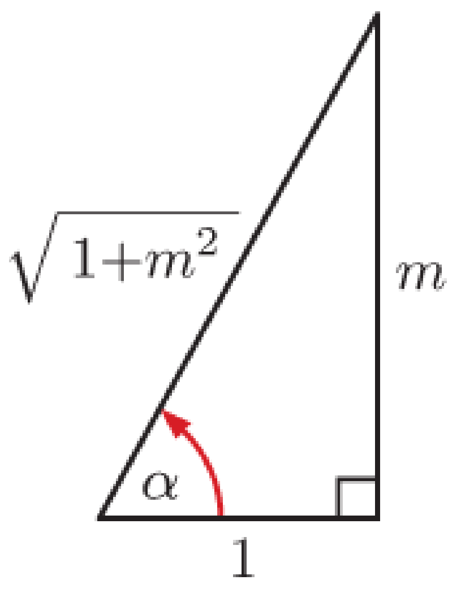
$$\therefore\ x' = \frac{x+4y}{17},\quad y' = \frac{4x+16y}{17}$$

$$\therefore\ (-1,\ 3) \to \left(\tfrac{-1+12}{17},\ \tfrac{-4+48}{17}\right)$$

that is, $(-1,\ 3) \to (\frac{11}{17},\ \frac{44}{17})$

$$\therefore\ \text{shortest distance} = \sqrt{\left(\tfrac{11}{17} - -1\right)^2 + \left(\tfrac{44}{17} - 3\right)^2}$$
$$= \sqrt{\left(\tfrac{28}{17}\right)^2 + \left(-\tfrac{7}{17}\right)^2}$$
$$= \tfrac{1}{17}\sqrt{28^2 + 7^2}$$
$$= \tfrac{1}{17}\sqrt{7^2(4^2 + 1^2)}$$
$$= \tfrac{1}{17} \times 7 \times \sqrt{17}$$
$$= \tfrac{7}{\sqrt{17}} \text{ units}$$

b For $y = mx$, $\tan\alpha = m$



$$\therefore\ \cos^2\alpha = \frac{1}{1+m^2},\quad \sin^2\alpha = \frac{m^2}{1+m^2}$$

$$\sin\alpha\cos\alpha = \frac{m}{1+m^2}$$

$$\therefore\ \mathbf{A} = \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix}$$

and so $x' = \dfrac{x+my}{1+m^2}$, $y' = \dfrac{mx+m^2y}{1+m^2}$

$$\therefore\ (h,\ k) \to \left(\frac{h+mk}{1+m^2},\ \frac{mh+m^2k}{1+m^2}\right)$$

$\therefore$ the shortest distance to $y = mx + c$

$$= \sqrt{\left(\frac{h+mk}{1+m^2} - h\right)^2 + \left(\frac{mh+m^2k}{1+m^2} - k\right)^2}$$
$$= \sqrt{\left(\frac{\cancel{h} + mk \cancel{-h} - hm^2}{1+m^2}\right)^2 + \left(\frac{mh + \cancel{m^2k} - k \cancel{-m^2k}}{1+m^2}\right)}$$
$$= \sqrt{\left(\frac{m(k-hm)}{1+m^2}\right)^2 + \left(\frac{mh-k}{1+m^2}\right)^2}$$
$$= \sqrt{\frac{m^2(mh-k)^2 + (mh-k)^2}{(1+m^2)^2}}$$
$$= \sqrt{\frac{(mh-k)^2(1+m^2)}{(1+m^2)^2}}$$
$$= \sqrt{\frac{(mh-k)^2}{1+m^2}}$$
$$= \frac{|mh-k|}{\sqrt{1+m^2}} \text{ units}$$

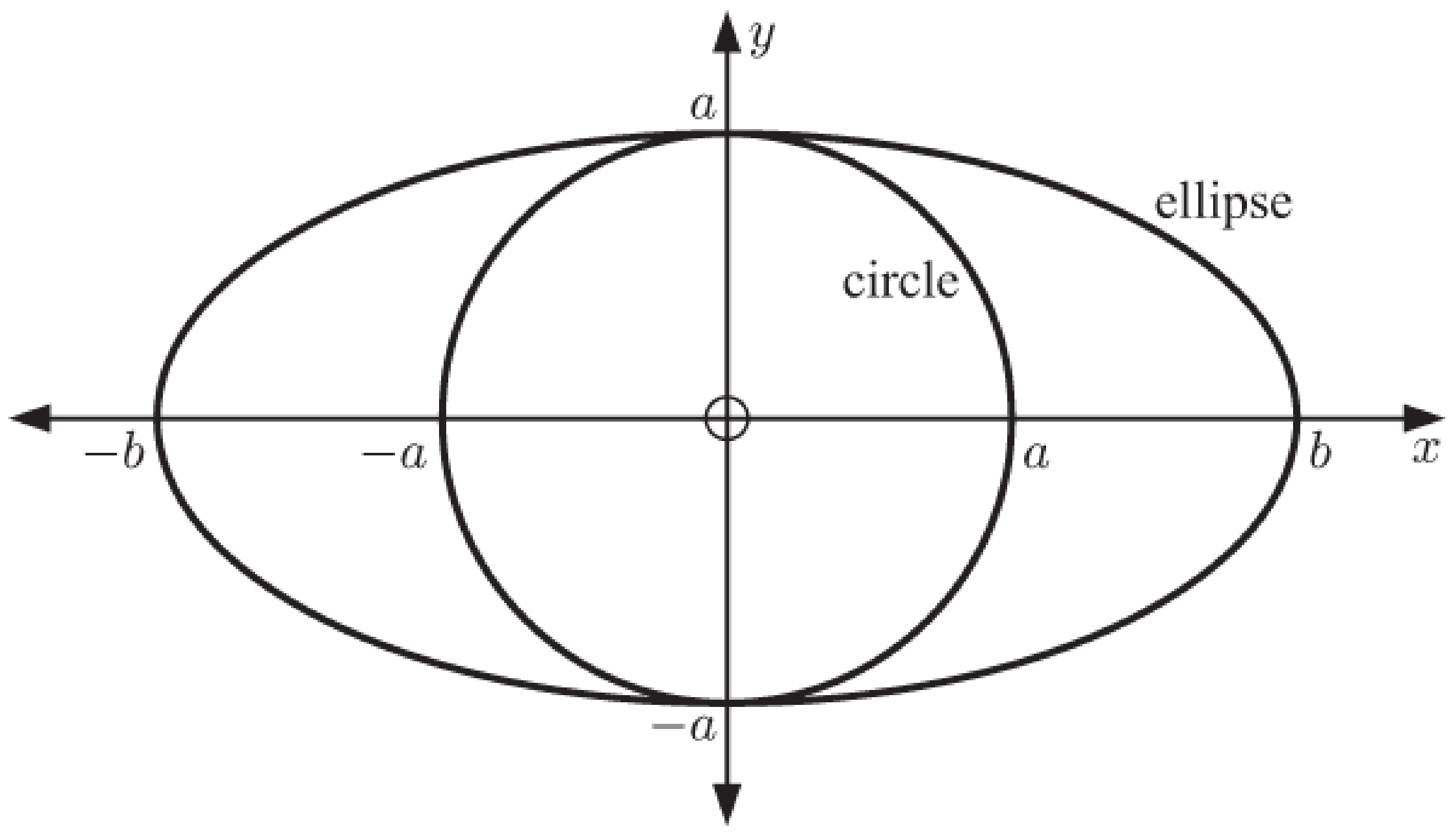
10 **a** For a horizontal stretch with $k = \dfrac{b}{a}$,

$$\mathbf{A} = \begin{pmatrix} \frac{b}{a} & 0 \\ 0 & 1 \end{pmatrix} \qquad \therefore\ x' = \frac{b}{a}x,\quad y' = y$$

The circle has the following axes intercepts, which we transform to obtain the axes intercepts of the image:

$$(a,\ 0) \to (b,\ 0)$$
$$(-a,\ 0) \to (-b,\ 0)$$
$$(0,\ a) \to (0,\ a)$$
$$(0,\ -a) \to (0,\ -a)$$

$\therefore$ the image is an ellipse.



b Area of circle $= \pi a^2$

and $\quad |\,\mathbf{A}\,| = \dfrac{b}{a}$

$\therefore \quad ||\,\mathbf{A}\,|| = \dfrac{b}{a} \quad \{a > 0,\ b > 0\}$

Area of image $=$ area of object $\times\ ||\,\mathbf{A}\,||$

$$= \pi a^2 \left(\frac{b}{a}\right)$$

$$= \pi ab$$

EXERCISE 1K.4

1 **a** $\mathrm{T_A}$ is a reflection in the x-axis.

$\mathrm{T_B}$ is an anticlockwise rotation of $\frac{\pi}{2}$ about O.

$\mathrm{T_A}$ has matrix $\quad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\mathrm{T_B}$ has matrix $\quad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Thus $\quad \mathbf{BA} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

which is the matrix of a reflection in $\ y = x$.

$\therefore \quad \mathrm{T_A}$ followed by $\mathrm{T_B}$ is a reflection in $\ y = x$.

b $\mathrm{T_A}$ is an anticlockwise rotation of $\frac{2\pi}{3}$ about O.

$\mathrm{T_B}$ is a reflection in $\ y = -x$.

As $\quad \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \quad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\mathrm{T_A}$ has matrix $\quad \mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$.

$\mathrm{T_B}$ has matrix $\quad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Thus $\quad \mathbf{BA} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

where $\quad |\,\mathbf{BA}\,| = |\,\mathbf{B}\,|\,|\,\mathbf{A}\,| = (-1)(1) = -1$

and the form is $\quad \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.

Hence $\mathrm{T_A}$ followed by $\mathrm{T_B}$ is a reflection in $\ y = (\tan\alpha)x$

where $\quad \cos 2\alpha = -\frac{\sqrt{3}}{2} \quad$ and $\quad \sin 2\alpha = \frac{1}{2}$.

$\therefore \quad 2\alpha = \frac{5\pi}{6}$

$\therefore \quad \alpha = \frac{5\pi}{12}$

$\therefore \quad \mathrm{T_A}$ followed by $\mathrm{T_B}$ is a reflection in $\ y = (\tan\frac{5\pi}{12})x$.

c $\mathrm{T_A}$ is a reflection in $\ y = \sqrt{3}x$.

$\mathrm{T_B}$ is a reflection in the y-axis.

For $\mathrm{T_A}$, $\quad \tan\alpha = \sqrt{3} \quad \therefore \quad \cos 2\alpha \quad$ and $\quad \sin 2\alpha$

$$= \frac{1-3}{1+3} \qquad = \frac{2\sqrt{3}}{1+3}$$

$$= -\frac{1}{2} \qquad = \frac{\sqrt{3}}{2}$$

$\mathrm{T_A}$ has matrix $\quad \mathbf{A} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

$\mathrm{T_B}$ has matrix $\quad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus $\quad \mathbf{BA} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

where $\quad |\,\mathbf{BA}\,| = |\,\mathbf{B}\,|\,|\,\mathbf{A}\,| = (-1)(-1) = 1$

and $\mathbf{BA}$ has form $\quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Hence $\mathrm{T_A}$ followed by $\mathrm{T_B}$ is a rotation about O through θ

where $\quad \cos\theta = \frac{1}{2}, \ \sin\theta = \frac{\sqrt{3}}{2} \quad \therefore \quad \theta = \frac{\pi}{3}$.

$\therefore \quad \mathrm{T_A}$ followed by $\mathrm{T_B}$ is a rotation about O through $\frac{\pi}{3}$.

d $\mathrm{T_A}$ is a reflection in $\ y = x$.

$\mathrm{T_B}$ is a reflection in $\ y = 3x$.

$\mathrm{T_A}$ has matrix $\quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For $\mathrm{T_B}$, $\quad \tan\alpha = 3 \quad \therefore \quad \cos 2\alpha \quad$ and $\quad \sin 2\alpha$

$$= \frac{1-9}{1+9} \qquad = \frac{6}{10}$$

$$= -\frac{4}{5} \qquad = \frac{3}{5}$$

$\therefore \quad \mathrm{T_B}$ has matrix $\quad \mathbf{B} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$.

Thus $\quad \mathbf{BA} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

where $\quad |\,\mathbf{BA}\,| = |\,\mathbf{B}\,|\,|\,\mathbf{A}\,| = (-1)(-1) = 1$

and $\mathbf{BA}$ has form $\quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

$\therefore \quad \mathrm{T_A}$ followed by $\mathrm{T_B}$ is a rotation about O through θ

where $\quad \cos\theta = \frac{3}{5} \quad$ and $\quad \sin\theta = \frac{4}{5}$

$\therefore \quad \theta = \arctan\left(\frac{4}{3}\right) \approx 0.927$

$\therefore \quad \mathrm{T_A}$ followed by $\mathrm{T_B}$ is a rotation about O through $\approx 0.927^c$.

2 **a** $\mathrm{T_1}$ is a reflection in $\ y = x$.

$\mathrm{T_2}$ is a rotation about O through $\frac{\pi}{3}$.

$\mathrm{T_1}$ has matrix $\quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$\mathrm{T_2}$ has matrix $\quad \mathbf{B} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

$$\{\cos\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2}, \quad \sin\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{2}\}$$

b **i** $\mathrm{T_1} \circ \mathrm{T_2}$ has matrix $\mathbf{AB}$ where

$$\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

and $\quad |\,\mathbf{AB}\,| = |\,\mathbf{A}\,|\,|\,\mathbf{B}\,| = -1 \times 1 = -1$

and $\mathbf{AB}$ has form $\quad \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.

$\therefore \quad \mathrm{T_1} \circ \mathrm{T_2}\ $ is a reflection in $\ y = (\tan\alpha)x$

where $\quad \cos 2\alpha = \frac{\sqrt{3}}{2}, \quad \sin 2\alpha = \frac{1}{2}$

$\therefore \quad 2\alpha = \frac{\pi}{6}$

$\therefore \quad \alpha = \frac{\pi}{12}$

$\therefore \quad \mathrm{T_1} \circ \mathrm{T_2}\ $ is a reflection in $\ y = (\tan\frac{\pi}{12})x$.

ii $T_2 \circ T_1$ has matrix $\mathbf{BA}$ where

$$\mathbf{BA} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

and $|\mathbf{BA}| = |\mathbf{B}|\,|\mathbf{A}| = 1 \times -1 = -1$

and $\mathbf{BA}$ has form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.

$\therefore$ $T_2 \circ T_1$ is a reflection in $y = (\tan\alpha)x$

where $\cos 2\alpha = -\frac{\sqrt{3}}{2}$, $\sin 2\alpha = \frac{1}{2}$

$\therefore$ $2\alpha = \frac{5\pi}{6}$

$\therefore$ $\alpha = \frac{5\pi}{12}$

$\therefore$ $T_2 \circ T_1$ is a reflection in $y = (\tan\frac{5\pi}{12})x$.

c As $T_1 \circ T_2$ and $T_2 \circ T_1$ represent different transformations, $T_1 \circ T_2 \neq T_2 \circ T_1$.

3 **a** T_A has matrix $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and

T_B has matrix $\mathbf{B} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

$\therefore$ $T_B \circ T_A$ has matrix $\mathbf{BA}$ where

$$\mathbf{BA} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\sin\theta\cos\phi - \cos\theta\sin\phi \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$

which is the matrix for a rotation about O through $(\theta + \phi)$.

b T_A is a reflection in $y = (\tan\alpha)x$ and
T_B is a reflection in $y = (\tan\beta)x$.

T_A has matrix $\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$.

T_B has matrix $\mathbf{B} = \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix}$.

Now $T_B \circ T_A$ has matrix $\mathbf{BA}$ where

$\mathbf{BA}$

$$= \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\beta\cos 2\alpha + \sin 2\beta\sin 2\alpha & \cos 2\beta\sin 2\alpha - \sin 2\beta\cos 2\alpha \\ \sin 2\beta\cos 2\alpha - \cos 2\beta\sin 2\alpha & \sin 2\beta\sin 2\alpha + \cos 2\beta\cos 2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\beta - 2\alpha) & -\sin(2\beta - 2\alpha) \\ \sin(2\beta - 2\alpha) & \cos(2\beta - 2\alpha) \end{pmatrix}$$

which is the matrix for a rotation about O through an angle of $2(\beta - \alpha)$.

$\therefore$ $T_B \circ T_A$ is a rotation about O through $2(\beta - \alpha)$.

4 Let T_A be a reflection in $y = (\tan\alpha)x$, and let T_B be a rotation about O through θ.

T_A has matrix $\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$.

T_B has matrix $\mathbf{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Now $T_B \circ T_A$ has matrix $\mathbf{BA}$ where

$\mathbf{BA}$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos 2\alpha - \sin\theta\sin 2\alpha & \cos\theta\sin 2\alpha + \sin\theta\cos 2\alpha \\ \sin\theta\cos 2\alpha + \cos\theta\sin 2\alpha & \sin\theta\sin 2\alpha - \cos\theta\cos 2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta + 2\alpha) & \sin(\theta + 2\alpha) \\ \sin(\theta + 2\alpha) & -\cos(\theta + 2\alpha) \end{pmatrix}$$

which is the matrix for a reflection in the line

$y = \left[\tan\left(\frac{\theta}{2} + \alpha\right)x\right]$

$\therefore$ $T_B \circ T_A$ is a reflection in $y = \left[\tan\left(\frac{\theta}{2} + \alpha\right)x\right]$.

5 Consider $T_1 \circ T_2 \circ T_3 \circ T_4 \circ \ldots\ldots \circ T_n$ where the T_i are reflections.

From **3 b**, $T_1 \circ T_2$ is a rotation.

$\therefore$ $(T_1 \circ T_2) \circ T_3$ is a reflection {from **4**}

$\therefore$ $(T_1 \circ T_2 \circ T_3) \circ T_4$ is a rotation and so on.

So, by the inductive process, the combined effect of:

- an even number of reflections is a rotation
- an odd number of reflections is a reflection.

6 T_A is unknown with matrix $\mathbf{A}$.

T_B is a clockwise rotation about O through $\frac{\pi}{2}$.

$T_B \circ T_A$ is a reflection in $y = \frac{1}{2}x$.

T_B has matrix $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

For $T_B \circ T_A$, $\tan\alpha = \frac{1}{2}$

$$\therefore \quad \cos 2\alpha = \frac{1 - \frac{1}{4}}{1 + \frac{1}{4}}, \qquad \sin 2\alpha = \frac{1}{1 + \frac{1}{4}}$$

$$= \frac{3}{5} \qquad\qquad = \frac{4}{5}$$

$\therefore$ $\mathbf{BA} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

Now $\mathbf{BA} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{A}$

$\therefore$ $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

$\therefore$ $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

$\therefore$ $\mathbf{A} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$ where

$|\mathbf{A}| = -1$ and $\mathbf{A}$ has form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$

$\therefore$ $\mathbf{A}$ is a reflection in $y = (\tan\alpha)x$

where $\cos 2\alpha = -\frac{4}{5}$ and $\sin 2\alpha = \frac{3}{5}$

$\therefore$ $\tan 2\alpha = -\frac{3}{4} = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

$\therefore$ $-3(1 - \tan^2\alpha) = 8\tan\alpha$

$\therefore$ $3\tan^2\alpha - 8\tan\alpha - 3 = 0$

$\therefore$ $(3\tan\alpha + 1)(\tan\alpha - 3) = 0$

$\therefore$ $\tan\alpha = -\frac{1}{3}$ or 3

But $\frac{\pi}{2} < 2\alpha < \pi \quad \{\cos 2\alpha < 0, \ \sin 2\alpha > 0\}$

$\therefore \ \frac{\pi}{4} < \alpha < \frac{\pi}{2}$

$\therefore$ $\tan\alpha > 0$ and so $\tan\alpha = 3$

$\therefore$ T_A is a reflection in $y = 3x$.

7 T_A is a reflection in $y = -x$.

T_B is a rotation of $\frac{3\pi}{4}$ about O.

T_A has matrix $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

T_B has matrix $\mathbf{B} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

$\{\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \quad \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}\}$

$T_B \circ T_A$ has matrix **BA** where

$$\mathbf{BA} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$T_B \circ T_A$ has equations $x' = \dfrac{x+y}{\sqrt{2}}$

$$y' = \frac{x-y}{\sqrt{2}}$$

and $(\mathbf{BA})^{-1} = \mathbf{BA}$ {as **BA** is a reflection}

$$\therefore \ x = \frac{x'+y'}{\sqrt{2}}, \quad y = \frac{x'-y'}{\sqrt{2}}$$

$$\therefore \ y = 2x - 1 \xrightarrow{T_B \circ T_A} \frac{x'-y'}{\sqrt{2}} = 2\left(\frac{x'+y'}{\sqrt{2}}\right) - 1$$

$$\Rightarrow \ x' - y' = 2x' + 2y' - \sqrt{2}$$

$$\Rightarrow \ x' + 3y' = \sqrt{2}$$

$$\therefore \ y = 2x - 1 \xrightarrow{T_B \circ T_A} x + 3y = \sqrt{2}$$

8 **a** $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

b T_A is a vertical stretch of factor 2.

T_B is a reflection in $y = -2x$.

T_A has matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

T_B has matrix $\mathbf{B} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$.

$\{\tan\alpha = -2 \quad \therefore \ \cos 2\alpha = \dfrac{1-4}{1+4} = -\frac{3}{5},$

$\sin 2\alpha = \dfrac{-4}{1+4} = -\frac{4}{5}\}$

$$T_B \circ T_A \text{ has matrix } \mathbf{BA} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{5} & -\frac{8}{5} \\ -\frac{4}{5} & \frac{6}{5} \end{pmatrix}$$

c $T_B \circ T_A$ has $|\,\mathbf{BA}\,| = |\,\mathbf{B}\,|\,|\,\mathbf{A}\,|$

$$= -1 \times 2$$

$$= -2$$

$\therefore$ $T_B \circ T_A$ reverses sense and doubles area.

EXERCISE 1L.1

1 **a** If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then

$$\begin{vmatrix} \lambda - 2 & 1 \\ -2 & \lambda - 5 \end{vmatrix} = 0$$

$$\therefore \ \lambda^2 - 7\lambda + 12 = 0$$

$$\therefore \ (\lambda - 3)(\lambda - 4) = 0$$

$$\therefore \ \lambda = 3 \text{ or } 4$$

Thus the eigenvalues are $\lambda = 3, 4$.

For $\lambda = 3$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \ a + b = 0$$

Letting $b = t$, $t \neq 0$, $a = -t$

$$\therefore \ \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

For $\lambda = 4$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \ 2a + b = 0$$

Letting $a = t$, $t \neq 0$, $b = -2t$

$$\therefore \ \mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} -1 \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 3.

Any vector of the form $\begin{pmatrix} 1 \\ -2 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 4.

b If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then

$$\begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = 0$$

$$\therefore \ \lambda^2 + 1 = 0$$

$$\therefore \ (\lambda + i)(\lambda - i) = 0$$

$$\therefore \ \lambda = \pm i$$

Thus the eigenvalues are $\lambda = i, -i$.

For $\lambda = i$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \ ai + b = 0$$

Letting $a = t$, $t \neq 0$, $b = -it$

$$\therefore \ \mathbf{x} = \begin{pmatrix} 1 \\ -i \end{pmatrix} t, \quad t \neq 0$$

For $\lambda = -i$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \ -ai + b = 0$$

Letting $a = t$, $t \neq 0$, $b = it$

$$\therefore \ \mathbf{x} = \begin{pmatrix} 1 \\ i \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} 1 \\ -i \end{pmatrix} t,\ \ t \neq 0$ is an eigenvector corresponding to the eigenvalue i.

Any vector of the form $\begin{pmatrix} 1 \\ i \end{pmatrix} t,\ \ t \neq 0$ is an eigenvector corresponding to the eigenvalue $-i$.

c If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then

$$\begin{vmatrix} \lambda - 1 & 0 \\ -2 & \lambda - 1 \end{vmatrix} = 0$$

$$\therefore\ \lambda^2 - 2\lambda + 1 = 0$$
$$\therefore\ (\lambda - 1)^2 = 0$$
$$\therefore\ \lambda = 1$$

Thus the eigenvalue is 1.

For $\lambda = 1$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ -2a = 0$$
$$\therefore\ a = 0$$
$$\therefore\ \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} 0 \\ 1 \end{pmatrix} t,\ \ t \neq 0$ is an eigenvector corresponding to the eigenvalue 1.

d If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then

$$\begin{vmatrix} \lambda - 2 & -1 \\ -4 & \lambda - 2 \end{vmatrix} = 0$$

$$\therefore\ \lambda^2 - 4\lambda = 0$$
$$\therefore\ \lambda(\lambda - 4) = 0$$
$$\therefore\ \lambda = 0 \text{ or } 4$$

Thus the eigenvalues are $\lambda = 0,\ 4$.

For $\lambda = 0$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ 2a + b = 0$$

Letting $a = t,\ \ t \neq 0,\ \ b = -2t$

$$\therefore\ \mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t, \quad t \neq 0$$

For $\lambda = 4$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ 2a - b = 0$$

Letting $a = t,\ \ t \neq 0,\ \ b = 2t$

$$\therefore\ \mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} 1 \\ -2 \end{pmatrix} t,\ \ t \neq 0$ is an eigenvector corresponding to the eigenvalue 0.

Any vector of the form $\begin{pmatrix} 1 \\ 2 \end{pmatrix} t,\ \ t \neq 0$ is an eigenvector corresponding to the eigenvalue 4.

e If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then

$$\begin{vmatrix} \lambda - 3 & -1 \\ -2 & \lambda + 1 \end{vmatrix} = 0$$

$$\therefore\ \lambda^2 - 2\lambda - 5 = 0$$
$$\therefore\ \lambda = \frac{2 \pm \sqrt{4 + 20}}{2}$$
$$\therefore\ \lambda = 1 \pm \sqrt{6}$$

Thus the eigenvalues are $\lambda = 1 + \sqrt{6},\ \ 1 - \sqrt{6}$.

For $\lambda = 1 + \sqrt{6}$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} -2 + \sqrt{6} & -1 \\ -2 & 2 + \sqrt{6} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ (-2 + \sqrt{6})a - b = 0$$

Letting $a = t,\ \ t \neq 0,\ \ b = (-2 + \sqrt{6})t$

$$\therefore\ \mathbf{x} = \begin{pmatrix} 1 \\ -2 + \sqrt{6} \end{pmatrix} t, \quad t \neq 0$$

For $\lambda = 1 - \sqrt{6}$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} -2 - \sqrt{6} & -1 \\ -2 & 2 - \sqrt{6} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ (-2 - \sqrt{6})a - b = 0$$

Letting $a = t,\ \ t \neq 0,\ \ b = (-2 - \sqrt{6})t$

$$\therefore\ \mathbf{x} = \begin{pmatrix} 1 \\ -2 - \sqrt{6} \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} 1 \\ -2 + \sqrt{6} \end{pmatrix} t,\ \ t \neq 0$ is an eigenvector corresponding to the eigenvalue $1 + \sqrt{6}$.

Any vector of the form $\begin{pmatrix} 1 \\ -2 - \sqrt{6} \end{pmatrix} t,\ \ t \neq 0$ is an eigenvector corresponding to the eigenvalue $1 - \sqrt{6}$.

2 **a** If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then

$$\begin{vmatrix} \lambda - 2 & -1 \\ -4 & \lambda + 1 \end{vmatrix} = 0$$

$$\therefore\ \lambda^2 - \lambda - 6 = 0$$
$$\therefore\ (\lambda + 2)(\lambda - 3) = 0$$
$$\therefore\ \lambda = -2 \text{ or } 3$$

Thus the eigenvalues are $\lambda = -2,\ 3$.

For $\lambda = -2$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} -4 & -1 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ 4a + b = 0$$

Letting $a = t,\ \ t \neq 0,\ \ b = -4t$

$$\therefore\ \mathbf{x} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} t, \quad t \neq 0$$

For $\lambda = 3$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ a - b = 0$$

Letting $b = t,\ \ t \neq 0,\ \ a = t$

$$\therefore\ \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix}1\\-4\end{pmatrix}t,\ \ t\neq 0$ is an eigenvector corresponding to the eigenvalue -2.

Any vector of the form $\begin{pmatrix}1\\1\end{pmatrix}t,\ \ t\neq 0$ is an eigenvector corresponding to the eigenvalue 3.

b The effect of $\mathbf{A}$ on eigenvectors $\begin{pmatrix}1\\-4\end{pmatrix}t,\ \ t\neq 0$ is to increase their length by factor 2, and reverse their direction.

The effect of $\mathbf{A}$ on eigenvectors $\begin{pmatrix}1\\1\end{pmatrix}t,\ t\neq 0$ is to increase their length by factor 3, and preserve their direction.

c $E_{-2} = \left\{\mathbf{x}\,\middle|\,\mathbf{x} = \begin{pmatrix}1\\-4\end{pmatrix}t,\ \ t\in\mathbb{R}\right\}$ is the eigenspace for $\mathbf{A}$ corresponding to $\lambda=-2$.

$E_3 = \left\{\mathbf{x}\,\middle|\,\mathbf{x} = \begin{pmatrix}1\\1\end{pmatrix}t,\ \ t\in\mathbb{R}\right\}$ is the eigenspace for $\mathbf{A}$ corresponding to $\lambda=3$.

$E = \left\{\begin{pmatrix}1\\-4\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ is an eigenbasis for $\mathbf{A}$.

3 The characteristic polynomial of $\mathbf{A}^{\mathrm{T}}$

$$\begin{aligned}
&= \left|\lambda\mathbf{I}-\mathbf{A}^{\mathrm{T}}\right| \\
&= \left|(\lambda\mathbf{I}-\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}\right| && \{\left|\mathbf{B}\right| = \left|\mathbf{B}^{\mathrm{T}}\right|\} \\
&= \left|(\lambda\mathbf{I})^{\mathrm{T}}-(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}\right| && \{(\mathbf{B}+\mathbf{C})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}+\mathbf{C}^{\mathrm{T}}\} \\
&= \left|\lambda\mathbf{I}-\mathbf{A}\right| && \{\mathbf{I}^{\mathrm{T}}=\mathbf{I},\ \ (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}=\mathbf{A}\} \\
&= \text{the characteristic polynomial of } \mathbf{A}
\end{aligned}$$

Since $\mathbf{A}^{\mathrm{T}}$ and $\mathbf{A}$ have the same characteristic polynomials, they have the same eigenvalues.

If $\mathbf{A} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ then $\mathbf{A}^{\mathrm{T}} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}$.

For $\lambda=\lambda_i$,

$$(\lambda\mathbf{I}-\mathbf{A})\mathbf{x} = \mathbf{0}$$

$$\therefore\ \begin{pmatrix}\lambda_i-a & -b\\ -c & \lambda_i-d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\therefore\ (\lambda_i-a)x-by=0$$

$$\therefore\ \mathbf{x} = \begin{pmatrix}1\\ \dfrac{\lambda_i-a}{b}\end{pmatrix}t,\quad t\neq 0$$

$$(\lambda\mathbf{I}-\mathbf{A}^{\mathrm{T}})\mathbf{x} = \mathbf{0}$$

$$\therefore\ \begin{pmatrix}\lambda_i-a & -c\\ -b & \lambda_i-d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\therefore\ (\lambda_i-a)x-cy=0$$

$$\therefore\ \mathbf{x} = \begin{pmatrix}1\\ \dfrac{\lambda_i-a}{c}\end{pmatrix}t,\quad t\neq 0$$

$\therefore$ $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ do not have the same corresponding eigenvectors.

4 **a** $\mathbf{A}^2 = \begin{pmatrix}1 & -3\\ -5 & 3\end{pmatrix}\begin{pmatrix}1 & -3\\ -5 & 3\end{pmatrix} = \begin{pmatrix}16 & -12\\ -20 & 24\end{pmatrix}$

$\mathbf{A}^{-1} = \frac{1}{-12}\begin{pmatrix}3 & 3\\ 5 & 1\end{pmatrix} = \begin{pmatrix}-\frac{1}{4} & -\frac{1}{4}\\ -\frac{5}{12} & -\frac{1}{12}\end{pmatrix}$

b $\operatorname{tr}(\mathbf{A}) = 4$ and $\left|\mathbf{A}\right| = -12$

The characteristic polynomial for $\mathbf{A}$ is

$$\begin{aligned}
\det(\lambda\mathbf{I}-\mathbf{A}) &= \lambda^2-\operatorname{tr}(\mathbf{A})\lambda+\left|\mathbf{A}\right| \\
&= \lambda^2-4\lambda-12 \\
&= (\lambda+2)(\lambda-6)
\end{aligned}$$

$\operatorname{tr}(\mathbf{A}^2) = 40$ and $\left|\mathbf{A}\right| = 144$

The characteristic polynomial for $\mathbf{A}^2$ is

$$\begin{aligned}
\det(\lambda\mathbf{I}-\mathbf{A}) &= \lambda^2-\operatorname{tr}(\mathbf{A}^2)\lambda+\left|\mathbf{A}^2\right| \\
&= \lambda^2-40\lambda+144 \\
&= (\lambda-4)(\lambda-36)
\end{aligned}$$

$\operatorname{tr}(\mathbf{A}^{-1}) = -\frac{1}{3}$ and $\left|\mathbf{A}\right| = -\frac{1}{12}$

The characteristic polynomial for $\mathbf{A}^{-1}$ is

$$\begin{aligned}
\det(\lambda\mathbf{I}-\mathbf{A}) &= \lambda^2-\operatorname{tr}(\mathbf{A}^{-1})\lambda+\left|\mathbf{A}^{-1}\right| \\
&= \lambda^2-(-\tfrac{1}{3})\lambda+\frac{1}{\left|\mathbf{A}\right|} \\
&= \lambda^2+\tfrac{1}{3}\lambda-\tfrac{1}{12} \\
&= (\lambda+\tfrac{1}{2})(\lambda-\tfrac{1}{6})
\end{aligned}$$

c $\mathbf{A}$ has eigenvalues -2 and 6, $\mathbf{A}^2$ has eigenvalues 4 and 36, and $\mathbf{A}^{-1}$ has eigenvalues $-\frac{1}{2}$ and $\frac{1}{6}$.

i The eigenvalues of $\mathbf{A}^2$ are the squares of the eigenvalues of $\mathbf{A}$.

ii The eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$.

5 **a** $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is given

$\therefore$ $\mathbf{A}^2\mathbf{x} = \mathbf{A}(\lambda\mathbf{x})$

$\therefore$ $\mathbf{A}^2\mathbf{x} = \lambda(\mathbf{A}\mathbf{x})$

$\therefore$ $\mathbf{A}^2\mathbf{x} = \lambda(\lambda\mathbf{x})$

$\therefore$ $\mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}$

$\therefore$ $\mathbf{A}^2$ has eigenvalue λ^2 and the corresponding eigenvector is $\mathbf{x}$.

b $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is given

$\therefore$ $\mathbf{A}^{-1}(\mathbf{A}\mathbf{x}) = \mathbf{A}^{-1}\lambda\mathbf{x}$

$\therefore$ $\mathbf{I}\mathbf{x} = \lambda\mathbf{A}^{-1}\mathbf{x}$

$\therefore$ $\mathbf{x} = \lambda\mathbf{A}^{-1}\mathbf{x}$

$\therefore$ $\mathbf{A}^{-1}\mathbf{x} = \dfrac{1}{\lambda}\mathbf{x}$

$\therefore$ $\mathbf{A}^{-1}$ has eigenvalue $\dfrac{1}{\lambda}$ with corresponding eigenvector $\mathbf{x}$.

c **Proof:** (By the Principle of Mathematical Induction)

P_n is that "$\mathbf{A}^n\mathbf{x} = \lambda^n\mathbf{x}$" for all $n\in\mathbb{Z}^+$.

(1) If $n=1$, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is given.

$\therefore$ P_1 is true.

(2) If P_k is true, then $\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{x}$, $k\in\mathbb{Z}^+$ .... $(*)$

Now
$$\begin{aligned}
\mathbf{A}^{k+1}\mathbf{x} &= \mathbf{A}(\mathbf{A}^k\mathbf{x}) \\
&= \mathbf{A}\lambda^k\mathbf{x} && \{\text{using } *\} \\
&= \lambda^k\mathbf{A}\mathbf{x} \\
&= \lambda^k\lambda\mathbf{x} \\
&= \lambda^{k+1}\mathbf{x}
\end{aligned}$$

Thus P_1 is true, and P_{k+1} is true whenever P_k is true.

$\therefore$ P_n is true for all $n\in\mathbb{Z}^+$.

$\therefore$ if $\mathbf{A}$ has eigenvalue λ with corresponding eigenvector $\mathbf{x}$, then $\mathbf{A}^n$ has eigenvalue λ^n with corresponding eigenvector $\mathbf{x}$.

6 Let 2 × 2 matrix $\mathbf{A}$ have eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent.

$\therefore$ $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$ has a non-trivial solution

$$\therefore \quad \mathbf{x}_2 = -\frac{c_1}{c_2}\mathbf{x}_1 \quad \{c_2 \neq 0\}$$

Now $\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$

$$\begin{aligned}
\therefore \quad \mathbf{A}\left(-\frac{c_1}{c_2}\mathbf{x}_1\right) &= \lambda_2\left(-\frac{c_1}{c_2}\mathbf{x}_1\right)\\
\therefore \quad -\frac{c_1}{c_2}\mathbf{A}\mathbf{x}_1 &= -\frac{c_1}{c_2}\lambda_2\mathbf{x}_1\\
\therefore \quad -\frac{c_1}{c_2}\lambda_1\mathbf{x}_1 &= -\frac{c_1}{c_2}\lambda_2\mathbf{x}_1\\
\therefore \quad \lambda_1 &= \lambda_2
\end{aligned}$$

$\therefore$ if the eigenvectors are linearly dependent, then the eigenvalues are not distinct.

$\therefore$ if the eigenvalues are distinct, then the eigenvectors are linearly independent.

7 **a** $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

$$\begin{aligned}
(\mathbf{A} + k\mathbf{I})\mathbf{x} &= \mathbf{A}\mathbf{x} + k\mathbf{I}\mathbf{x}\\
&= \lambda\mathbf{x} + k\mathbf{x}\\
&= (\lambda + k)\mathbf{x}
\end{aligned}$$

$\therefore$ $\mathbf{A} + k\mathbf{I}$ has eigenvalue $\lambda + k$ and corresponding eigenvector $\mathbf{x}$.

So, $\mathbf{A}$ and $\mathbf{A} + k\mathbf{I}$ have the same eigenvector.

b $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

$$\begin{aligned}
(\mathbf{A}^2 + 4\mathbf{A})\mathbf{x} &= \mathbf{A}^2\mathbf{x} + 4\mathbf{A}\mathbf{x}\\
&= \lambda^2\mathbf{x} + 4\lambda\mathbf{x} \quad \{\text{from } \mathbf{5\ a}\}\\
&= (\lambda^2 + 4\lambda)\mathbf{x}
\end{aligned}$$

$\therefore$ $\mathbf{A}^2 + 4\mathbf{A}$ has eigenvalue $\lambda^2 + 4\lambda$ with corresponding eigenvector $\mathbf{x}$.

8 Consider $|\lambda\mathbf{I} - \mathbf{A}\mathbf{B}|$

$$\begin{aligned}
&= \left|(\lambda\mathbf{I} - \mathbf{A}\mathbf{B})^T\right| && \{|\mathbf{C}| = \left|\mathbf{C}^T\right|\}\\
&= \left|(\lambda\mathbf{I})^T - (\mathbf{A}\mathbf{B})^T\right| && \{(\mathbf{C} + \mathbf{D})^T = \mathbf{C}^T + \mathbf{D}^T\}\\
&= \left|\lambda\mathbf{I}^T - \mathbf{B}^T\mathbf{A}^T\right| && \{(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T\}\\
&= |\lambda\mathbf{I} - \mathbf{B}\mathbf{A}| && \{\mathbf{I}^T = \mathbf{I},\ \mathbf{A} \text{ and } \mathbf{B} \text{ are symmetric}\}
\end{aligned}$$

$\therefore$ $\mathbf{A}\mathbf{B}$ and $\mathbf{B}\mathbf{A}$ have the same characteristic polynomial.

$\therefore$ they have the same eigenvalues.

9

$$\begin{aligned}
\mathbf{x}^T\mathbf{A}\mathbf{x} &> 0 \quad \text{for all } \mathbf{x} \neq \mathbf{0}\\
\therefore \quad \mathbf{x}^T\lambda\mathbf{x} &> 0\\
\therefore \quad \lambda\mathbf{x}^T\mathbf{x} &> 0\\
\therefore \quad \lambda\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n \end{pmatrix} &> 0\\
\therefore \quad \lambda({x_1}^2 + {x_2}^2 + \dots + {x_n}^2) &> 0
\end{aligned}$$

$\therefore$ $\lambda > 0$ for all $\mathbf{x} \neq \mathbf{0}$

$\{({x_1}^2 + {x_2}^2 + \dots + {x_n}^2) > 0$ as not all $x_i = 0\}$

$\therefore$ the eigenvalues of $\mathbf{A}$ are positive.

10 Consider ${\mathbf{x}_1}^T\mathbf{A}\mathbf{x}_2 = {\mathbf{x}_1}^T\lambda_2\mathbf{x}_2$

$$\begin{aligned}
&= \lambda_2{\mathbf{x}_1}^T\mathbf{x}_2\\
&= \lambda_2(\mathbf{x}_1 \bullet \mathbf{x}_2) \quad \dots\dots (1)
\end{aligned}$$

But

$$\begin{aligned}
{\mathbf{x}_1}^T\mathbf{A}\mathbf{x}_2 &= {\mathbf{x}_1}^T\mathbf{A}^T\mathbf{x}_2 && \{\mathbf{A} \text{ is symmetric}\}\\
&= (\mathbf{A}\mathbf{x}_1)^T\mathbf{x}_2 && \{\mathbf{B}^T\mathbf{A}^T = (\mathbf{A}\mathbf{B})^T\}\\
&= (\lambda_1\mathbf{x}_1)^T\mathbf{x}_2\\
&= \lambda_1{\mathbf{x}_1}^T\mathbf{x}_2\\
&= \lambda_1(\mathbf{x}_1 \bullet \mathbf{x}_2) \quad \dots\dots (2)
\end{aligned}$$

From (1) and (2), $\lambda_1(\mathbf{x}_1 \bullet \mathbf{x}_2) = \lambda_2(\mathbf{x}_1 \bullet \mathbf{x}_2)$

where λ_1, λ_2 were given unequal

$$\begin{aligned}
\therefore \quad \mathbf{x}_1 \bullet \mathbf{x}_2 &= 0\\
\therefore \quad \mathbf{x}_1 &\perp \mathbf{x}_2
\end{aligned}$$

$\therefore$ $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

EXERCISE 1L.2

1 **a** If $|\lambda\mathbf{I} - \mathbf{A}| = 0$ then

$$\begin{vmatrix} \lambda - 8 & -3\\ -2 & \lambda - 7 \end{vmatrix} = 0$$

$$\begin{aligned}
\therefore \quad \lambda^2 - 15\lambda + 50 &= 0\\
\therefore \quad (\lambda - 5)(\lambda - 10) &= 0\\
\therefore \quad \lambda &= 5, 10
\end{aligned}$$

For $\lambda = 5$, $|\lambda\mathbf{I} - \mathbf{A}|\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} -3 & -3\\ -2 & -2 \end{pmatrix}\begin{pmatrix} a\\ b \end{pmatrix} = \begin{pmatrix} 0\\ 0 \end{pmatrix}$

$$\therefore \quad a + b = 0$$

Letting $a = t$, $t \neq 0$, $b = -t$

$$\therefore \quad \mathbf{x} = \begin{pmatrix} 1\\ -1 \end{pmatrix}t, \quad t \neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 1\\ -1 \end{pmatrix}$ $\{$choosing $t = 1\}$

For $\lambda = 10$, $|\lambda\mathbf{I} - \mathbf{A}|\mathbf{x} = \mathbf{0}$

becomes $\begin{pmatrix} 2 & -3\\ -2 & 3 \end{pmatrix}\begin{pmatrix} a\\ b \end{pmatrix} = \begin{pmatrix} 0\\ 0 \end{pmatrix}$

$$\therefore \quad 2a - 3b = 0$$

Letting $a = t$, $t \neq 0$, $b = \frac{2}{3}t$

$$\therefore \quad \mathbf{x} = \begin{pmatrix} 1\\ \frac{2}{3} \end{pmatrix}t, \quad t \neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 3\\ 2 \end{pmatrix}$ $\{$choosing $t = 3\}$

$\therefore$ the eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 10$ with

corresponding eigenvectors $\begin{pmatrix} 1\\ -1 \end{pmatrix}$, $\begin{pmatrix} 3\\ 2 \end{pmatrix}$.

b $\mathbf{P} = (\mathbf{x}_1\,|\,\mathbf{x}_2) = \begin{pmatrix} 1 & 3\\ -1 & 2 \end{pmatrix}$

i

$$\begin{aligned}
\mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \tfrac{1}{5}\begin{pmatrix} 2 & -3\\ 1 & 1 \end{pmatrix}\begin{pmatrix} 8 & 3\\ 2 & 7 \end{pmatrix}\begin{pmatrix} 1 & 3\\ -1 & 2 \end{pmatrix}\\
&= \tfrac{1}{5}\begin{pmatrix} 10 & -15\\ 10 & 10 \end{pmatrix}\begin{pmatrix} 1 & 3\\ -1 & 2 \end{pmatrix}\\
&= \begin{pmatrix} 2 & -3\\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3\\ -1 & 2 \end{pmatrix}\\
&= \begin{pmatrix} 5 & 0\\ 0 & 10 \end{pmatrix}\\
&= \begin{pmatrix} \lambda_1 & 0\\ 0 & \lambda_2 \end{pmatrix}
\end{aligned}$$

ii $$\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} = \tfrac{1}{5}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 70 & 45 \\ 30 & 55 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 14 & 9 \\ 6 & 11 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 10 & -15 \\ 20 & 20 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 25 & 0 \\ 0 & 100 \end{pmatrix}$$
$$= \begin{pmatrix} 5^2 & 0 \\ 0 & 10^2 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1{}^2 & 0 \\ 0 & \lambda_2{}^2 \end{pmatrix}$$

2 **a** If $|\lambda\mathbf{I} - \mathbf{A}| = 0$ then
$$\begin{vmatrix} \lambda - 3 & -4 \\ -5 & \lambda - 2 \end{vmatrix} = 0$$
$$\therefore\ \lambda^2 - 5\lambda - 14 = 0$$
$$\therefore\ (\lambda + 2)(\lambda - 7) = 0$$
$$\therefore\ \lambda = -2,\ 7$$

For $\lambda = -2$, $|\lambda\mathbf{I} - \mathbf{A}|\,\mathbf{x} = \mathbf{0}$

becomes $$\begin{pmatrix} -5 & -4 \\ -5 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ 5a + 4b = 0$$

Letting $a = t,\ t \neq 0,\ b = -\tfrac{5}{4}t$
$$\therefore\ \mathbf{x} = \begin{pmatrix} 1 \\ -\frac{5}{4} \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ {choosing $t = 4$}

For $\lambda = 7$, $|\lambda\mathbf{I} - \mathbf{A}|\,\mathbf{x} = \mathbf{0}$

becomes $$\begin{pmatrix} 4 & -4 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ a - b = 0$$

Letting $a = t,\ t \neq 0,\ b = t$
$$\therefore\ \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ {choosing $t = 1$}

$\therefore$ the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 7$ with

corresponding eigenvectors $\begin{pmatrix} 4 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b $\mathbf{P}$ could be $\begin{pmatrix} 4 & 1 \\ -5 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 4 \\ 1 & -5 \end{pmatrix}$.

c For $\mathbf{P} = \begin{pmatrix} 4 & 1 \\ -5 & 1 \end{pmatrix}$, $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -2 & 0 \\ 0 & 7 \end{pmatrix}$

and $\mathbf{A}^3 = \mathbf{P}\mathbf{D}^3\mathbf{P}^{-1}$ {power of a matrix theorem}
$$\therefore\ \mathbf{P}^{-1}\mathbf{A}^3\mathbf{P} = \mathbf{P}^{-1}\mathbf{P}\mathbf{D}^3\mathbf{P}^{-1}\mathbf{D}$$
$$= \mathbf{D}^3$$
$$= \begin{pmatrix} -8 & 0 \\ 0 & 343 \end{pmatrix}$$

For $\mathbf{P} = \begin{pmatrix} 1 & 4 \\ 1 & -5 \end{pmatrix}$, $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 7 & 0 \\ 0 & -2 \end{pmatrix}$
$$\therefore\ \mathbf{P}^{-1}\mathbf{A}^3\mathbf{P} = \mathbf{D}^3$$
$$= \begin{pmatrix} 343 & 0 \\ 0 & -8 \end{pmatrix}$$

3 If $|\lambda\mathbf{I} - \mathbf{A}| = 0$ then
$$\begin{vmatrix} \lambda + 1 & -1 \\ -2 & \lambda - 1 \end{vmatrix} = 0$$
$$\therefore\ \lambda^2 - 3 = 0$$
$$\therefore\ (\lambda + \sqrt{3})(\lambda - \sqrt{3}) = 0$$
$$\therefore\ \lambda = \sqrt{3},\ -\sqrt{3}$$

When $\lambda = \sqrt{3}$, $|\lambda\mathbf{I} - \mathbf{A}|\,\mathbf{x} = \mathbf{0}$

becomes $$\begin{pmatrix} 1 + \sqrt{3} & -1 \\ -2 & \sqrt{3} - 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ (1 + \sqrt{3})a - b = 0$$

Letting $a = t,\ t \neq 0,\ b = (1 + \sqrt{3})t$
$$\therefore\ \mathbf{x} = \begin{pmatrix} 1 \\ 1 + \sqrt{3} \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 1 \\ 1 + \sqrt{3} \end{pmatrix}$ {choosing $t = 1$}

When $\lambda = -\sqrt{3}$, $|\lambda\mathbf{I} - \mathbf{A}|\,\mathbf{x} = \mathbf{0}$

becomes $$\begin{pmatrix} 1 - \sqrt{3} & -1 \\ -2 & -\sqrt{3} - 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore\ (1 - \sqrt{3})a - b = 0$$

Letting $a = t,\ t \neq 0,\ b = (1 - \sqrt{3})t$
$$\therefore\ \mathbf{x} = \begin{pmatrix} 1 \\ 1 - \sqrt{3} \end{pmatrix} t, \quad t \neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 1 \\ 1 - \sqrt{3} \end{pmatrix}$ {choosing $t = 1$}

$\therefore$ $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 + \sqrt{3} & 1 - \sqrt{3} \end{pmatrix}$ diagonalises $\mathbf{A}$

and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$.

$\mathbf{A}^{60}$
$$= \mathbf{P}\begin{pmatrix} \sqrt{3}^{60} & 0 \\ 0 & (-\sqrt{3})^{60} \end{pmatrix}\mathbf{P}^{-1} \quad \text{\{power of a matrix theorem\}}$$
$$= \begin{pmatrix} 1 & 1 \\ 1 + \sqrt{3} & 1 - \sqrt{3} \end{pmatrix}\begin{pmatrix} 3^{30} & 0 \\ 0 & 3^{30} \end{pmatrix}\tfrac{1}{-2\sqrt{3}}\begin{pmatrix} 1 - \sqrt{3} & -1 \\ -1 - \sqrt{3} & 1 \end{pmatrix}$$
$$= -\tfrac{1}{2\sqrt{3}}\begin{pmatrix} 3^{30} & 3^{30} \\ 3^{30}(1 + \sqrt{3}) & 3^{30}(1 - \sqrt{3}) \end{pmatrix}\begin{pmatrix} 1 - \sqrt{3} & -1 \\ -1 - \sqrt{3} & 1 \end{pmatrix}$$
$$= -\tfrac{1}{2\sqrt{3}}\begin{pmatrix} 3^{30}(1 - \sqrt{3} - 1 - \sqrt{3}) & -3^{30} + 3^{30} \\ 3^{30}(1 - 3 - 1 + 3) & 3^{30}(-1 - \sqrt{3} + 1 - \sqrt{3}) \end{pmatrix}$$
$$= \tfrac{1}{2\sqrt{3}}\begin{pmatrix} -2\sqrt{3} \times 3^{30} & 0 \\ 0 & -2\sqrt{3} \times 3^{30} \end{pmatrix}$$
$$= \begin{pmatrix} 3^{30} & 0 \\ 0 & 3^{30} \end{pmatrix}$$
$$= 3^{30}\mathbf{I}$$

4 If $|\lambda\mathbf{I}-\mathbf{A}|=0$ then

$$\begin{vmatrix} \lambda & -2 \\ -1 & \lambda-1 \end{vmatrix} = 0$$

$$\therefore\ \lambda^2-\lambda-2=0$$

$$\therefore\ (\lambda+1)(\lambda-2)=0$$

$$\therefore\ \lambda=-1,\ 2$$

When $\lambda=-1$, $|\lambda\mathbf{I}-\mathbf{A}|\,\mathbf{x}=\mathbf{0}$

becomes $\begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ a+2b=0$$

Letting $b=t,\ t\neq 0,\ a=-2t$

$$\therefore\ \mathbf{x}=\begin{pmatrix} -2 \\ 1 \end{pmatrix}t,\quad t\neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ {choosing $t=1$}

When $\lambda=2$, $|\lambda\mathbf{I}-\mathbf{A}|\,\mathbf{x}=\mathbf{0}$

becomes $\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ a-b=0$$

Letting $b=t,\ t\neq 0,\ a=t$

$$\therefore\ \mathbf{x}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}t,\quad t\neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ {choosing $t=1$}

$\therefore$ $\mathbf{P}=\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$ diagonalises $\mathbf{C}$

and $\mathbf{P}^{-1}\mathbf{C}\mathbf{P}=\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.

$$\mathbf{C}^{2015}=\mathbf{P}\begin{pmatrix} (-1)^{2015} & 0 \\ 0 & 2^{2015} \end{pmatrix}\mathbf{P}^{-1}$$

{power of a matrix theorem}

$$=\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 2^{2015} \end{pmatrix}\left(-\tfrac{1}{3}\right)\begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}$$

$$=\tfrac{1}{3}\begin{pmatrix} 2 & 2^{2015} \\ -1 & 2^{2015} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$=\tfrac{1}{3}\begin{pmatrix} -2+2^{2015} & 2+2^{2016} \\ 1+2^{2015} & -1+2^{2016} \end{pmatrix}$$

5 $\mathbf{A}=\begin{pmatrix} k & 1 \\ 0 & k^2 \end{pmatrix}$

If $|\lambda\mathbf{I}-\mathbf{A}|=0$ then $\begin{vmatrix} \lambda-k & -1 \\ 0 & \lambda-k^2 \end{vmatrix}=0$

$$\therefore\ (\lambda-k)(\lambda-k^2)=0$$

$$\therefore\ \lambda=k,\ k^2$$

For $\lambda=k$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$

becomes $\begin{pmatrix} 0 & -1 \\ 0 & k-k^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ b=0$$

$$\therefore\ \mathbf{x}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}t,\quad t\neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ {choosing $t=1$}

For $\lambda=k^2$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$

becomes $\begin{pmatrix} k^2-k & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ k(k-1)a-b=0$$

Letting $a=t,\ t\neq 0,\ b=k(k-1)t$

$$\therefore\ \mathbf{x}=\begin{pmatrix} 1 \\ k(k-1) \end{pmatrix}t,\quad t\neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 1 \\ k(k-1) \end{pmatrix}$ {choosing $t=1$}

$\therefore$ $\mathbf{P}=\begin{pmatrix} 1 & 1 \\ 0 & k(k-1) \end{pmatrix}$ or $\mathbf{P}=\begin{pmatrix} 1 & 1 \\ k(k-1) & 0 \end{pmatrix}$

diagonalises $\mathbf{A}$ if $\mathbf{P}^{-1}$ exists.

$\mathbf{P}^{-1}$ exists if $|\mathbf{P}|\neq 0$

$\therefore$ $\begin{vmatrix} 1 & 1 \\ 0 & k(k-1) \end{vmatrix}\neq 0$ *or* $\begin{vmatrix} 1 & 1 \\ k(k-1) & 0 \end{vmatrix}\neq 0$

$\therefore\ k(k-1)\neq 0$ $\qquad$ $\therefore\ -k(k-1)\neq 0$

$\therefore\ k\neq 0$ or 1 $\qquad$ $\therefore\ k\neq 0$ or 1

$\therefore$ $\mathbf{A}=\begin{pmatrix} k & 1 \\ 0 & k^2 \end{pmatrix}$ is diagonalisable for k, $k\neq 0,\ 1$.

6 Let $\mathbf{A}=\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$, where $\mathbf{B}^2=\mathbf{A}$.

If $|\lambda\mathbf{I}-\mathbf{A}|=0$ then $\begin{vmatrix} \lambda-1 & 0 \\ -2 & \lambda-4 \end{vmatrix}=0$

$$\therefore\ (\lambda-1)(\lambda-4)=0$$

$$\therefore\ \lambda=1,\ 4$$

For $\lambda=1$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$

becomes $\begin{pmatrix} 0 & 0 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ 2a+3b=0$$

Letting $a=t,\ t\neq 0,\ b=-\frac{2}{3}t$

$$\therefore\ \mathbf{x}=\begin{pmatrix} 1 \\ -\frac{2}{3} \end{pmatrix}t,\quad t\neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ {choosing $t=3$}

For $\lambda=4$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$

becomes $\begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore\ a=0$$

$$\therefore\ \mathbf{x}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}t,\quad t\neq 0$$

$\therefore$ an eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ {choosing $t=1$}

Thus $\mathbf{P}=\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}$ will diagonalise $\mathbf{A}$, and

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\therefore\ \mathbf{P}^{-1}\mathbf{B}^2\mathbf{P}=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\therefore\ \mathbf{B}^2=\mathbf{P}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{P}^{-1}$$

$$\therefore \quad \mathbf{B}^2 = \mathbf{PC}^2\mathbf{P}^{-1} \quad \text{where} \quad \mathbf{C} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 2 \end{pmatrix}$$

$$\therefore \quad \mathbf{B}^2 = (\mathbf{PCP}^{-1})(\mathbf{PCP}^{-1})$$

$$\therefore \quad \mathbf{B} = \mathbf{PCP}^{-1}$$

$$\text{If} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\tfrac{1}{3}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 2 \end{pmatrix}$$

$$\text{If} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$$

$$\text{If} \quad \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 2 & 2 \end{pmatrix}$$

$$\text{If} \quad \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ -\frac{2}{3} & -2 \end{pmatrix}$$

$\therefore$ the 4 solutions to $\mathbf{B}^2 = \mathbf{A}$ are

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -\frac{2}{3} & -2 \end{pmatrix}$$

7 **a** Consider $(\mathbf{A}^2 - (\lambda_1+\lambda_2)\mathbf{A} + \lambda_1\lambda_2\mathbf{I})\mathbf{x}_1$

$$\begin{aligned}
&= \mathbf{A}^2\mathbf{x}_1 - (\lambda_1+\lambda_2)\mathbf{A}\mathbf{x}_1 + \lambda_1\lambda_2\mathbf{I}\mathbf{x}_1 \\
&= \lambda_1{}^2\mathbf{x}_1 - (\lambda_1+\lambda_2)\lambda_1\mathbf{x}_1 + \lambda_1\lambda_2\mathbf{x}_1 \\
&= (\lambda_1{}^2 - \lambda_1{}^2 - \lambda_1\lambda_2 + \lambda_1\lambda_2)\mathbf{x}_1 \\
&= 0\mathbf{x}_1 \\
&= \mathbf{0}
\end{aligned}$$

Also, $(\mathbf{A}^2 - (\lambda_1+\lambda_2)\mathbf{A} + \lambda_1\lambda_2\mathbf{I})\mathbf{x}_2$

$$\begin{aligned}
&= \mathbf{A}^2\mathbf{x}_2 - (\lambda_1+\lambda_2)\mathbf{A}\mathbf{x}_2 + \lambda_1\lambda_2\mathbf{I}\mathbf{x}_2 \\
&= \lambda_2{}^2\mathbf{x}_2 - (\lambda_1+\lambda_2)\lambda_2\mathbf{x}_2 + \lambda_1\lambda_2\mathbf{x}_2 \\
&= (\lambda_2{}^2 - \lambda_1\lambda_2 - \lambda_2{}^2 + \lambda_1\lambda_2)\mathbf{x}_2 \\
&= 0\mathbf{x}_2 \\
&= \mathbf{0}
\end{aligned}$$

$$\therefore \quad \mathbf{A}^2 - (\lambda_1+\lambda_2)\mathbf{A} + \lambda_1\lambda_2\mathbf{I} = \mathbf{O}$$

b **i** $\mathbf{A}^2 = (\lambda_1+\lambda_2)\mathbf{A} - \lambda_1\lambda_2\mathbf{I}$

$$\begin{aligned}
\mathbf{A}^3 &= \mathbf{A}^2\mathbf{A} \\
&= [(\lambda_1+\lambda_2)\mathbf{A} - \lambda_1\lambda_2\mathbf{I}]\mathbf{A} \\
&= (\lambda_1+\lambda_2)\mathbf{A}^2 - \lambda_1\lambda_2\mathbf{IA} \\
&= (\lambda_1+\lambda_2)[(\lambda_1+\lambda_2)\mathbf{A} - \lambda_1\lambda_2\mathbf{I}] - \lambda_1\lambda_2\mathbf{A} \\
&= (\lambda_1+\lambda_2)^2\mathbf{A} - (\lambda_1+\lambda_2)\lambda_1\lambda_2\mathbf{I} - \lambda_1\lambda_2\mathbf{A} \\
&= [(\lambda_1+\lambda_2)^2 - \lambda_1\lambda_2]\mathbf{A} - \lambda_1\lambda_2(\lambda_1+\lambda_2)\mathbf{I}
\end{aligned}$$

ii

$$\begin{aligned}
\mathbf{A}^2 &= (\lambda_1+\lambda_2)\mathbf{A} - \lambda_1\lambda_2\mathbf{I} \\
\therefore \quad \mathbf{A}^2\mathbf{A}^{-1} &= [(\lambda_1+\lambda_2)\mathbf{A} - \lambda_1\lambda_2\mathbf{I}]\mathbf{A}^{-1} \\
\therefore \quad \mathbf{A} &= (\lambda_1+\lambda_2)\mathbf{I} - \lambda_1\lambda_2\mathbf{A}^{-1} \\
\therefore \quad \lambda_1\lambda_2\mathbf{A}^{-1} &= -\mathbf{A} + (\lambda_1+\lambda_2)\mathbf{I} \\
\therefore \quad \mathbf{A}^{-1} &= \frac{-1}{\lambda_1\lambda_2}\mathbf{A} + \frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}\mathbf{I}
\end{aligned}$$

8

$$\begin{aligned}
&|\,\lambda\mathbf{I} - \mathbf{B}\,| \\
&= \left|\,\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}\,\right| \\
&= \left|\,\lambda\mathbf{P}^{-1}\mathbf{IP} - \mathbf{P}^{-1}\mathbf{AP}\,\right| \\
&= \left|\,\mathbf{P}^{-1}\lambda\mathbf{IP} - \mathbf{P}^{-1}\mathbf{AP}\,\right| \\
&= \left|\,\mathbf{P}^{-1}(\lambda\mathbf{I} - \mathbf{A})\mathbf{P}\,\right| \\
&= \left|\,\mathbf{P}^{-1}\,\right|\,|\lambda\mathbf{I} - \mathbf{A}|\,|\,\mathbf{P}\,| \\
&= \frac{1}{\cancelto{1}{|\mathbf{P}|}}\,|\lambda\mathbf{I} - \mathbf{A}|\,\cancelto{1}{|\mathbf{P}|} \\
&= |\,\lambda\mathbf{I} - \mathbf{A}\,|
\end{aligned}$$

A and **B** have the same characteristic polynomial.

$\therefore$ they have the same eigenvalues.

9 **a** Let $\mathbf{A} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$

$$\text{If} \quad |\,\lambda\mathbf{I} - \mathbf{A}\,| = 0 \quad \text{then} \quad \begin{vmatrix} \lambda - a & -b & -c \\ 0 & \lambda - d & -e \\ 0 & 0 & \lambda - f \end{vmatrix} = 0$$

$$\therefore \quad (\lambda - a)\begin{vmatrix} \lambda - d & -e \\ 0 & \lambda - f \end{vmatrix} + b\begin{vmatrix} 0 & -e \\ 0 & \lambda - f \end{vmatrix} - c\begin{vmatrix} 0 & \lambda - d \\ 0 & 0 \end{vmatrix} = 0$$

$$\therefore \quad (\lambda - a)(\lambda - d)(\lambda - f) + 0 + 0 = 0$$

$$\therefore \quad (\lambda - a)(\lambda - d)(\lambda - f) = 0$$

$$\therefore \quad \lambda = a,\ d,\ f$$

$\therefore$ the eigenvalues are the elements of the main diagonal.

b Let $\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$

$$\text{If} \quad |\,\lambda\mathbf{I} - \mathbf{A}\,| = 0 \quad \text{then} \quad \begin{vmatrix} \lambda - a & 0 & 0 \\ -b & \lambda - c & 0 \\ -d & -e & \lambda - f \end{vmatrix} = 0$$

$$\therefore \quad (\lambda - a)\begin{vmatrix} \lambda - c & 0 \\ -e & \lambda - f \end{vmatrix} + 0 + 0 = 0$$

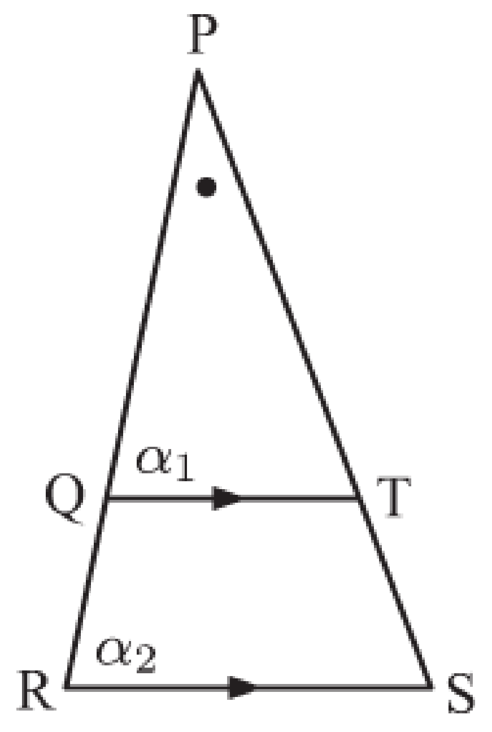
$$\therefore \quad (\lambda - a)(\lambda - c)(\lambda - f) = 0$$

$$\therefore \quad \lambda = a,\ c,\ f$$

$\therefore$ yes, the eigenvalues of a 3×3 lower triangular matrix are the elements of its main diagonal.

EXERCISE 2A

1 **a**



i $\widehat{QPT} = \widehat{RPS}$

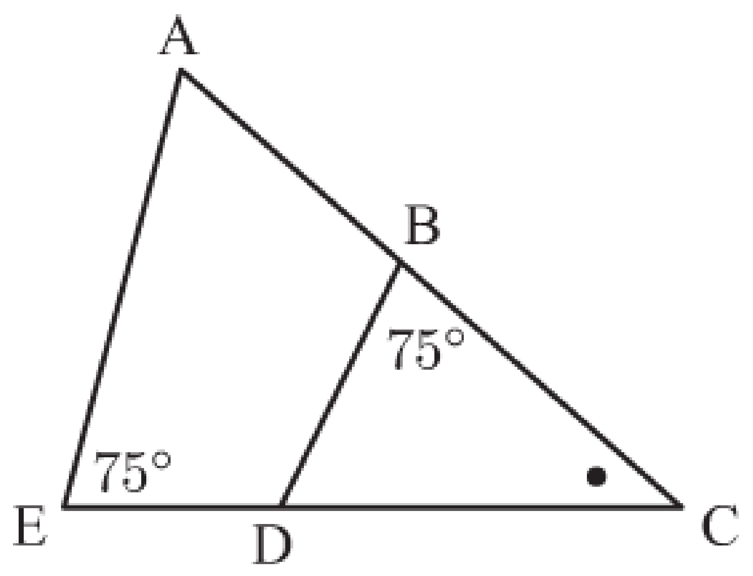
and $\alpha_1 = \alpha_2$

{equal corresponding angles}

$\therefore$ $\triangle$s PQT and PRS are equiangular, and therefore similar.

ii $\dfrac{PQ}{PR} = \dfrac{QT}{RS} = \dfrac{PT}{PS}$

b



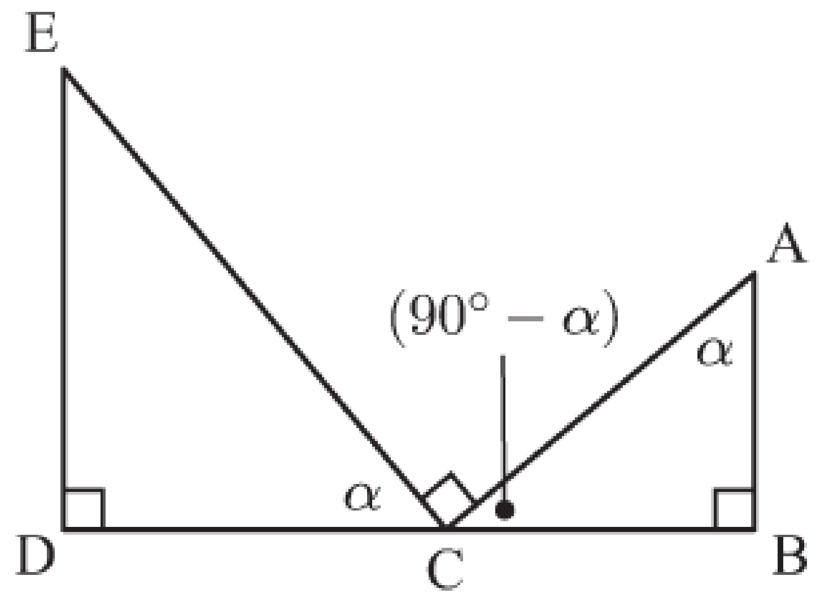
i $\widehat{ACE} = \widehat{DCB}$

and $\widehat{AEC} = \widehat{DBC}$ {given}

$\therefore$ $\triangle$s AEC and DBC are equiangular, and therefore similar.

ii $\dfrac{AC}{DC} = \dfrac{AE}{DB} = \dfrac{EC}{BC}$

c



i Let $\widehat{BAC} = \alpha$

$\therefore$ $\widehat{ACB} = 90° - \alpha$ {angles of a triangle}

$\therefore$ $\widehat{DCE} = \alpha$ {angles on a line}

$\therefore$ $\widehat{BAC} = \widehat{DCE}$

and $\widehat{ABC} = \widehat{CDE}$ {given}

$\therefore$ $\triangle$s ABC and CDE are equiangular, and therefore similar.

ii $\dfrac{AB}{CD} = \dfrac{AC}{CE} = \dfrac{BC}{DE}$

2 $\dfrac{PT}{TS} = \dfrac{PQ}{QR}$ {parallel lines within a triangle theorem}

$\therefore$ $\dfrac{PT}{2 \text{ cm}} = \dfrac{5 \text{ cm}}{3 \text{ cm}}$

$\therefore$ $PT = \frac{10}{3}$ cm

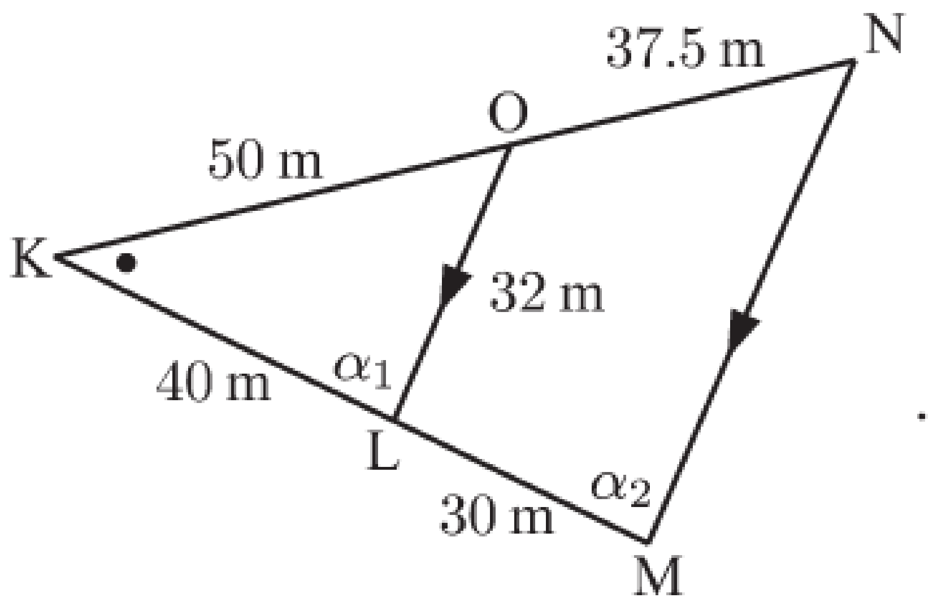
$\therefore$ $PT \approx 3.33$ cm

3 **a** $\dfrac{KL}{LM} = \dfrac{40 \text{ m}}{30 \text{ m}} = \dfrac{4}{3}$ and $\dfrac{KO}{ON} = \dfrac{50 \text{ m}}{37.5 \text{ m}} = \dfrac{4}{3}$

$\therefore$ $\dfrac{KL}{LM} = \dfrac{KO}{ON}$

$\therefore$ [OL] $\|$ [NM] {converse to parallel lines within a triangle theorem}

b



$\widehat{LKO} = \widehat{MKN}$

and $\alpha_1 = \alpha_2$

{equal corresponding angles}

$\therefore$ $\triangle$s KLO and KMN are equiangular, and therefore similar.

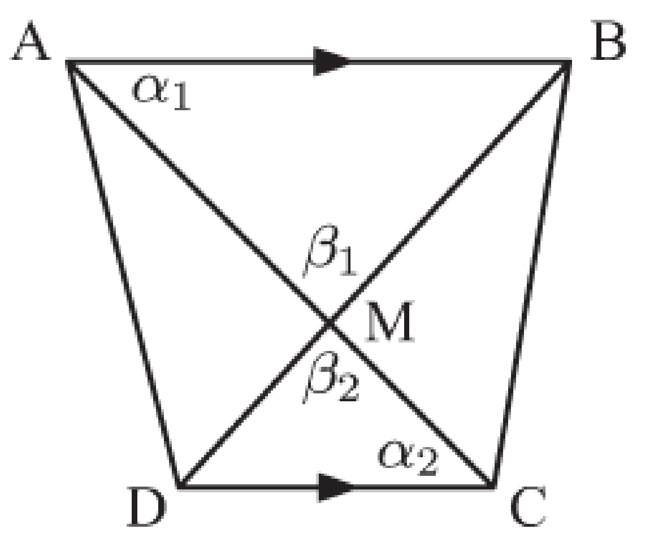
$\therefore$ $\dfrac{MN}{LO} = \dfrac{MK}{LK}$

$\therefore$ $\dfrac{MN}{32 \text{ m}} = \dfrac{70 \text{ m}}{40 \text{ m}}$

$\therefore$ $MN = \dfrac{32 \times 70}{40}$ m

$\therefore$ $MN = 56$ m

4



$\alpha_1 = \alpha_2$

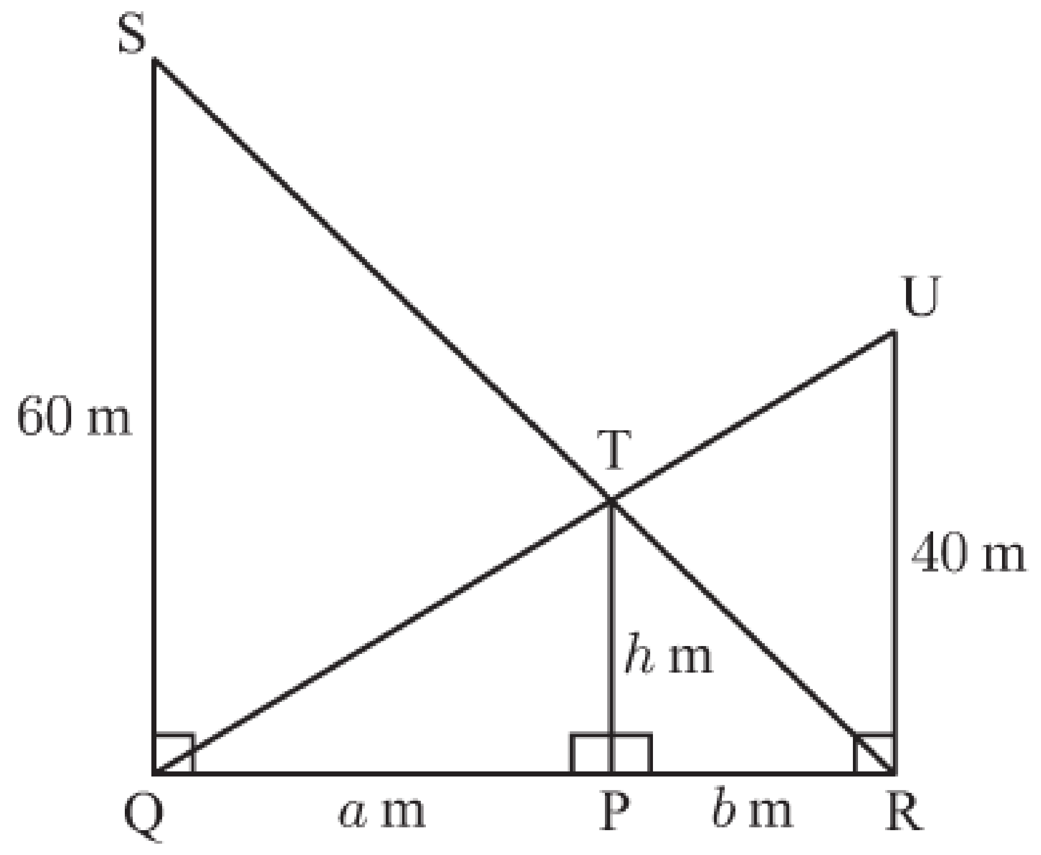
{equal alternate angles}

and $\beta_1 = \beta_2$

{vertically opposite angles}

$\therefore$ $\triangle$s ABM and CDM are equiangular, and therefore similar.

5



Let $PT = h$ m, $QP = a$ m, and $PR = b$ m.

$\widehat{TQP} = \widehat{UQR}$

and $\widehat{TPQ} = \widehat{URQ}$ {given}

$\therefore$ $\triangle$s TQP and UQR are equiangular, and therefore similar.

$\therefore$ $\dfrac{h}{40} = \dfrac{a}{a+b}$ .... (1)

Likewise, $\triangle$s TRP and SRQ are equiangular and therefore similar.

$\therefore$ $\dfrac{h}{60} = \dfrac{b}{a+b}$ .... (2)

Adding (1) and (2), $\dfrac{h}{40} + \dfrac{h}{60} = \dfrac{a+b}{a+b}$

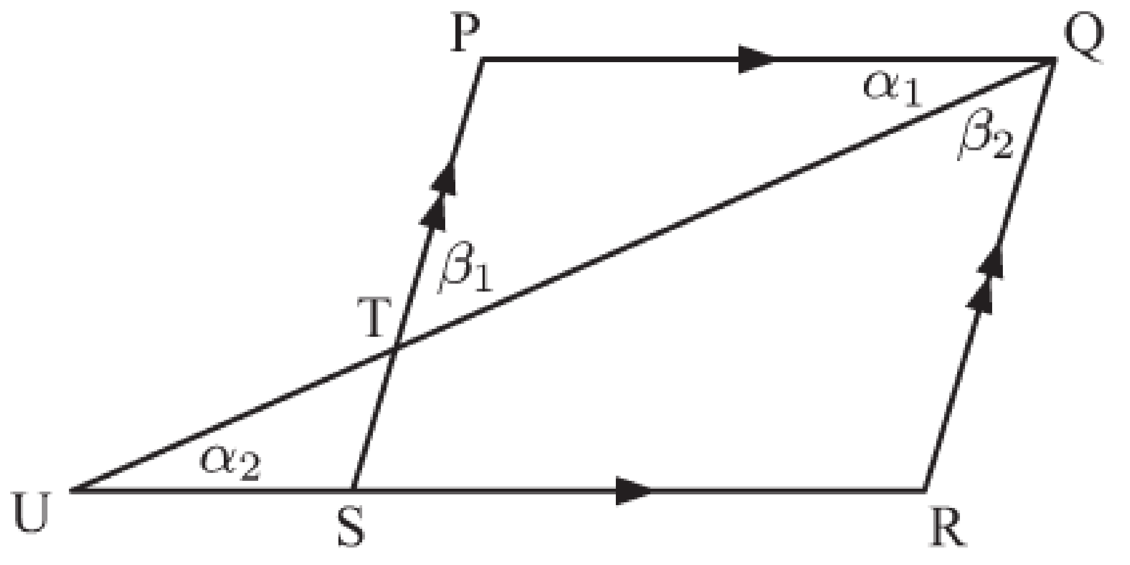
$\therefore$ $\dfrac{3h + 2h}{120} = 1$

$\therefore$ $5h = 120$

$\therefore$ $h = 24$

$\therefore$ the tree is 24 m high.

6



Consider $\triangle$s PQT, RUQ:

$\alpha_1 = \alpha_2$ {equal alternate angles}

and $\beta_1 = \beta_2$ {equal alternate angles}

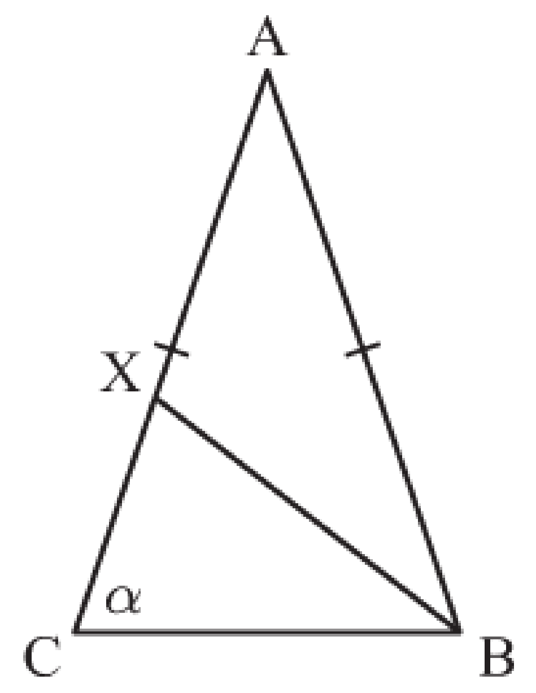
$\therefore$ $\triangle$s PQT, RUQ are equiangular, and therefore similar.

$\therefore$ $\dfrac{QT}{UQ} = \dfrac{PT}{RQ}$

$\therefore$ $\dfrac{QT}{QU} = \dfrac{PT}{PS}$ {RQ = PS, parallelogram theorem}

$\therefore$ QT.PS = QU.PT

7



We are given $CB^2 = CX.CA$ and $AB = AC$.

Now $\dfrac{CB}{CA} = \dfrac{CX}{CB}$ .... (1)

Consider $\triangle$s CBA, CXB.

These $\triangle$s have two sides in proportion and share an included angle of α at C.

Thus $\triangle$s CBA, CXB are similar.

Consequently $\triangle$CXB is also isosceles.

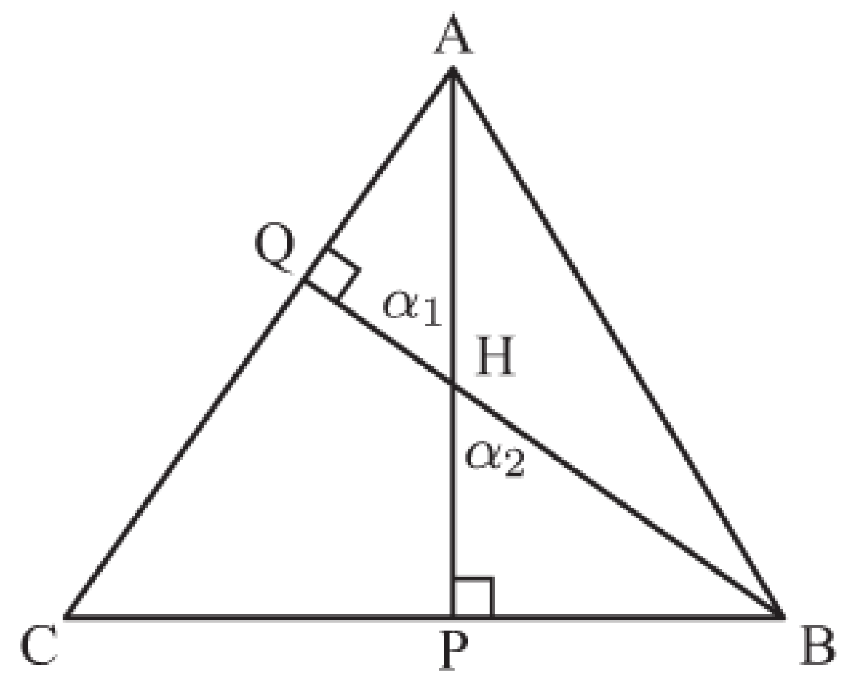
As the angles at B and C are equal,

{$\triangle$ABC is isosceles with $AB = AC$}

the angles at C and X in $\triangle$CXB are equal.

$\therefore$ $BX = BC$.

8



Consider $\triangle$s AHQ, BHP:

$\alpha_1 = \alpha_2$ {vertically opposite angles}

and $\widehat{AQH} = \widehat{BPH} = 90°$ {given}

$\therefore$ $\triangle$s AHQ and BHP are equiangular, and therefore similar.

$\therefore$ $\dfrac{AH}{BH} = \dfrac{HQ}{HP}$

$\therefore$ $AH.HP = BH.HQ$

9 We are given that $\dfrac{AX}{AY} = \dfrac{BX}{CY}$

$\therefore$ $AX.CY = AY.BX$

$\therefore$ $AX.CY + AX.AY = AY.BX + AX.AY$

{adding $AX.AY$ to both sides}

$\therefore$ $AX(CY + AY) = AY(AX + BX)$

$\therefore$ $AX.AC = AY.AB$

$\therefore$ $\dfrac{AX}{AB} = \dfrac{AY}{AC}$

$\therefore$ in $\triangle$s AXY and ABC, two sides are in the same ratio and the included angle at A is common to both.

$\therefore$ $\triangle$s AXY and ABC are similar

$\therefore$ $\widehat{AXY} = \widehat{ABC}$ {equal corresponding angles}

$\therefore$ $[XY] \parallel [BC]$ {converse of corresponding angles}

EXERCISE 2B

1 **a** Yes, using SAS.

b No, as the equal angles are not the included angles.

c Yes, using SSS.

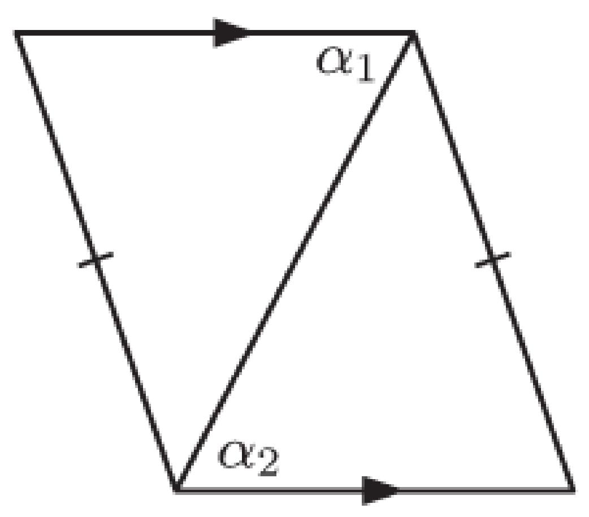
d Yes, using SAS {the altitude is common}.

e No, as the equal angles are not the included angles.

f Yes, using RHS.

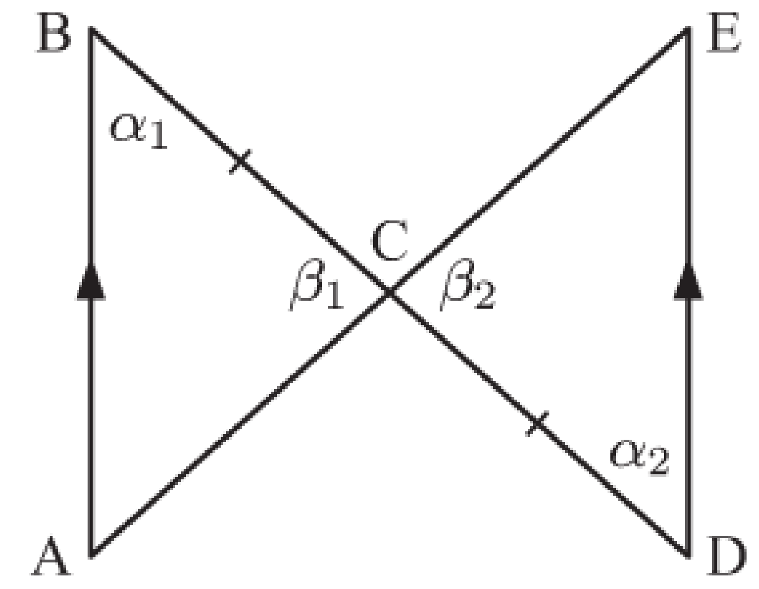
g Yes, using AAcorS.

h



No, as the equal angles are not the included angles.

2



a We observe that:

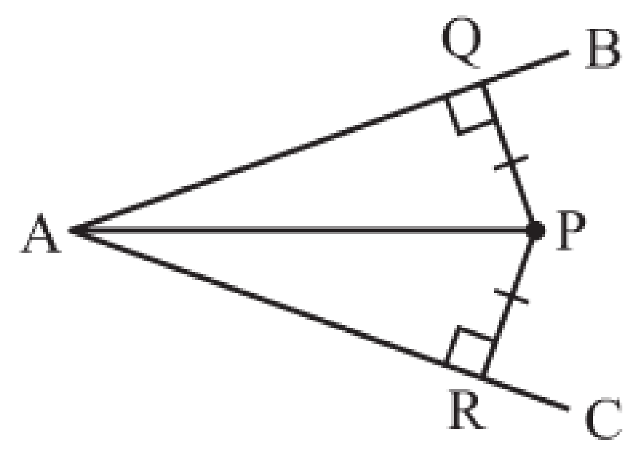
- $\alpha_1 = \alpha_2$ {equal alternate angles}
- $\beta_1 = \beta_2$ {vertically opposite angles}
- $BC = DC$ {given}

$\therefore$ $\triangle$s ABC and EDC are congruent {AAcorS}.

b **i** Consequently, $AC = CE = 6$ cm

ii $\widehat{DEC} = \widehat{BAC} = 42°$

3



From P we draw perpendiculars to [AB] and [AC] meeting at Q and R respectively. Join [AP].

We observe that:

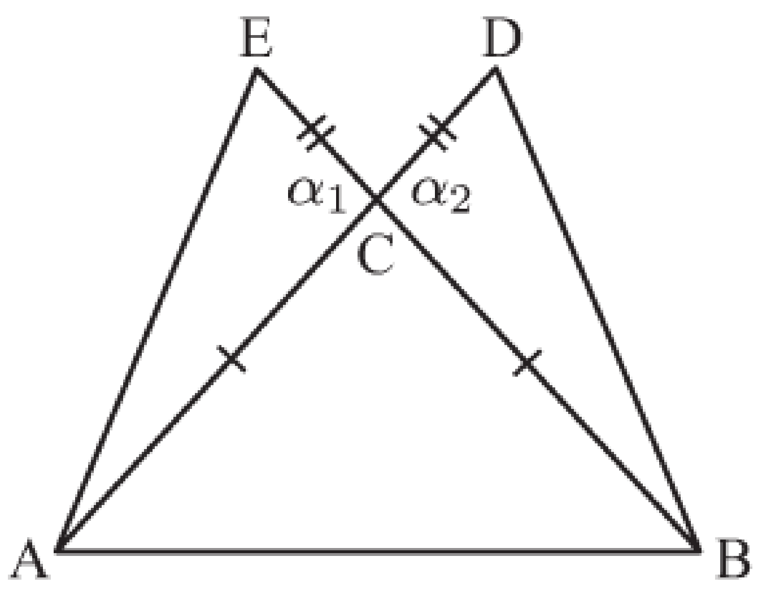
- $\widehat{AQP} = \widehat{ARP} = 90°$
- $PQ = PR$ {given}
- [AP] is common

$\therefore$ $\triangle$s AQP and ARP are congruent {RHS}.

Consequently, $\widehat{QAP} = \widehat{RAP}$.

That is, [AP] bisects $\widehat{BAC}$.

4



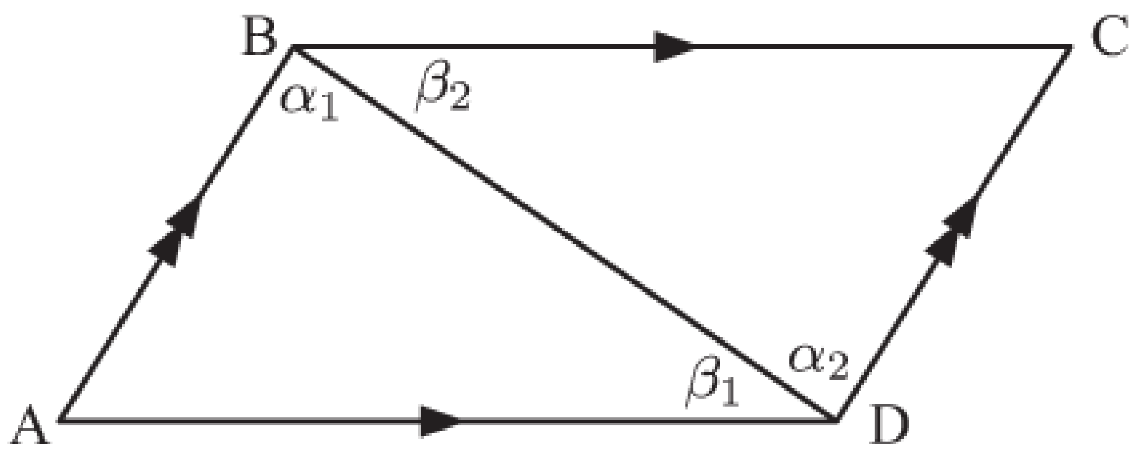
We observe that:

- $\alpha_1 = \alpha_2$ {vertically opposite}
- $EC = DC$ {given}
- $AC = BC$ {given}

$\therefore$ $\triangle$s AEC and BDC are congruent {SAS}.

Consequently, $AE = BD$.

5



We draw diagonal [DB].

We observe that:

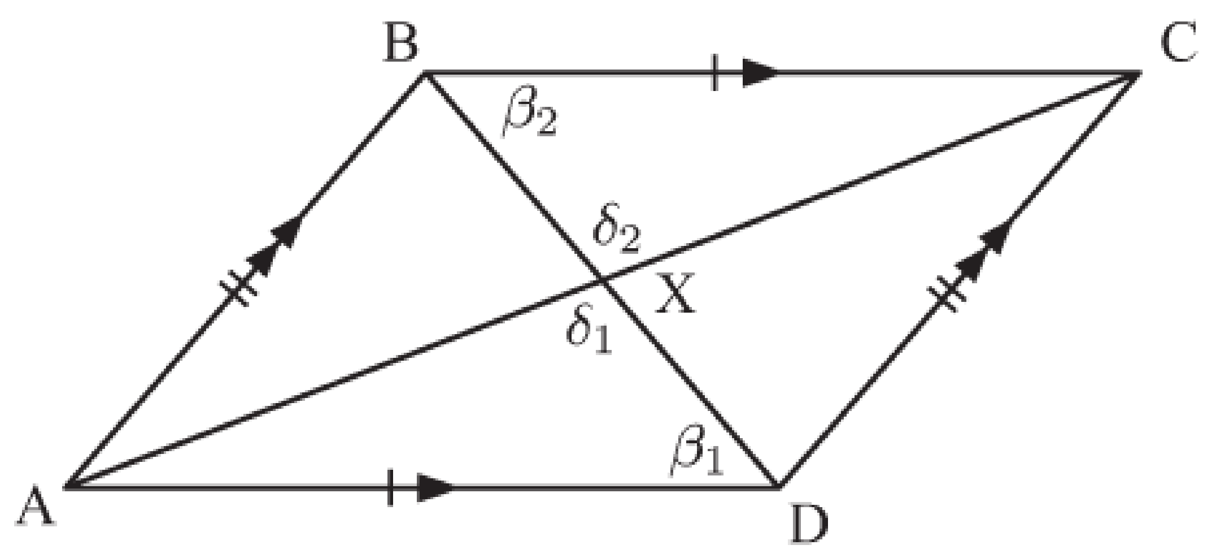
- $\alpha_1 = \alpha_2$ {equal alternate angles}
- $\beta_1 = \beta_2$ {equal alternate angles}
- [BD] is common

$\therefore$ $\triangle$s ABD and CDB are congruent {AAcorS}.

a Consequently, $AB = CD$ and $AD = CB$ thus opposite sides are equal in length.

b Also DAB = BCD (and $\widehat{ADC} = \widehat{ABC} = \alpha + \beta$) thus opposite angles are equal.

c



In $\triangle$s AXD, CXB, we observe that:

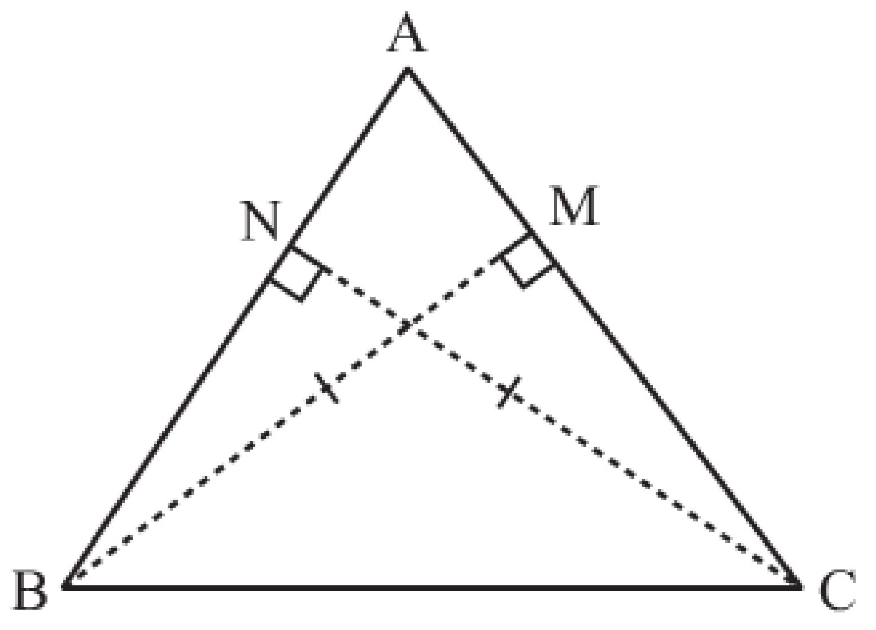
- AD = CB
- $\beta_1 = \beta_2$ {equal alternate angles}
- $\delta_1 = \delta_2$ {vertically opposite angles}

$\therefore$ $\triangle$s AXD and CXB are congruent {AAcorS}.

Consequently, AX = CX and DX = BX.

$\therefore$ the diagonals bisect each other.

6



a In $\triangle$s BCM, CBN, we observe that:

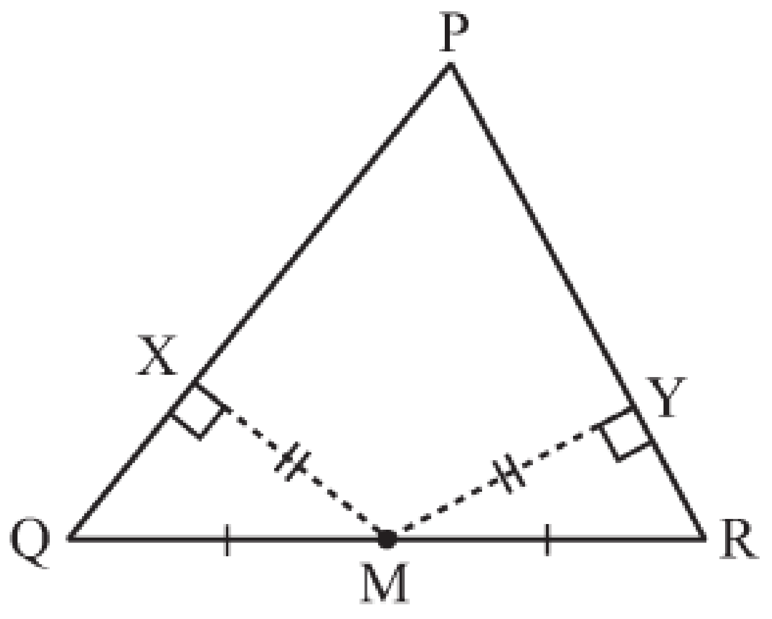
- BM = CN {given}
- $\widehat{BMC} = \widehat{CNB} = 90°$ {given}
- [BC] is common

$\therefore$ $\triangle$s BCM and CBN are congruent {RHS}.

b Consequently, $\widehat{NBC} = \widehat{MCB}$ which are the base angles of $\triangle$ABC.

Thus, $\triangle$ABC is isosceles.

7



a In $\triangle$s MQX, MRY, we observe that:

- QM = RM {given}
- MX = MY {given}
- $\widehat{QXM} = \widehat{RYM} = 90°$ {given}

$\therefore$ $\triangle$s MQX and MRY are congruent {RHS}.

b Consequently, $\widehat{XQM} = \widehat{YRM}$

$\therefore$ $\widehat{PQR} = \widehat{PRQ}$

$\therefore$ $\triangle$PQR is isosceles {equal base angles}

EXERCISE 2C

1 From Euclid's theorem,

a $\text{QR}^2 = \text{QS.QP}$

$\therefore\ 5^2 = \text{QS} \times 13$

$\therefore\ \text{QS} = \frac{25}{13}$

$\therefore\ \text{QS} \approx 1.92$ cm

b $\text{RS}^2 = \text{QS.SP}$

$\therefore\ \text{RS}^2 = \frac{25}{13} \times (13 - \frac{25}{13})$

$\therefore\ \text{RS} = \sqrt{\frac{25}{13} \times \frac{144}{13}}$

$\therefore\ \text{RS} = \frac{60}{13}$

$\therefore\ \text{RS} \approx 4.62$ cm

(*Check*: Equating areas, $\frac{1}{2} \times 13 \times \text{RS} = \frac{1}{2} \times 5 \times 12$

$\therefore\ \text{RS} = \frac{60}{13} \approx 4.62$)

2 From Euclid's theorem,

a $\text{KN}^2 = \text{LN.NM}$

$\therefore\ \text{KN}^2 = 3 \times 7$

$\therefore\ \text{KN} = \sqrt{21}$

$\therefore\ \text{KN} \approx 4.58$ m

b $\text{KL}^2 = \text{LN.LM}$

$\therefore\ \text{KL}^2 = 3 \times 10$

$\therefore\ \text{KL} = \sqrt{30}$

$\therefore\ \text{KL} \approx 5.48$ m

c $\text{KM}^2 = \text{NM.LM}$

$\therefore\ \text{KM}^2 = 7 \times 10$

$\therefore\ \text{KM} = \sqrt{70}$

$\therefore\ \text{KM} \approx 8.37$ m

3 **a** $\text{BC}^2 = 8^2 + 15^2$ {Pythagoras}

$\therefore\ \text{BC} = \sqrt{289} = 17$ cm

b From Euclid's theorem,

$\text{AC}^2 = \text{DC.BC}$

$\therefore\ 15^2 = \text{DC} \times 17$

$\therefore\ \text{DC} = \frac{225}{17} \approx 13.2$ cm

c From Euclid's theorem,

$\text{AD}^2 = \text{BD.DC}$

$\therefore\ \text{AD}^2 = (17 - \frac{225}{17}) \times \frac{225}{17}$

$\therefore\ \text{AD} = \sqrt{\frac{64}{17} \times \frac{225}{17}}$

$\therefore\ \text{AD} = \frac{120}{17} \approx 7.06$ cm

4 **a** In $\triangle$ABC, from Euclid's theorem,

$\text{BQ}^2 = \text{AQ.CQ}$

$\therefore\ 8^2 = (x+3)5$

$\therefore\ x + 3 = 12.8$

$\therefore\ x = 9.8$

b In $\triangle$ACD, from Euclid's theorem,

$\text{DP}^2 = \text{AP.PC}$

$\therefore\ y^2 = 3(x+5)$

$\therefore\ y^2 = 3 \times 14.8$

$\therefore\ y = \sqrt{44.4} \approx 6.66$

c From Euclid's theorem,

$$\begin{aligned}\text{AB} &= \sqrt{\text{AQ.AC}}\\ &= \sqrt{(3+x)(8+x)}\\ &= \sqrt{12.8 \times 17.8}\\ &= \sqrt{227.84}\end{aligned}$$

$$\begin{aligned}\text{BC} &= \sqrt{\text{CQ.CA}}\\ &= \sqrt{5(8+x)}\\ &= \sqrt{5 \times 17.8}\\ &= \sqrt{89}\end{aligned}$$

$$\begin{aligned}\text{DA} &= \sqrt{\text{AP.AC}}\\ &= \sqrt{3(8+x)}\\ &= \sqrt{3 \times 17.8}\\ &= \sqrt{53.4}\end{aligned}$$

$$\begin{aligned}\text{DC} &= \sqrt{\text{CP.CA}}\\ &= \sqrt{(x+5)(x+8)}\\ &= \sqrt{14.8 \times 17.8}\\ &= \sqrt{263.44}\end{aligned}$$

$\therefore$ perimeter $= \sqrt{227.84} + \sqrt{89} + \sqrt{53.4} + \sqrt{263.44}$

≈ 48.1 cm

5 By Euclid's theorem, $a^2 = p(p+q)$

and $b^2 = q(q+p)$

$$\begin{aligned}\therefore\ a^2 + b^2 &= p(p+q) + q(p+q)\\ &= (p+q)[p+q]\\ &= (p+q)^2\end{aligned}$$

Thus establishing Pythagoras' theorem.

6 a i $\quad \text{ACB} = \text{ECD}$

and $\widehat{\text{ABC}} = \widehat{\text{EDC}}$ {equal corresponding angles}

$\therefore$ $\triangle$s ABC and EDC are equiangular, and therefore similar.

$$\frac{\text{area of } \triangle\text{DEC}}{\text{area of } \triangle\text{ABC}} = \frac{\text{DC}^2}{\text{BC}^2} \quad \text{\{area comparison theorem\}}$$

$$= \frac{4^2}{7^2}$$

$\therefore$ area of $\triangle$DEC : area of $\triangle$ABC = 16 : 49

ii area of $\triangle$DEC : area of ABDE = 16 : (49 − 16)

$= 16 : 33$

b

$$\frac{\text{area of } \triangle\text{DEC}}{\text{area of ABDE}} = \frac{16}{33}$$

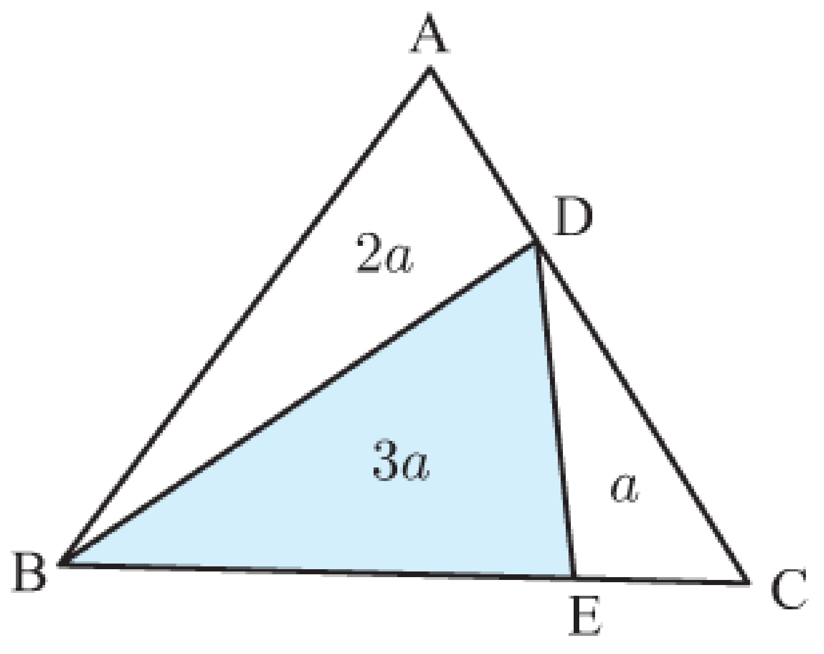
$$\therefore \frac{\text{area of } \triangle\text{DEC}}{6 \text{ cm}^2} = \frac{16}{33}$$

$\therefore$ area of $\triangle\text{DEC} = \frac{32}{11}$ cm^2

area of $\triangle\text{ABC} = 6 \text{ cm}^2 + \frac{32}{11} \text{ cm}^2$

$= \frac{98}{11} \text{ cm}^2 \approx 8.91 \text{ cm}^2$

7



Let area of $\triangle$DEC $= a$

$$\frac{\text{area of } \triangle\text{BED}}{\text{area of } \triangle\text{DEC}} = \frac{3}{1} \quad \text{\{equal altitudes\}}$$

$\therefore$ area of $\triangle$BED $= 3a$

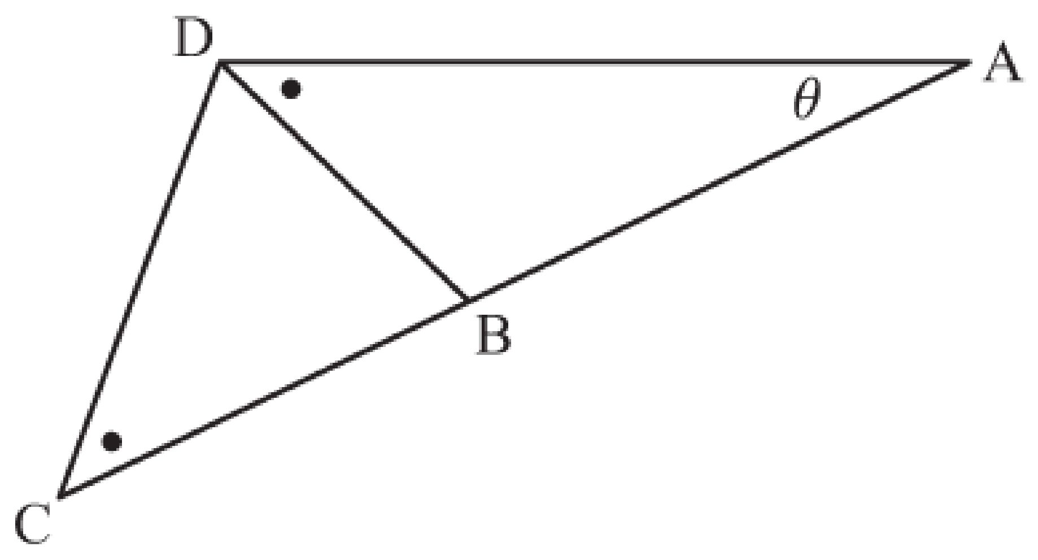
$\therefore$ area of $\triangle$BDC $= 4a$

$$\frac{\text{area of } \triangle\text{ABD}}{\text{area of } \triangle\text{BDC}} = \frac{1}{2} \quad \text{\{equal altitudes\}}$$

$\therefore$ area of $\triangle$ABD $= 2a$

$$\therefore \frac{\text{area of } \triangle\text{BDE}}{\text{area of } \triangle\text{ABC}} = \frac{3a}{6a} = \frac{1}{2}$$

8



Consider $\triangle$s ADB, ACD:

$\widehat{\text{ADB}} = \widehat{\text{ACD}}$ {given}

and $\widehat{\text{DAB}} = \widehat{\text{CAD}} = \theta$

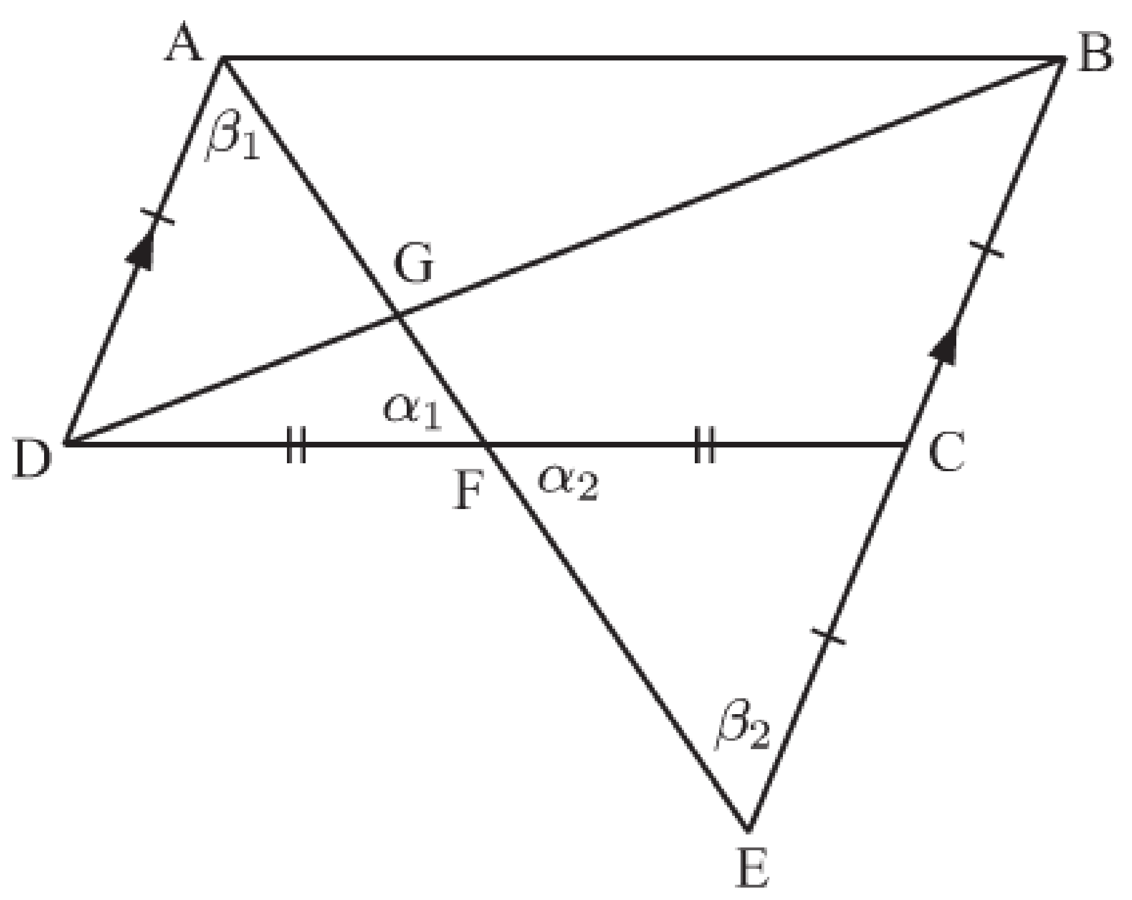
$\therefore$ $\triangle$s ADB and ACD are equiangular, and therefore similar.

$$\therefore \frac{\text{area of } \triangle\text{ADB}}{\text{area of } \triangle\text{ACD}} = \frac{\text{AD}^2}{\text{AC}^2} \quad \text{\{area comparison theorem\}}$$

$$\therefore \frac{\cancel{\tfrac{1}{2}}\,\cancel{\text{AD}}.\text{AB}\,\cancel{\sin\theta}}{\cancel{\tfrac{1}{2}}\,\cancel{\text{AD}}.\text{AC}\,\cancel{\sin\theta}} = \frac{\text{AD}^2}{\text{AC}^2}$$

$$\therefore \frac{\text{AD}^2}{\text{AC}^2} = \frac{\text{AB}}{\text{AC}}$$

9

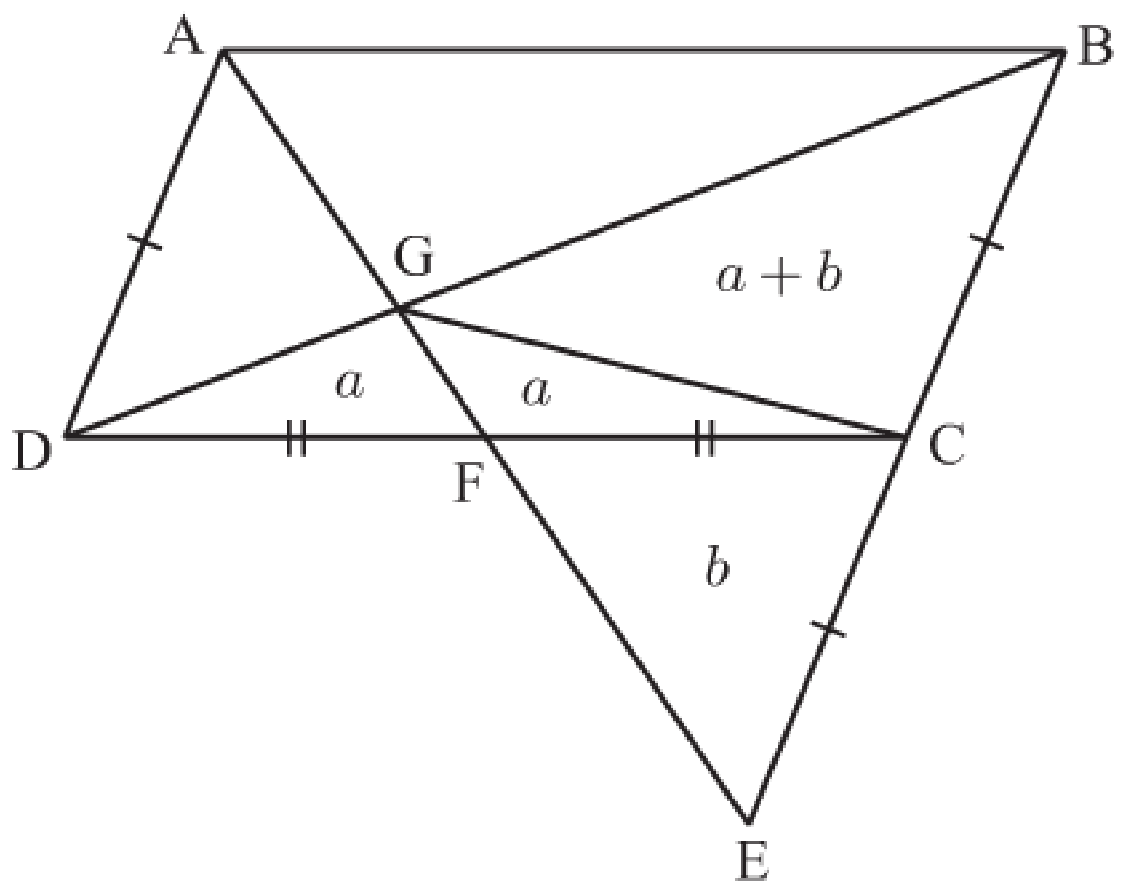


In $\triangle$s ADF, ECF, we observe that:

- $\alpha_1 = \alpha_2$ {vertically opposite angles}
- $\beta_1 = \beta_2$ {equal alternate angles}
- AD = BC = CE

$\therefore$ $\triangle$s ADF and ECF are congruent. {AAcorS}

Consequently DF = CF.



We join [GC].

Let area of $\triangle$DGF $= a$ and area of $\triangle$FCE $= b$.

$$\therefore \frac{\text{area of } \triangle\text{GFC}}{\text{area of } \triangle\text{DGF}} = \frac{1}{1} \quad \text{\{equal bases, equal altitudes\}}$$

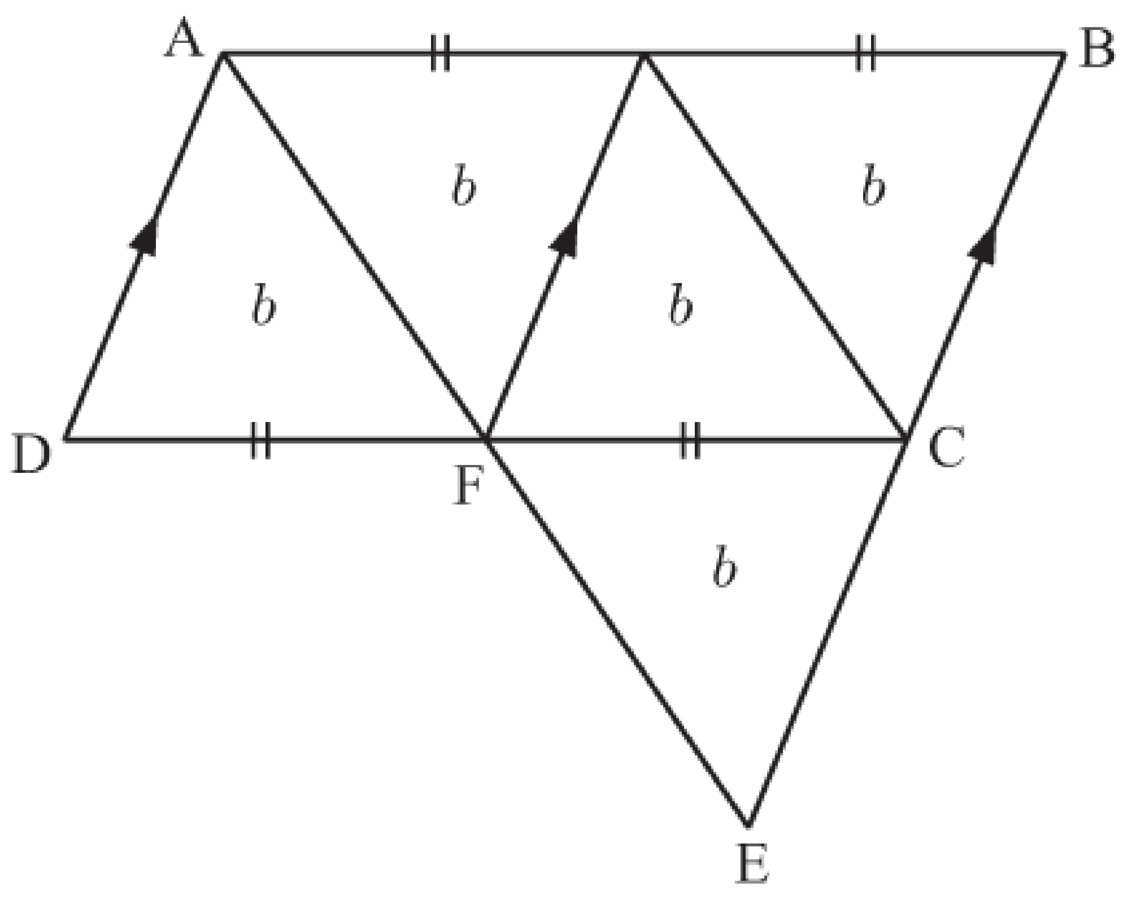
$\therefore$ area of $\triangle$GFC $= a$

$$\frac{\text{area of } \triangle\text{BCG}}{\text{area of } \triangle\text{ECG}} = \frac{1}{1} \quad \text{\{equal bases, equal altitudes\}}$$

$\therefore$ area of $\triangle$BCG $= a + b$

$\therefore$ area of $\triangle$BCD $= 3a + b$

$\therefore$ area of ABCD $= 6a + 2b$



But area of $\triangle$ADF = area of $\triangle$ECF $= b$ {congruent triangles}

$\therefore$ area of ABCD $= 4b$

Equating area of ABCD: $\quad 4b = 6a + 2b$

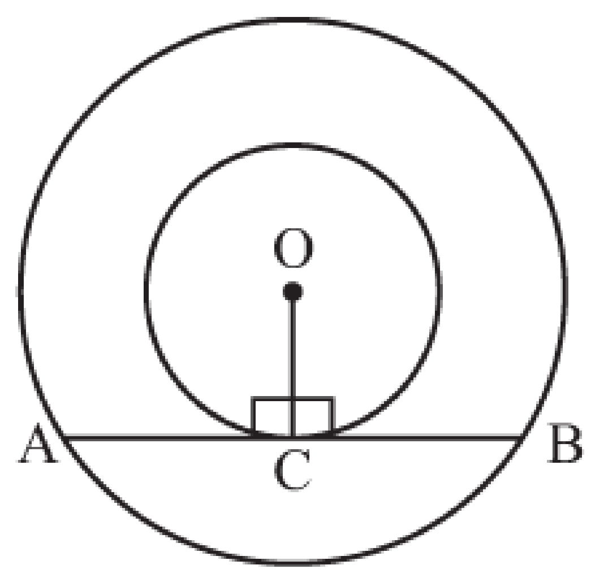
$\therefore \quad b = 3a$

$\therefore$ area of ABCD $= 12a$

$$\therefore \frac{\text{area of } \triangle\text{DGF}}{\text{area of ABCD}} = \frac{a}{12a} = \frac{1}{12}$$

EXERCISE 2D

1

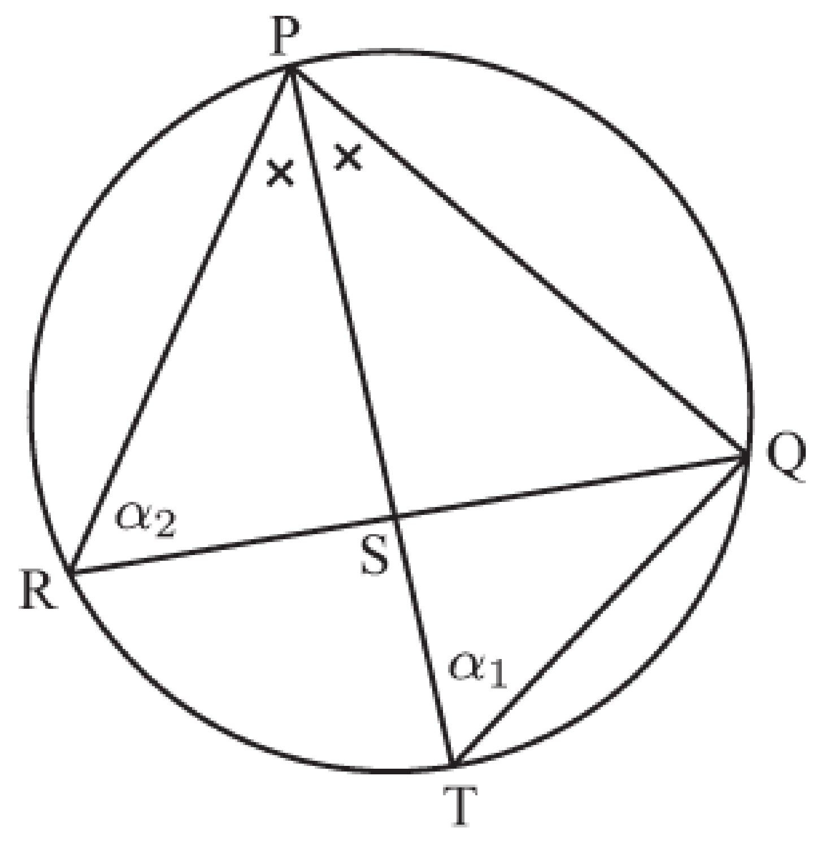


We join [OA], [OB], and [OC].

$\widehat{OCA} = \widehat{OCB} = 90°$ {radius-tangent theorem}

$\therefore$ AC = BC {chord of a circle theorem}

2



We join [QT].

In △s PQT, PSR:

$\widehat{QPT} = \widehat{SPR}$ {given}

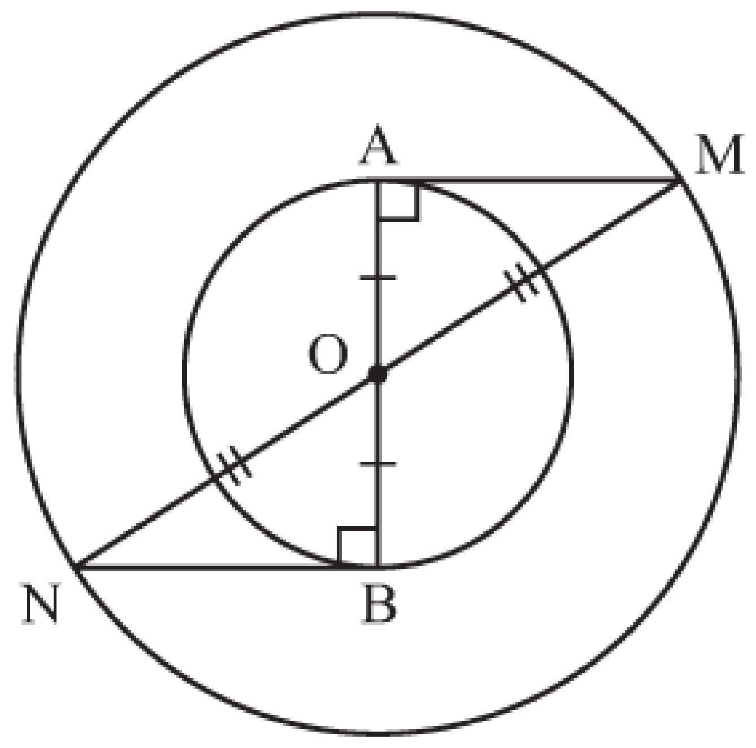
and $\alpha_1 = \alpha_2$ {angles subtended by the same arc}

$\therefore$ △s PQT and PSR are equiangular, and therefore similar.

Hence, $\dfrac{PQ}{PS} = \dfrac{PT}{PR}$

$\therefore$ PQ.PR = PS.PT

3



We join [OM] and [ON].

We observe that:

- $\widehat{OAM} = \widehat{OBN} = 90°$ {radius-tangent theorem}
- OA = OB {equal radii}
- OM = ON {equal radii}

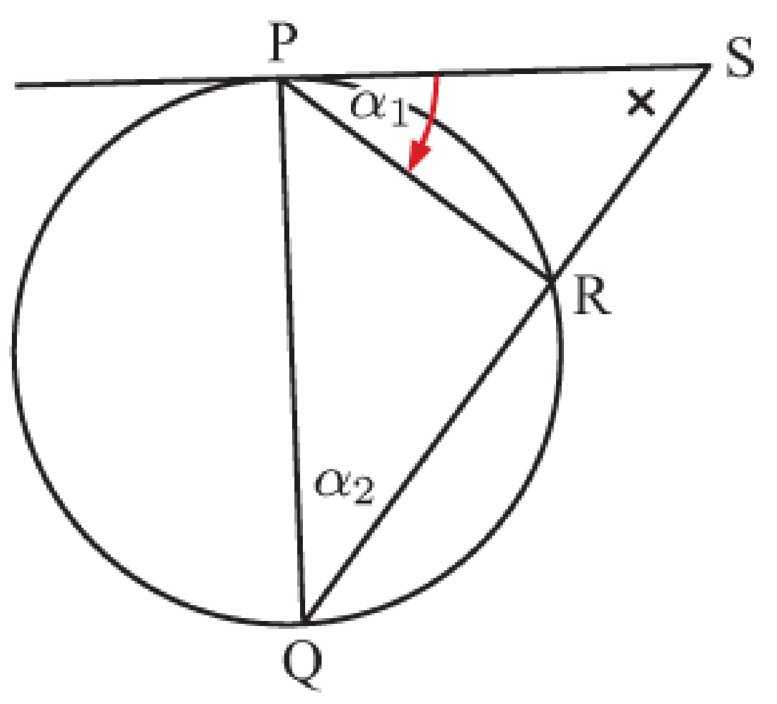
$\therefore$ △s OAM and OBN are congruent {RHS}.

Consequently, AM = BN.

Note: Also $\widehat{NOB} = \widehat{MOA}$

$\therefore$ MON is a straight line.

4



We join [PQ] and [PR].

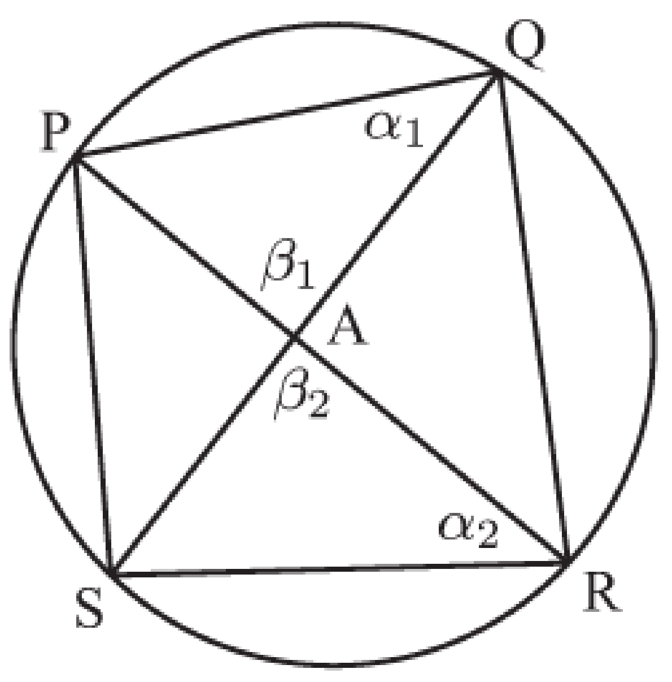
In △s SPQ, SRP:

$\widehat{PSQ} = \widehat{RSP}$

and $\alpha_1 = \alpha_2$ {angle between tangent and chord}

$\therefore$ △s SPQ and SRP are equiangular, and therefore similar.

5



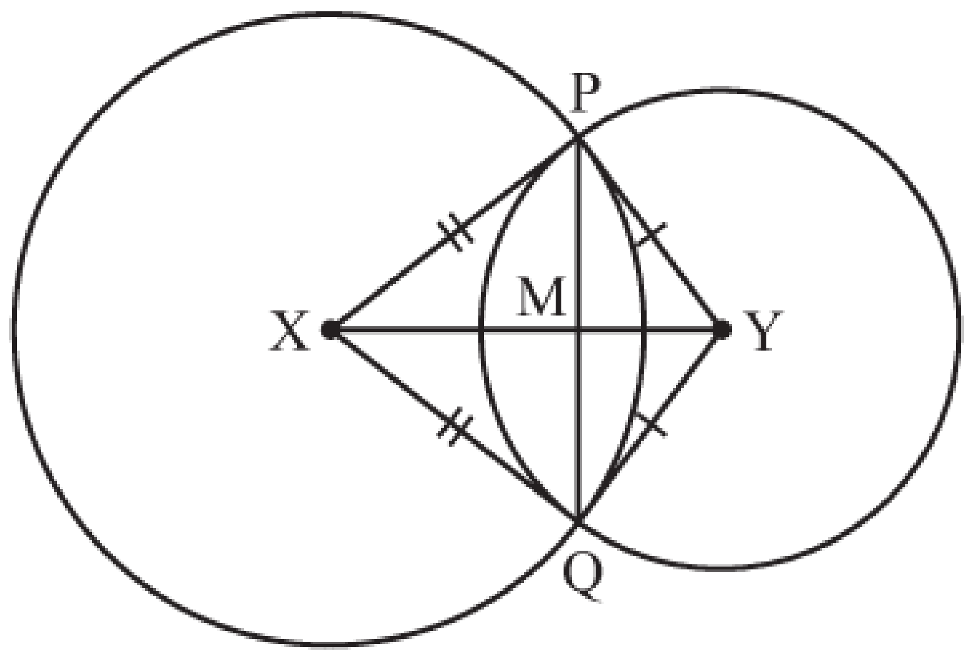
In △s PQA, SRA:

$\alpha_1 = \alpha_2$ {angles subtended by the same arc}

$\beta_1 = \beta_2$ {vertically opposite}

$\therefore$ △s PQA and SRA are equiangular, and therefore similar.

6



We join [XP], [XQ], [PY], and [QY].

Let [PQ] meet [XY] at M.

In △s XPY, XQY, we observe that:

- PX = QX {equal radii}
- PY = QY {equal radii}
- [XY] is common.

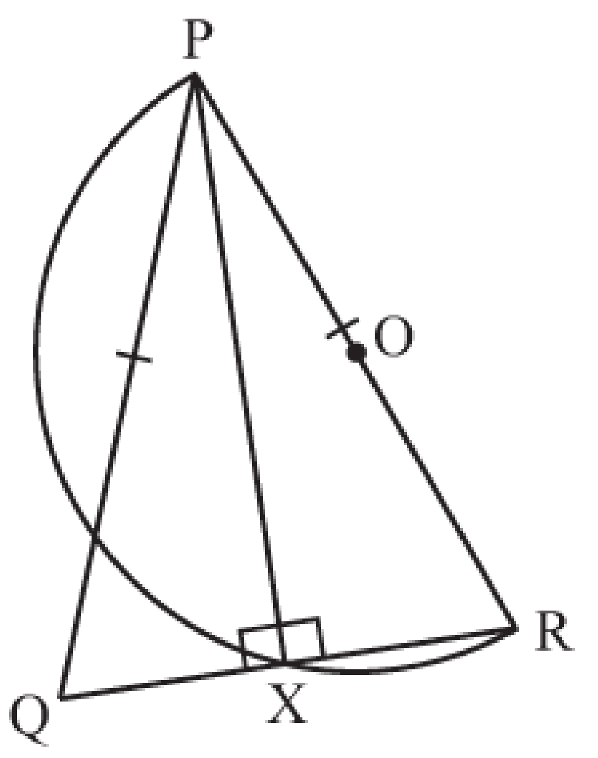
$\therefore$ △s XPY and XQY are congruent {SSS}.

Consequently, $\widehat{PXY} = \widehat{QXY}$.

$\therefore$ [XM] bisects the angle at X of isosceles △XPQ.

$\therefore$ [XY] bisects [PQ] at right angles. {converse of isosceles triangle theorem}

7



We join [PX].

$\widehat{PXR} = 90°$ {angle in semi-circle}

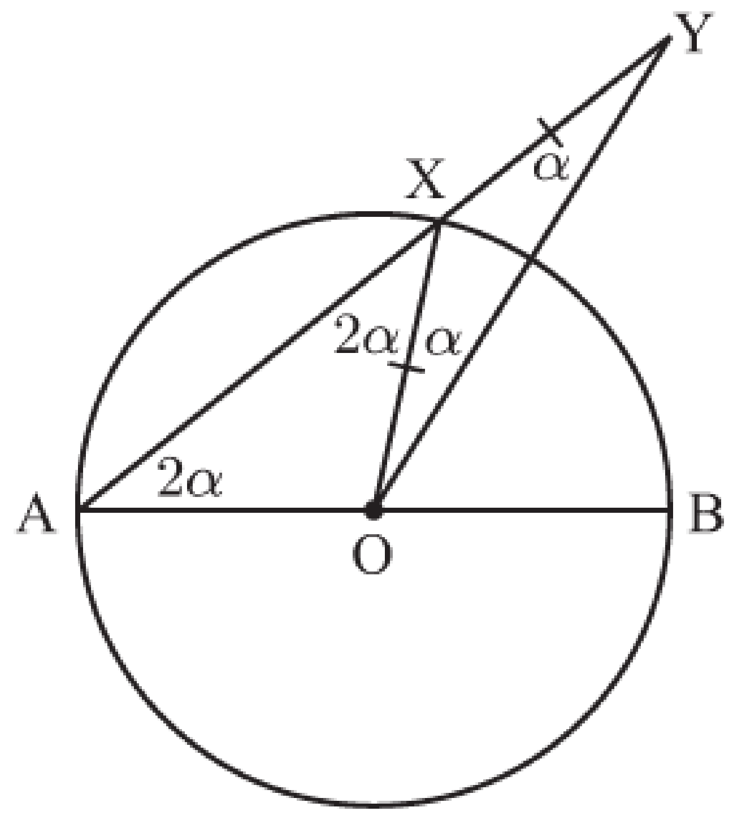
$\therefore$ $\widehat{PXQ} = 90°$

But △PQR is isosceles.

$\therefore$ the line from the apex, perpendicular to the base, bisects the base.

$\therefore$ X is the midpoint of [QR].

8



Let $\widehat{XOY} = \alpha$

$\therefore$ $\widehat{XYO} = \alpha$ {isosceles triangle}

$\therefore$ $\widehat{OXA} = 2\alpha$ {exterior angle of a triangle}

But OA = OX {equal radii}

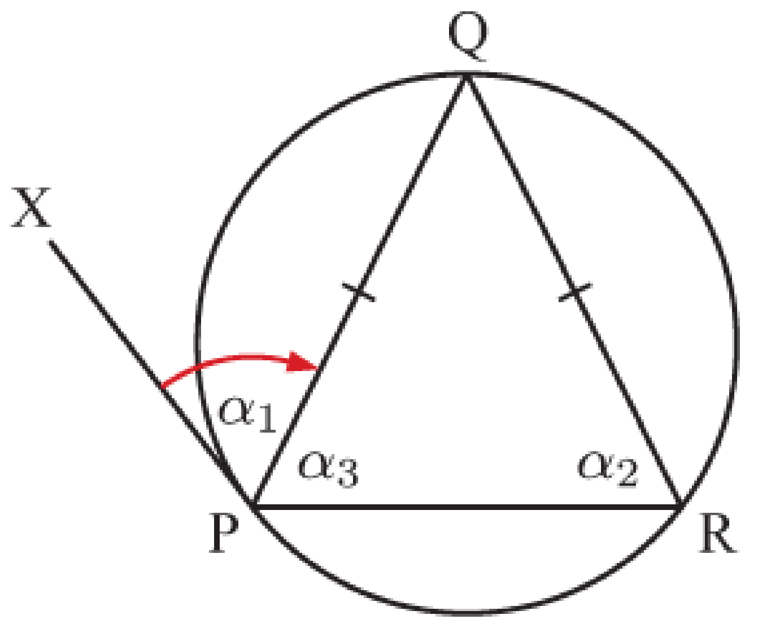
$\therefore$ $\widehat{OAX} = 2\alpha$ {isosceles triangle}

Thus $\widehat{XOB} = 4\alpha$ {angle at centre theorem}

$\therefore$ $\widehat{YOB} = 4\alpha - \alpha = 3\alpha$

$\therefore$ $\widehat{YOB} = 3(\widehat{XOY})$

9



Let $\widehat{XPQ} = \alpha_1$

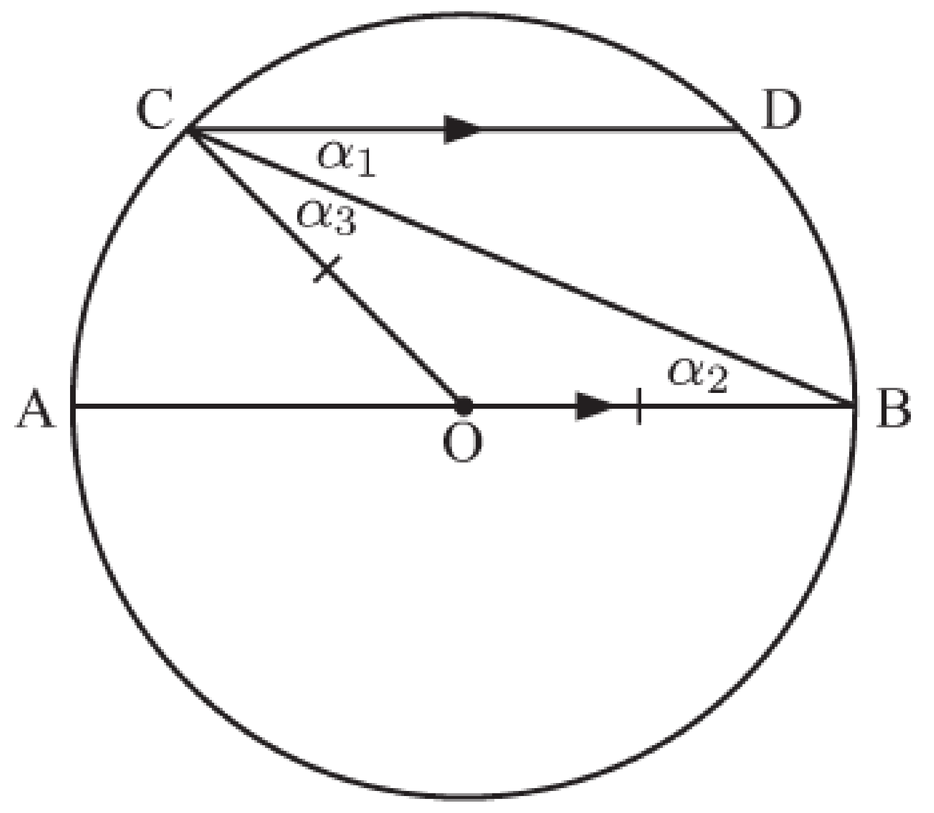
$\alpha_1 = \alpha_2$ {angle between tangent and chord}

$\therefore$ $\alpha_2 = \alpha_3$ {isosceles triangle}

$\therefore$ $\alpha_1 = \alpha_3$

$\therefore$ [QP] bisects $\widehat{XPR}$.

10



Let $\widehat{DCB} = \alpha_1$

$\alpha_1 = \alpha_2$ {equal alternate angles}

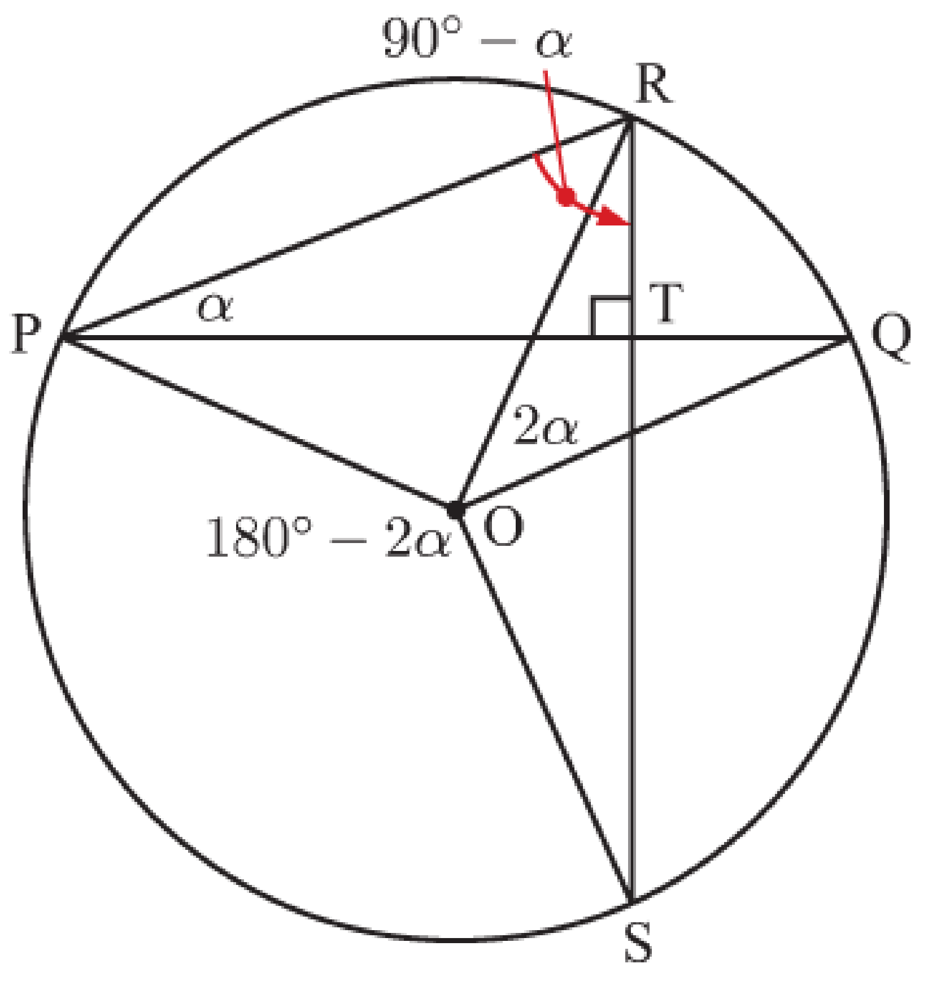
OB = OC {equal radii}

$\therefore$ $\alpha_2 = \alpha_3$ {isosceles triangle}

$\therefore$ $\alpha_1 = \alpha_3$

$\therefore$ [BC] bisects $\widehat{DCO}$.

11



$90° - \alpha$
R
P
α
T
Q
2α
$180° - 2\alpha$
O
S

We join [OP], [OQ], [OR], [OS], and [PR].

Let [PQ] meet [RS] at T.

Let $\widehat{RPQ} = \alpha$

$\therefore$ $\widehat{QOR} = 2\alpha$

{angle at centre theorem}

But $\widehat{PRT} = 90° - \alpha$

{angles of a triangle}

$\therefore$ $\widehat{POS} = 2(90° - \alpha)$

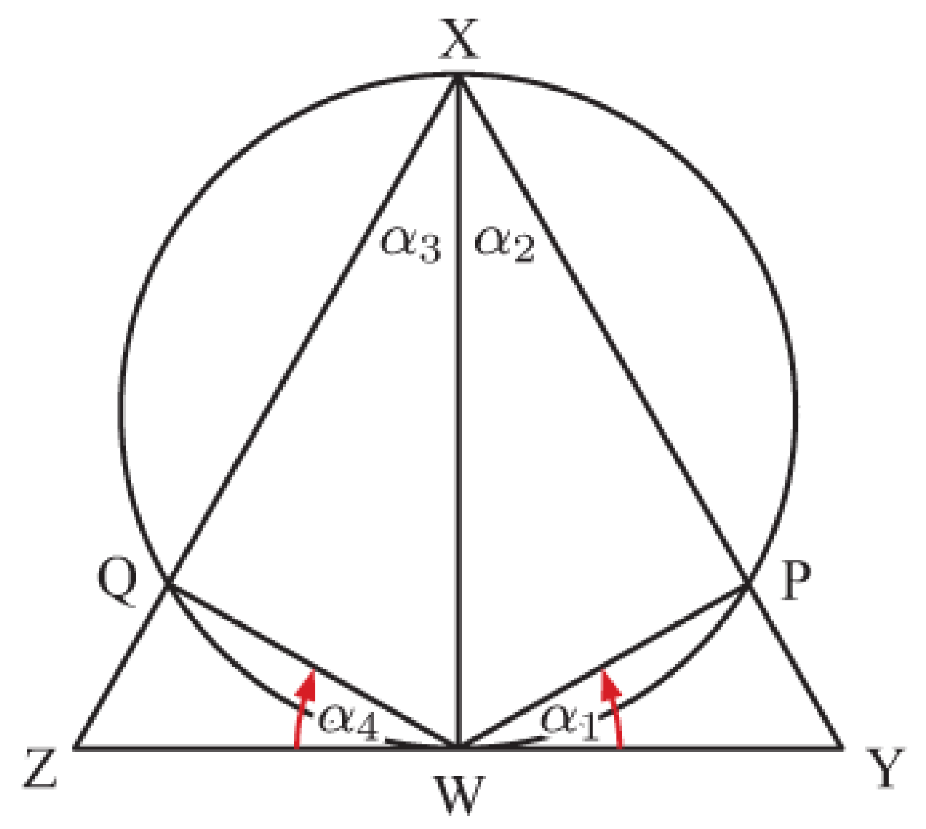
$= 180° - 2\alpha$

{angle at centre theorem}

$\therefore$ $\widehat{QOR} + \widehat{POS} = 2\alpha + 180° - 2\alpha = 180°$

$\therefore$ $\widehat{POS}$ and $\widehat{QOR}$ are supplementary.

12



X
α_3
α_2
Q
P
α_4
α_1
Z
W
Y

We join [WP] and [WQ].

Let $\widehat{YWP} = \alpha_1$

$\alpha_1 = \alpha_2$

{angle between tangent and chord}

and $\alpha_2 = \alpha_3$ {given}

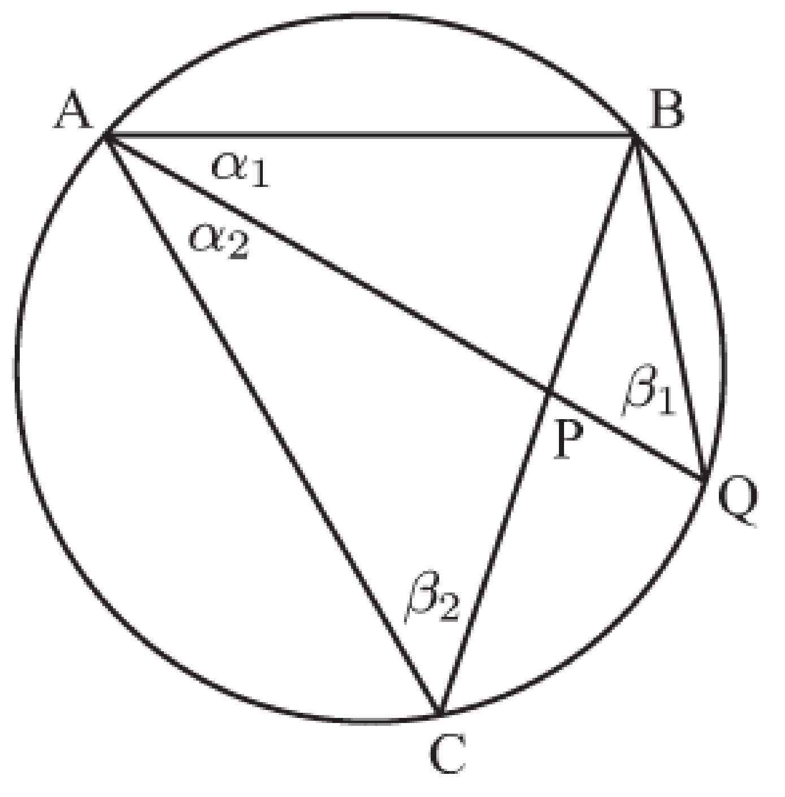
and $\alpha_3 = \alpha_4$

{angle between tangent and chord}

$\therefore$ $\alpha_1 = \alpha_4$

$\therefore$ $\widehat{YWP} = \widehat{ZWQ}$

13



A
B
α_1
α_2
β_1
P
Q
β_2
C

We join [BQ].

Let $\widehat{BAQ} = \alpha_1$, $\widehat{BQA} = \beta_1$

Now $\alpha_1 = \alpha_2$ {given}

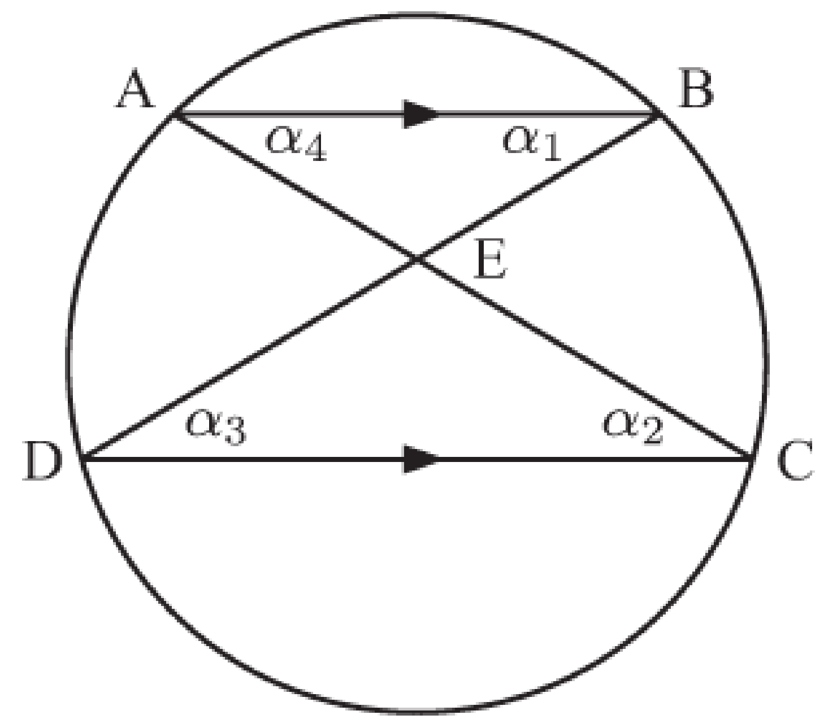
and $\beta_1 = \beta_2$

{angles subtended by the same arc}

$\therefore$ $\triangle$s ABQ and APC are equiangular, and therefore similar.

Hence, $\widehat{APC} = \widehat{ABQ}$.

14



A
B
α_4
α_1
E
D
α_3
α_2
C

a Let $\widehat{ABE} = \alpha_1$

$\alpha_1 = \alpha_2$ {angles subtended by the same arc}

and $\alpha_1 = \alpha_3$ {equal alternate angles}

But $\alpha_2 = \alpha_4$ {equal alternate angles}

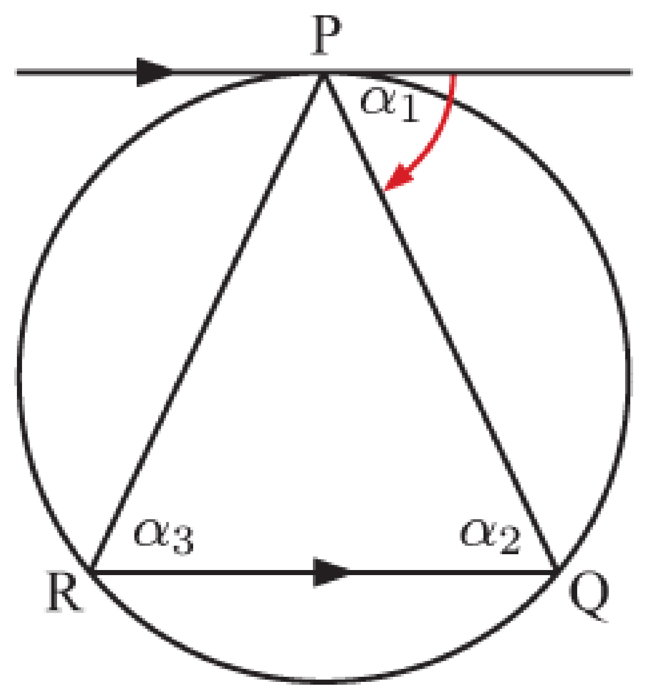
As $\triangle$ABE and $\triangle$CDE each have two equal angles, they are both isosceles. {converse of isosceles triangle theorem}

b Consequently, DE = CE and AE = BE

$\therefore$ AE + CE = BE + DE

$\therefore$ AC = BD

15



P
α_1
α_3
α_2
R
Q

$\alpha_1 = \alpha_2$

{equal alternate angles}

and $\alpha_1 = \alpha_3$

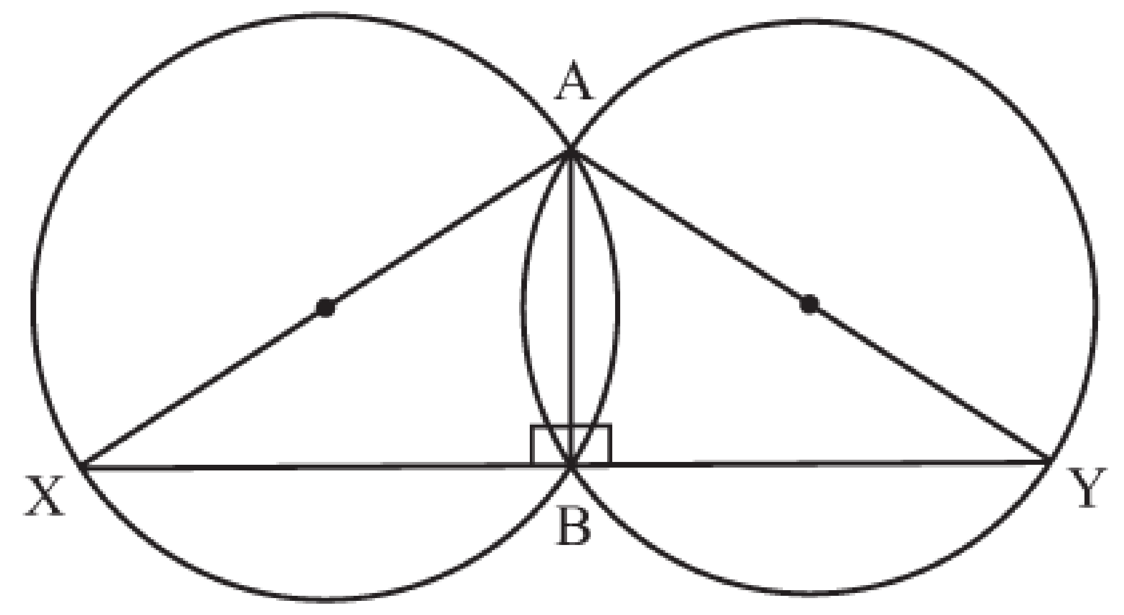
{angle between tangent and chord}

$\therefore$ $\alpha_2 = \alpha_3$

Thus $\triangle$PQR is isosceles.

{converse of isosceles triangle theorem}

16



We join [XB], [AB], and [BY].

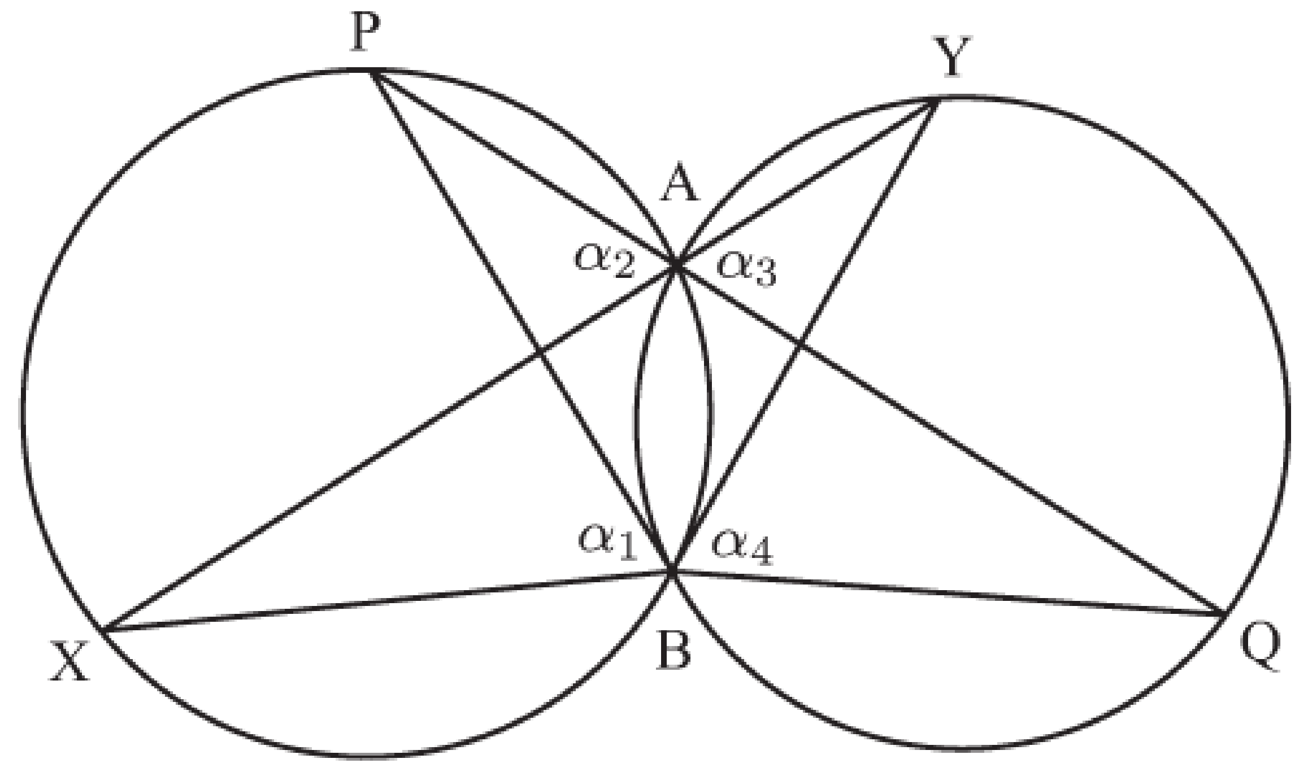
$\widehat{ABX}$ and $\widehat{ABY}$ are right angles {angle in a semi-circle theorem}

$\therefore$ $\widehat{XBY} = 90° + 90° = 180°$

$\therefore$ $\widehat{XBY}$ is a straight angle

$\therefore$ X, B, and Y are collinear.

17



P
Y
A
α_2
α_3
α_1
α_4
X
B
Q

We join [XB], [PB], [BY], and [BQ], and let $\widehat{PBX} = \alpha_1$.

Now $\alpha_1 = \alpha_2$ {angles subtended by the same arc}

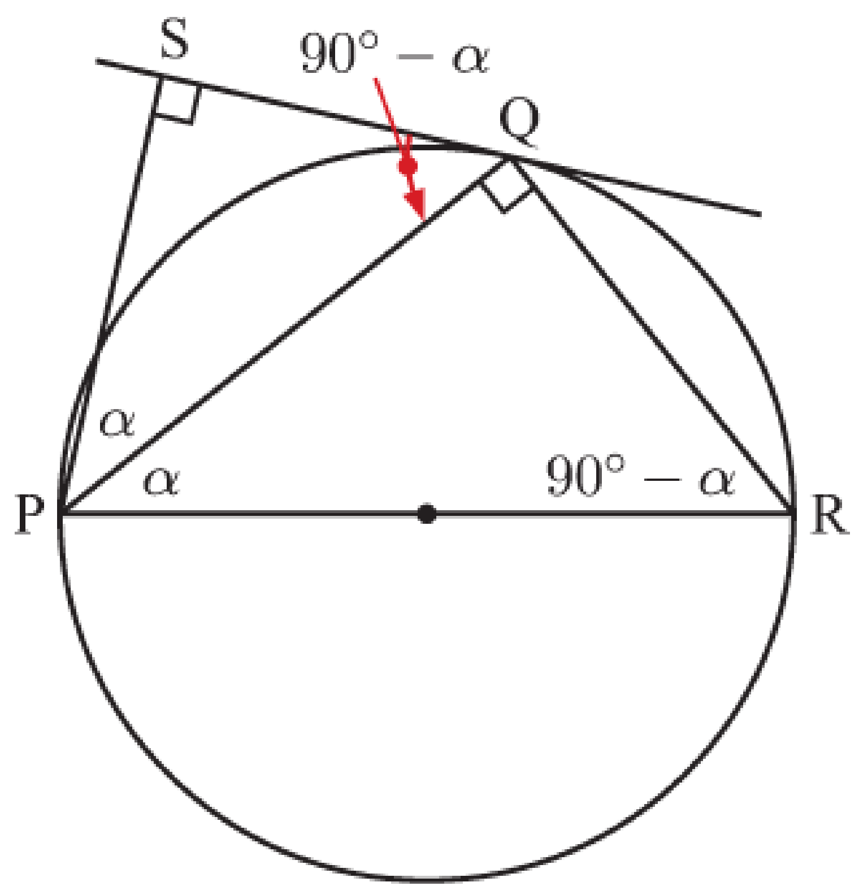
$\alpha_2 = \alpha_3$ {vertically opposite}

and $\alpha_3 = \alpha_4$ {angles subtended by the same arc}

$\therefore$ $\alpha_1 = \alpha_4$

$\therefore$ $\widehat{XBP} = \widehat{YBQ}$

18



S
$90° - \alpha$
Q
α
α
$90° - \alpha$
P
R

Let $\widehat{SPQ} = \alpha$

$\therefore$ $\widehat{SQP} = 90° - \alpha$

{angles of a triangle}

$\therefore$ $\widehat{QRP} = 90° - \alpha$

{angle between tangent and chord}

But $\widehat{PQR} = 90°$

{angle in a semi-circle}

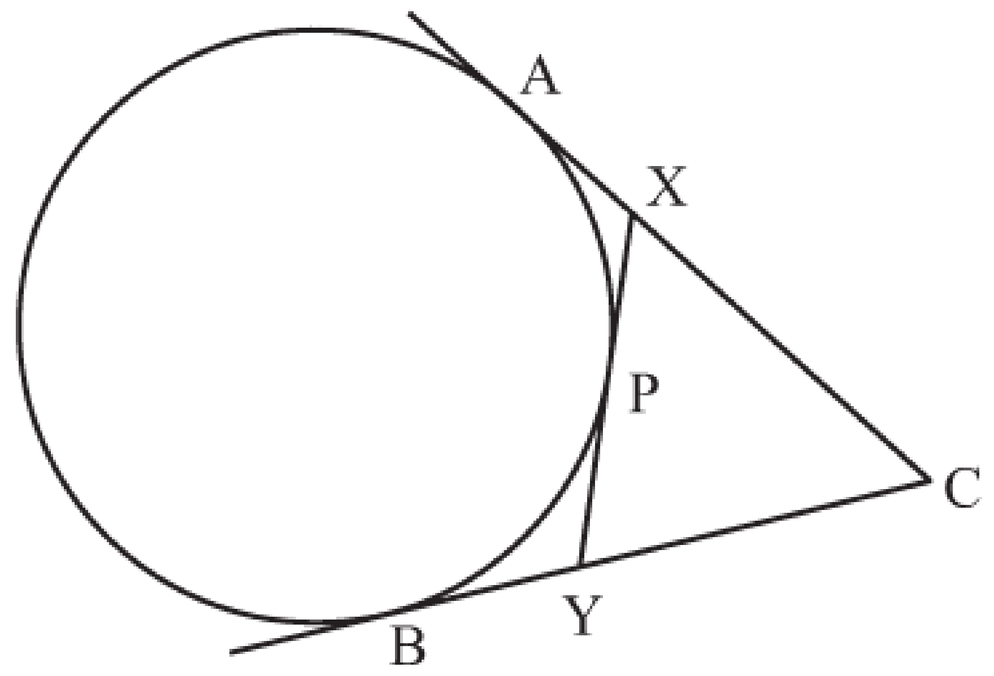
$\therefore$ $\widehat{RPQ} = \alpha$

{angles of a triangle}

Thus $\widehat{SPQ} = \widehat{RPQ} = \alpha$

$\therefore$ [PQ] bisects $\widehat{SPR}$.

19



Let P be the point of contact with the circle.

Now PX = AX and PY = BY {tangents from external point}

$\therefore$ perimeter of $\triangle$XYC $=$ CX $+$ CY $+$ XY

$=$ CX $+$ CY $+$ PX $+$ PY

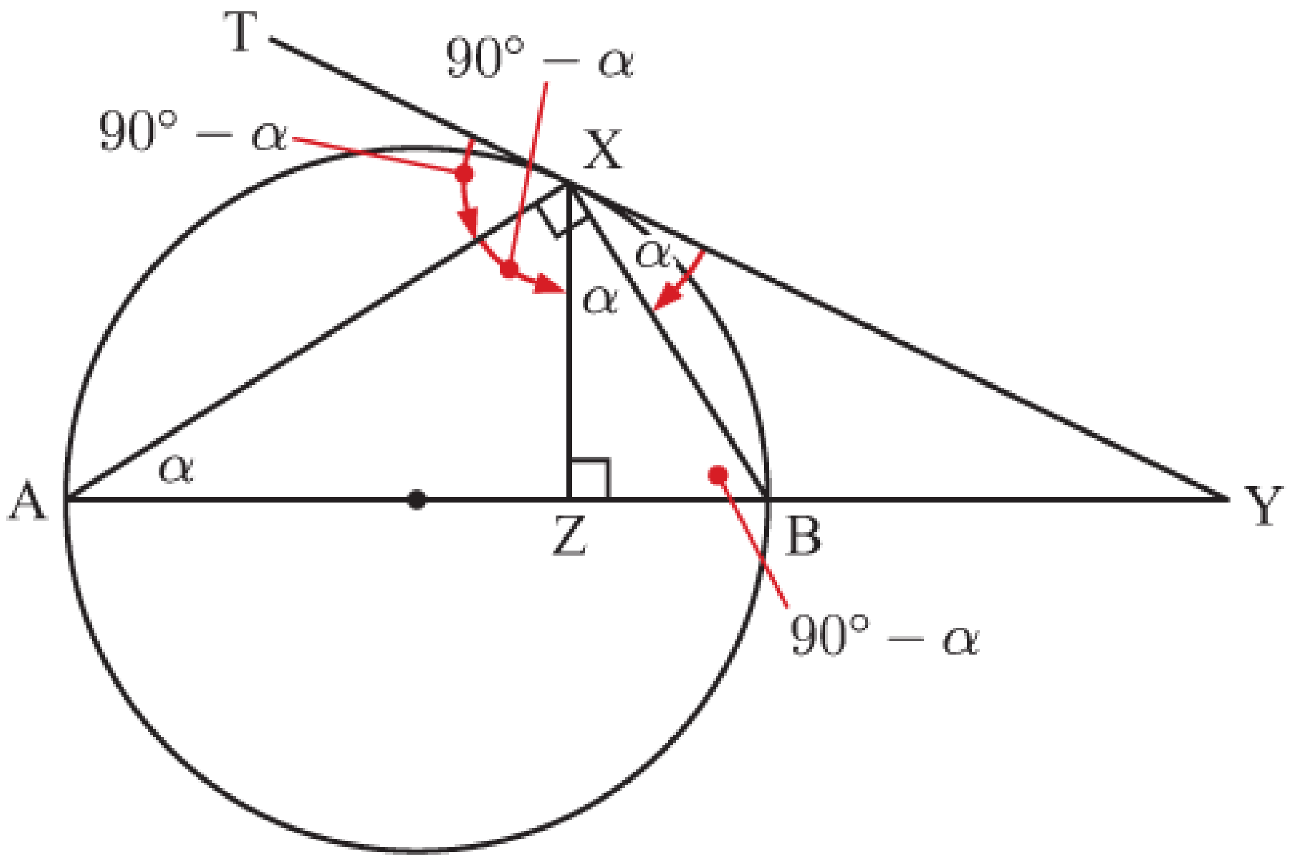
$=$ CX $+$ CY $+$ AX $+$ BY

$=$ (CX $+$ AX) $+$ (CY $+$ BY)

$=$ CA $+$ CB

which is constant as A, B, and C are fixed.

20



We join [XA] and [XB].

Let $Z\widehat{X}B = \alpha$

$A\widehat{X}B = 90°$ {angle in a semi-circle}

$\therefore$ $Z\widehat{X}A = 90° - \alpha$

$\therefore$ $X\widehat{A}Z = 180° - 90° - (90° - \alpha) = \alpha$

{angles in a triangle}

$\therefore$ $B\widehat{X}Y = \alpha$ {angle between tangent and chord}

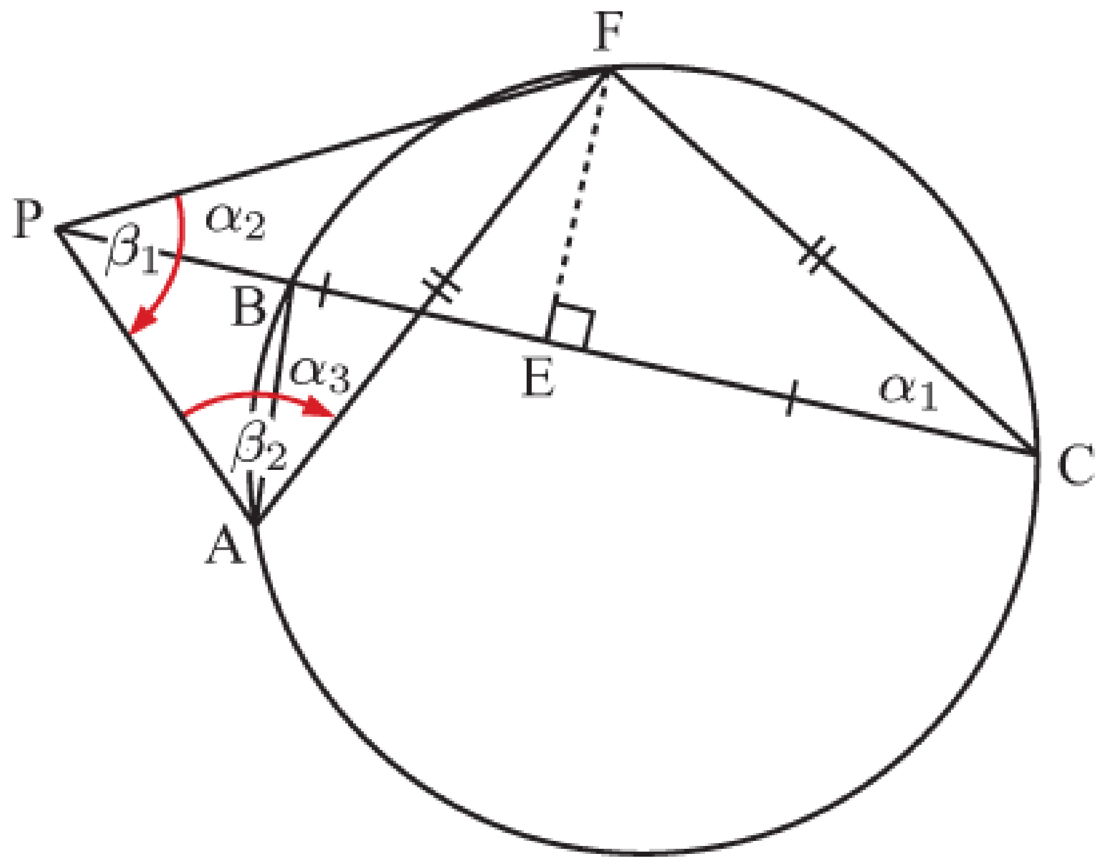
$Z\widehat{B}X = 180° - 90° - \alpha = 90° - \alpha$ {angles in a triangle}

$\therefore$ $A\widehat{X}T = 90° - \alpha$ {angle between tangent and chord}

$\therefore$ $Z\widehat{X}B = B\widehat{X}Y = \alpha$ and $Z\widehat{X}A = A\widehat{X}T = 90° - \alpha$

$\therefore$ [XB] bisects $Z\widehat{X}Y$ and [XA] bisects $Z\widehat{X}T$.

21



a F lies on the perpendicular bisector of [PC].

$\therefore$ $\triangle$PFC is isosceles

{converse of isosceles triangle theorem}

$\therefore$ PF $=$ FC

But AF $=$ FC {given}

$\therefore$ PF $=$ AF

$\therefore$ $\triangle$FPA is isosceles.

b Consequently $\beta_1 = \beta_2$ {isosceles triangle theorem}

But $\alpha_1 = \alpha_2$ {isosceles triangle theorem}

and $\alpha_1 = \alpha_3$ {angles subtended by the same arc}

$\therefore$ $B\widehat{P}A = \beta - \alpha = B\widehat{A}P$

$\therefore$ $\triangle$BPA is isosceles

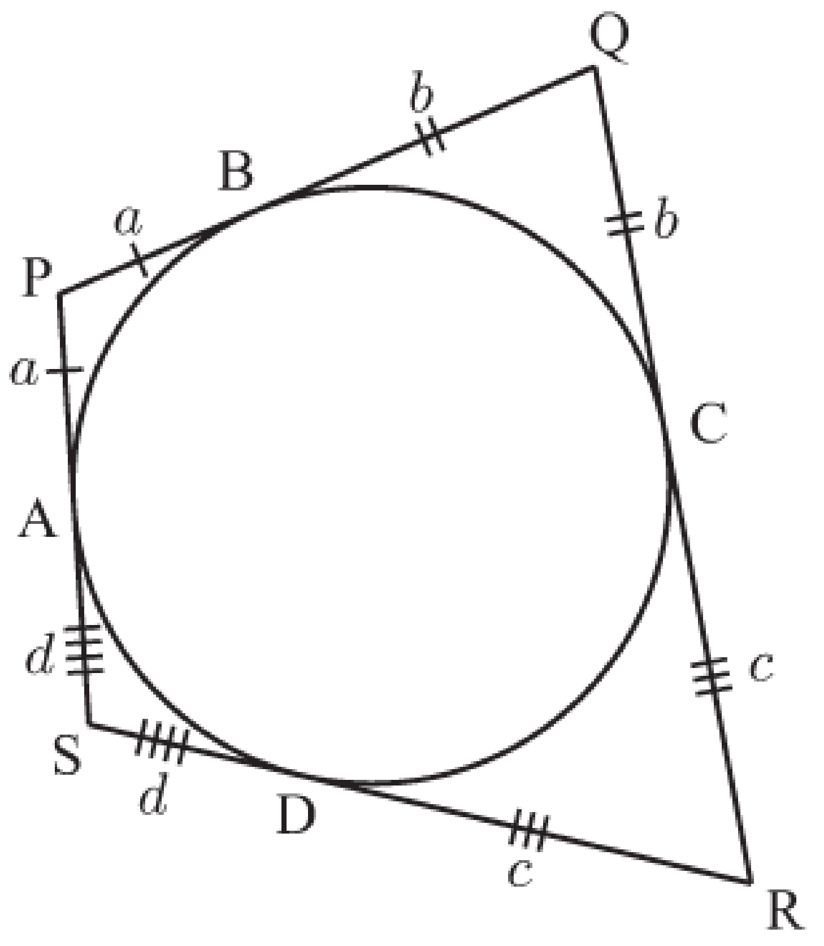
{converse of isosceles triangle theorem}

$\therefore$ PB $=$ AB

But PB $+$ BE $=$ EC {given}

$\therefore$ AB $+$ BE $=$ EC

22



Let PA $=$ PB $= a$,

QB $=$ QC $= b$,

RC $=$ RD $= c$,

SD $=$ SA $= d$

{tangents from an external point}

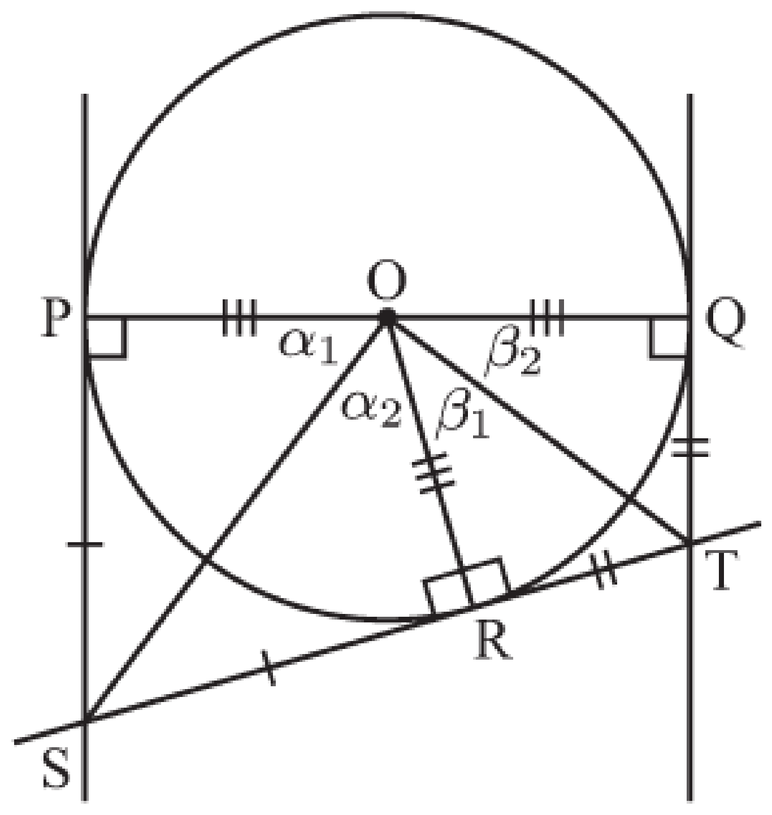
$\therefore$ PQ $+$ RS $= a + b + c + d$

$= (b + c) + (a + d)$

$=$ QR $+$ PS

So, the sum of the lengths of one pair of opposite sides equals the sum of the lengths of the other pair of opposite sides.

23



We join [OR] and mark right angles $O\widehat{P}S$, $O\widehat{Q}T$, $O\widehat{R}S$, and $O\widehat{R}T$.

{radius-tangent theorem}

SP $=$ SR and TR $=$ TQ

{tangents from an external point}

OP $=$ OR $=$ OQ

{equal radii}

Now $\triangle$s SPO and SRO are congruent {SSS}

$\therefore$ $\alpha_1 = \alpha_2$

and $\triangle$s QOT and ROT are congruent {SSS}

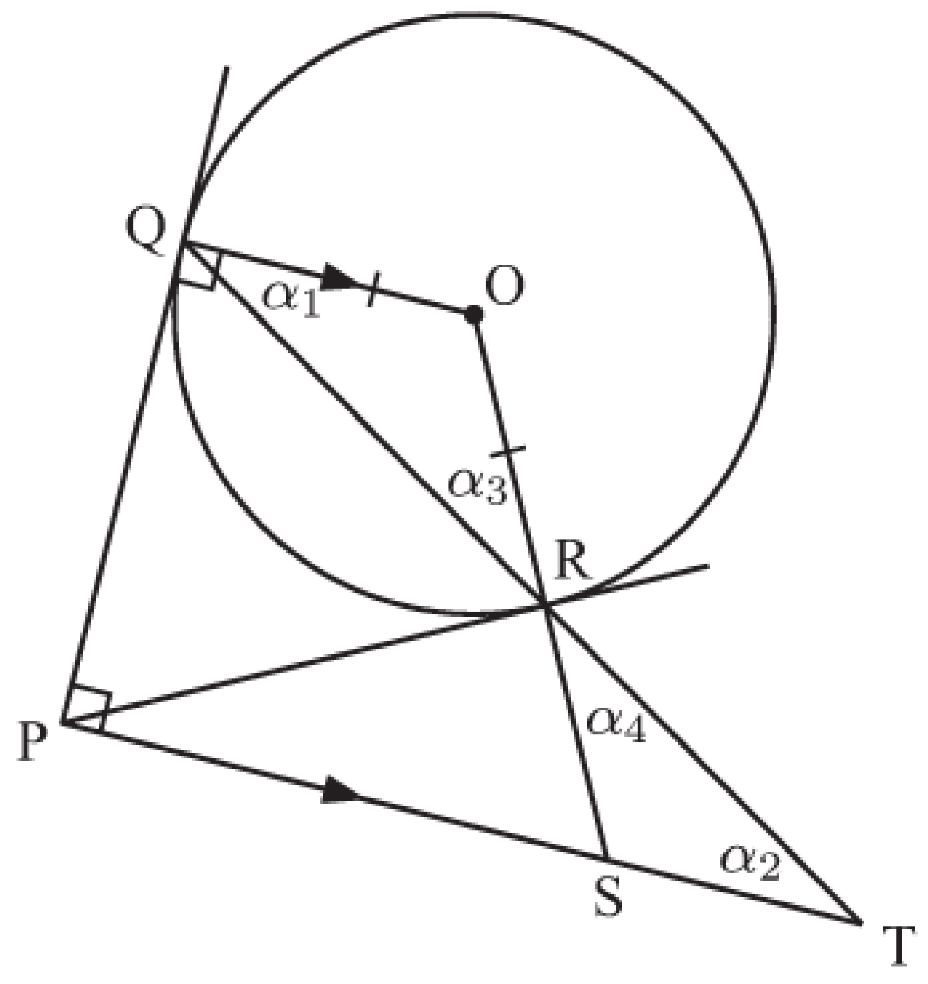
$\therefore$ $\beta_1 = \beta_2$

$\therefore$ $2\alpha + 2\beta = 180°$ {angles on a line}

$\therefore$ $\alpha + \beta = 90°$

$\therefore$ $S\widehat{O}T$ is a right angle.

24



We join [OQ]. $O\widehat{Q}P = 90°$ {radius-tangent}

Now [OQ] $\parallel$ [PT] {radius-tangent, converse of supplementary allied angles}

$\therefore$ $\alpha_1 = \alpha_2$ {equal alternate angles}

But OQ = OR {equal radii}

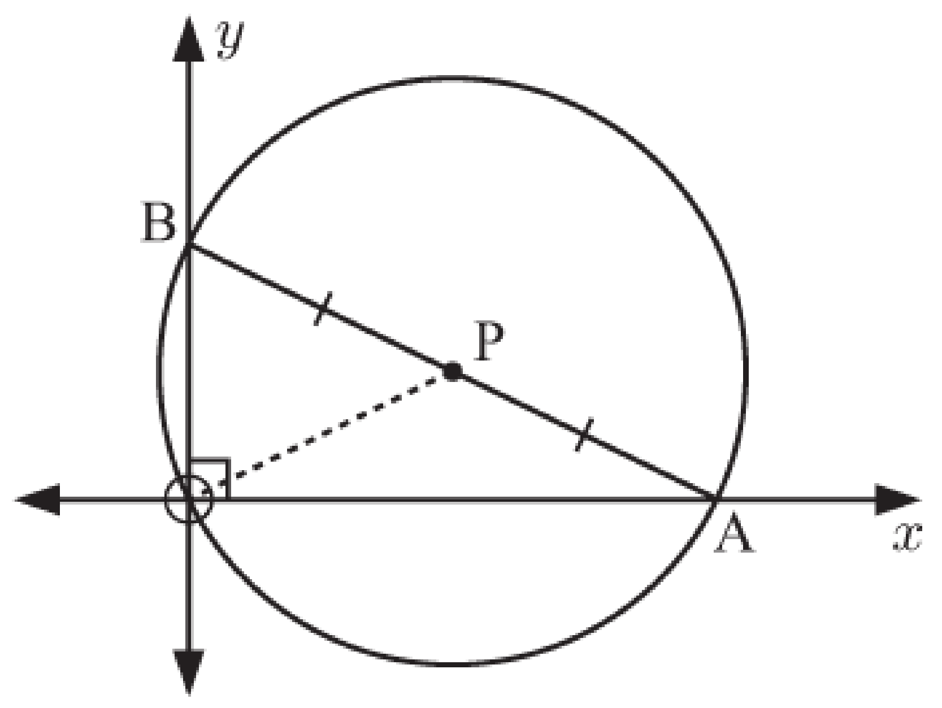
$\therefore$ $\alpha_1 = \alpha_3$ {isosceles triangle theorem}

But $\alpha_3 = \alpha_4$ {vertically opposite angles}

$\therefore$ $\alpha_2 = \alpha_4$

$\therefore$ $\triangle$STR is isosceles {converse of isosceles triangle theorem}

25



$\widehat{AOB} = 90°$ and P is the midpoint of [AB]

$\therefore$ we can draw a circle through A, O, and B with centre P and diameter [AB] {converse of angle in a semi-circle}

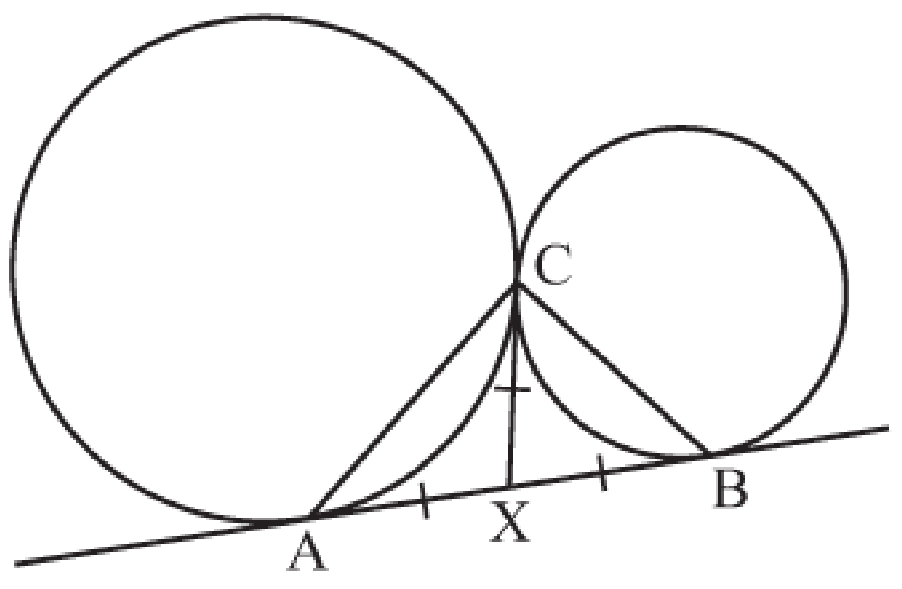
This circle through A, O, and B has diameter [AB] of fixed length AB.

$\therefore$ its radius [OP] has fixed length $\frac{1}{2} \times$ AB.

Point O is fixed

$\therefore$ the light source P traces out a circle with centre O and radius of length $\frac{1}{2} \times$ AB.

26



a We join [AC] and [BC] and draw the common tangent at C.

Let the common tangent at C meet [AB] at X.

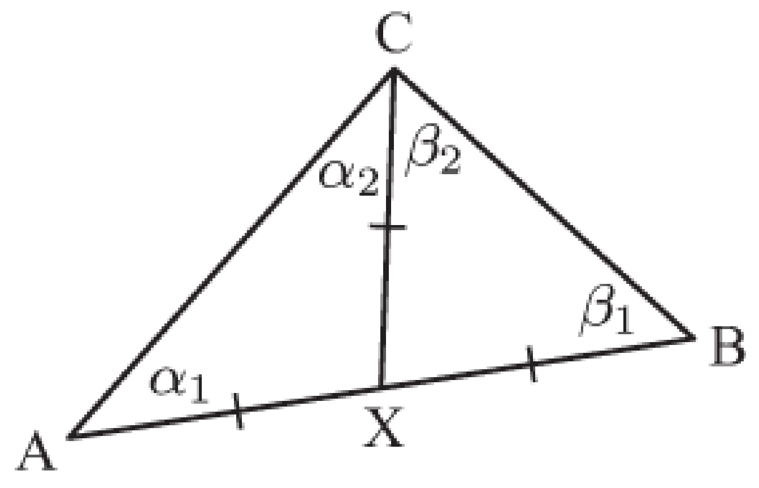
Then XA = XC and XC = XB

{tangents from external point}

$\therefore$ XA = XB

$\therefore$ X bisects [AB].

b



$\triangle$s AXC, BXC are isosceles.

$\therefore$ $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {isosceles triangle theorem}

But $\alpha_1 + \alpha_2 + \beta_2 + \beta_1 = 180°$ {angles of a triangle}

$\therefore$ $2\alpha + 2\beta = 180°$

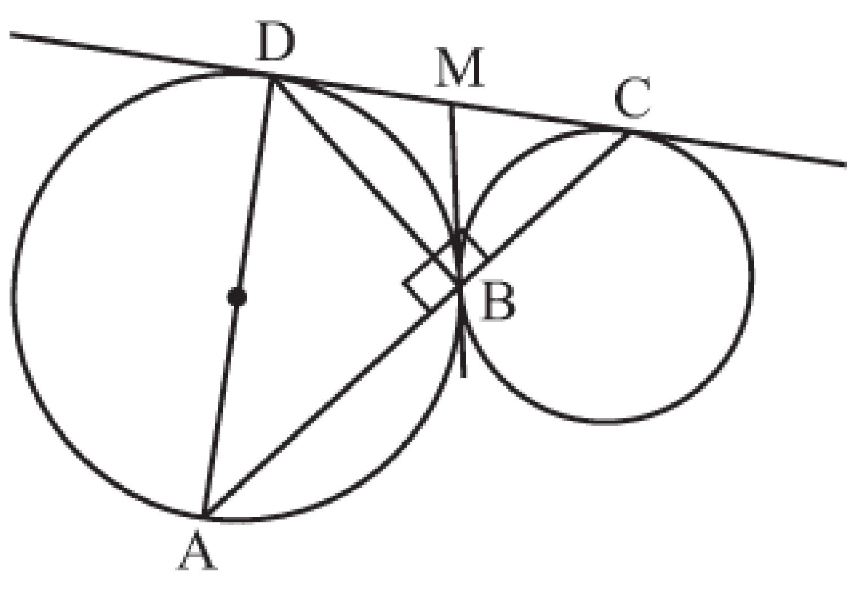
$\therefore$ $\alpha + \beta = 90°$

$\therefore$ $\widehat{ACB}$ is a right angle.

Note: As XA = XB = XC, a semi-circle could be drawn through A, C, and B.

$\therefore$ $\widehat{ACB}$ is a right angle.

27



We join [AB], [BC], and [BD] and draw the tangent at B.

Let the common tangent at B meet [DC] at M.

$\widehat{DBA}$ is a right angle. {angle in a semi-circle}

DM = BM = CM {tangents from an external point}

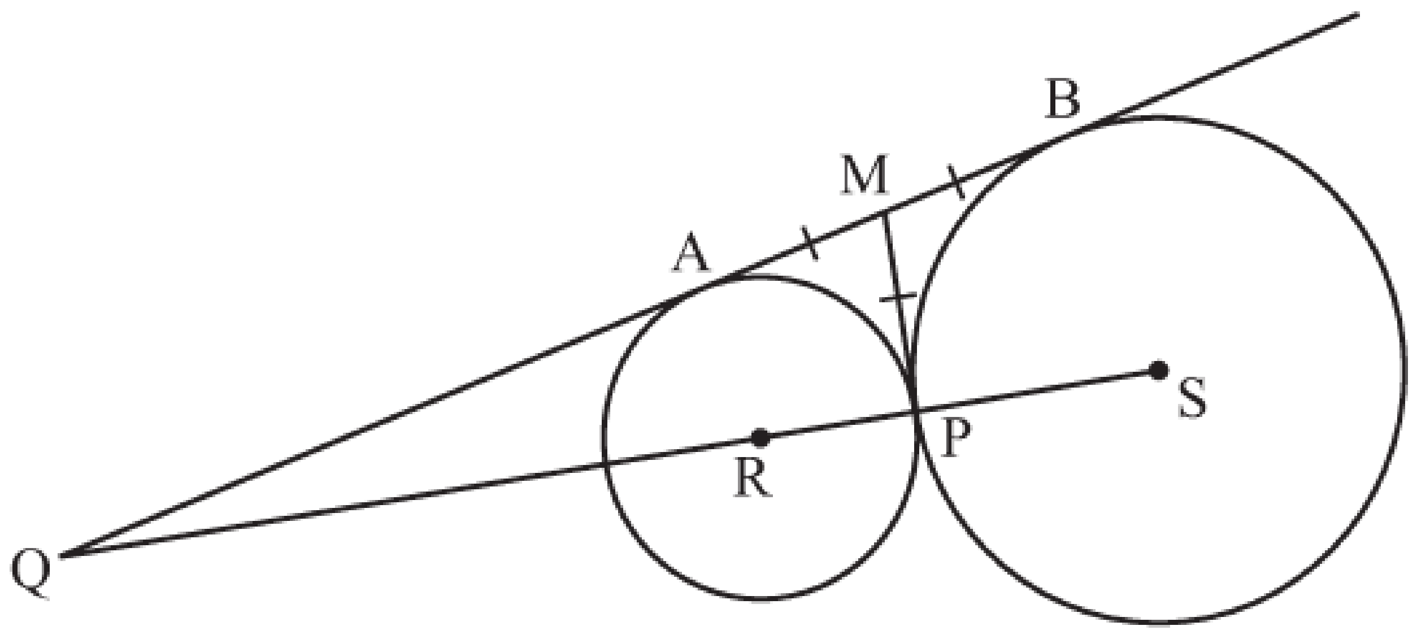
$\therefore$ a semi-circle can be drawn with diameter [DC] passing through B

$\therefore$ $\widehat{DBC}$ is also a right angle {angle in a semi-circle}

$\therefore$ $\widehat{ABC}$ is a straight angle

$\therefore$ A, B, and C are collinear.

28



Draw the common tangent at P to meet [AB] at M.

AM = PM = BM {tangents from an external point}

$\widehat{MPR} = 90°$ {radius-tangent}

$$\begin{aligned} QP^2 &= QM^2 - PM^2 && \text{\{Pythagoras\}} \\ &= (QM - PM)(QM + PM) \\ &= (QM - AM)(QM + BM) && \text{\{AM = PM = BM\}} \\ &= QA.QB \end{aligned}$$

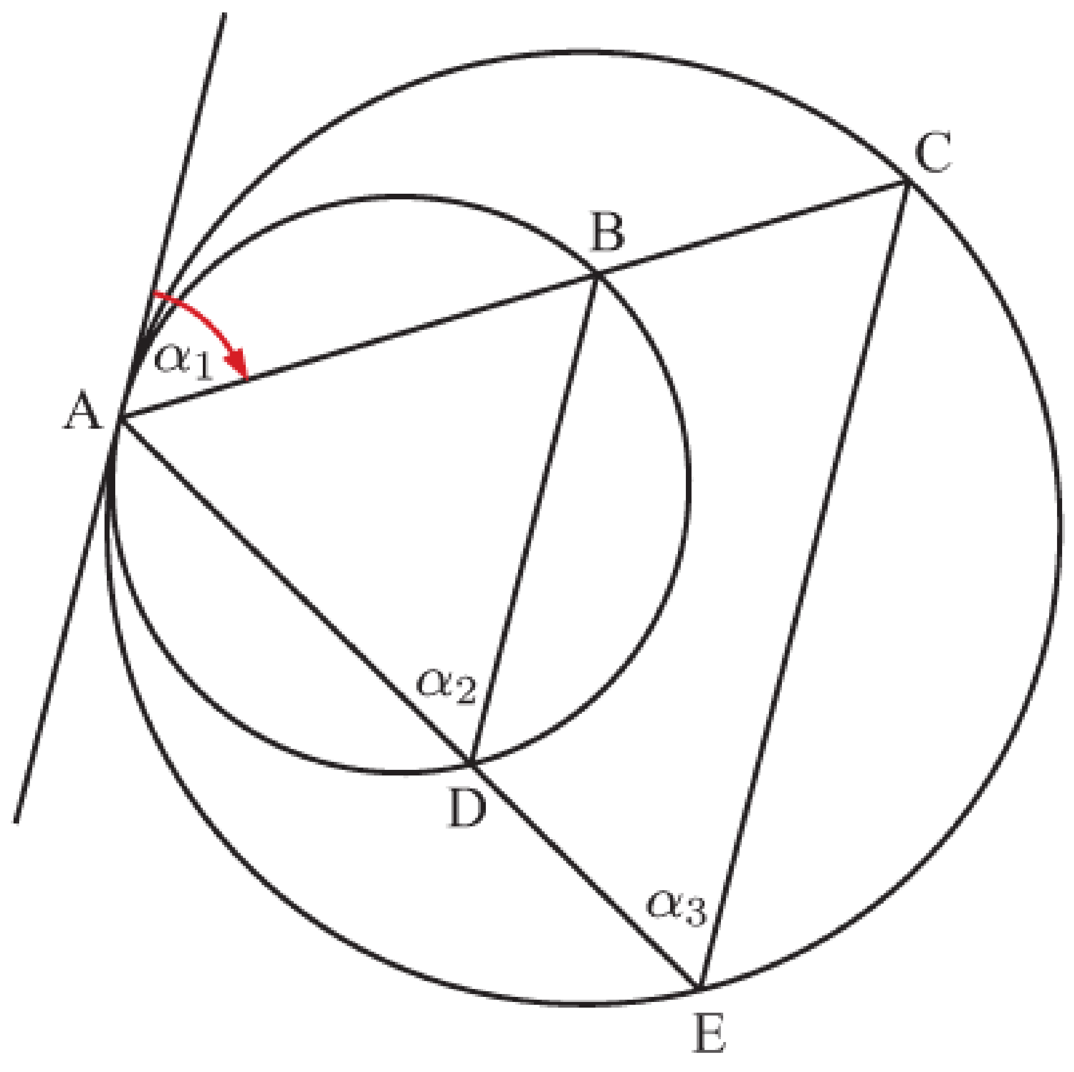
29 We draw the tangent at A, which is the tangent to both circles.

$\alpha_1 = \alpha_2$

and $\alpha_1 = \alpha_3$

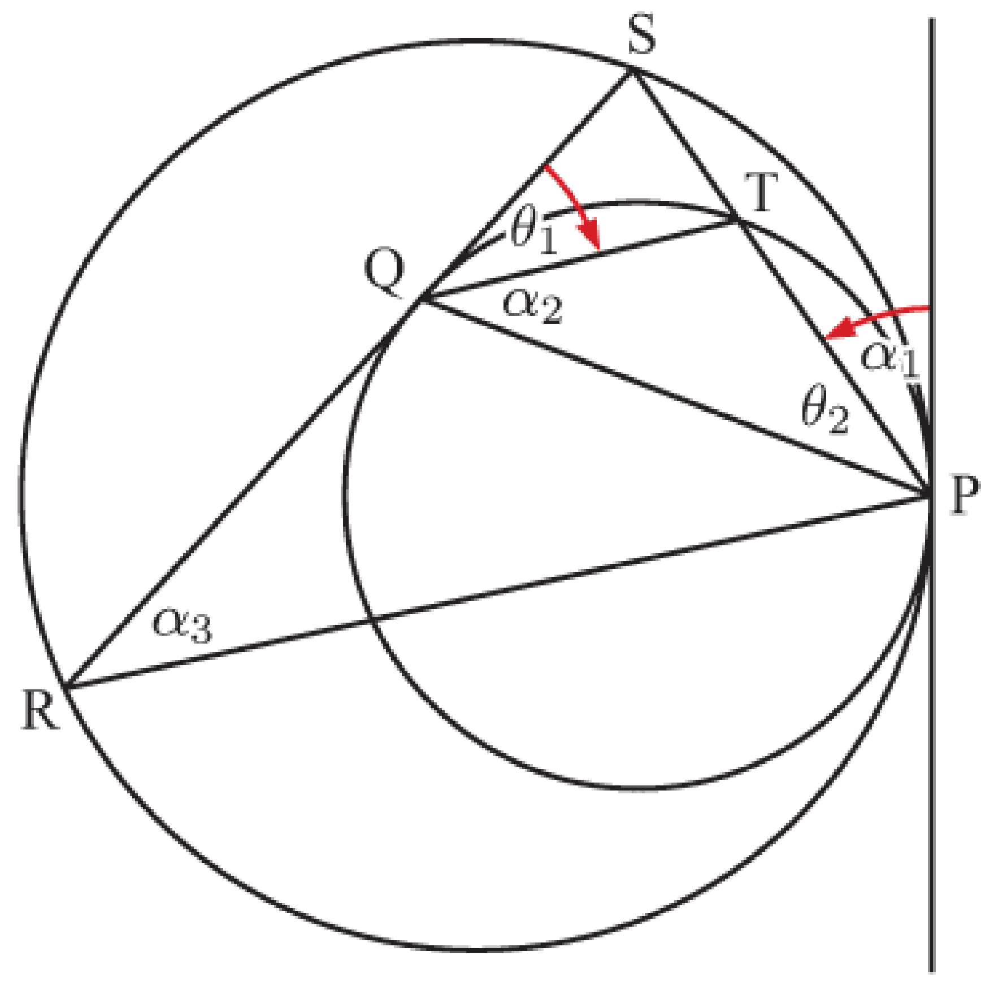
{angle between tangent and chord}

$\therefore$ $\alpha_2 = \alpha_3$



$\therefore$ [BD] $\parallel$ [CE] {converse of corresponding angles}

30



Let [SP] cut the smaller circle at T and join [QT].

$\alpha_1 = \alpha_2$ and $\alpha_1 = \alpha_3$ {angle between tangent and chord}

$\theta_1 = \theta_2$ {angle between tangent and chord}

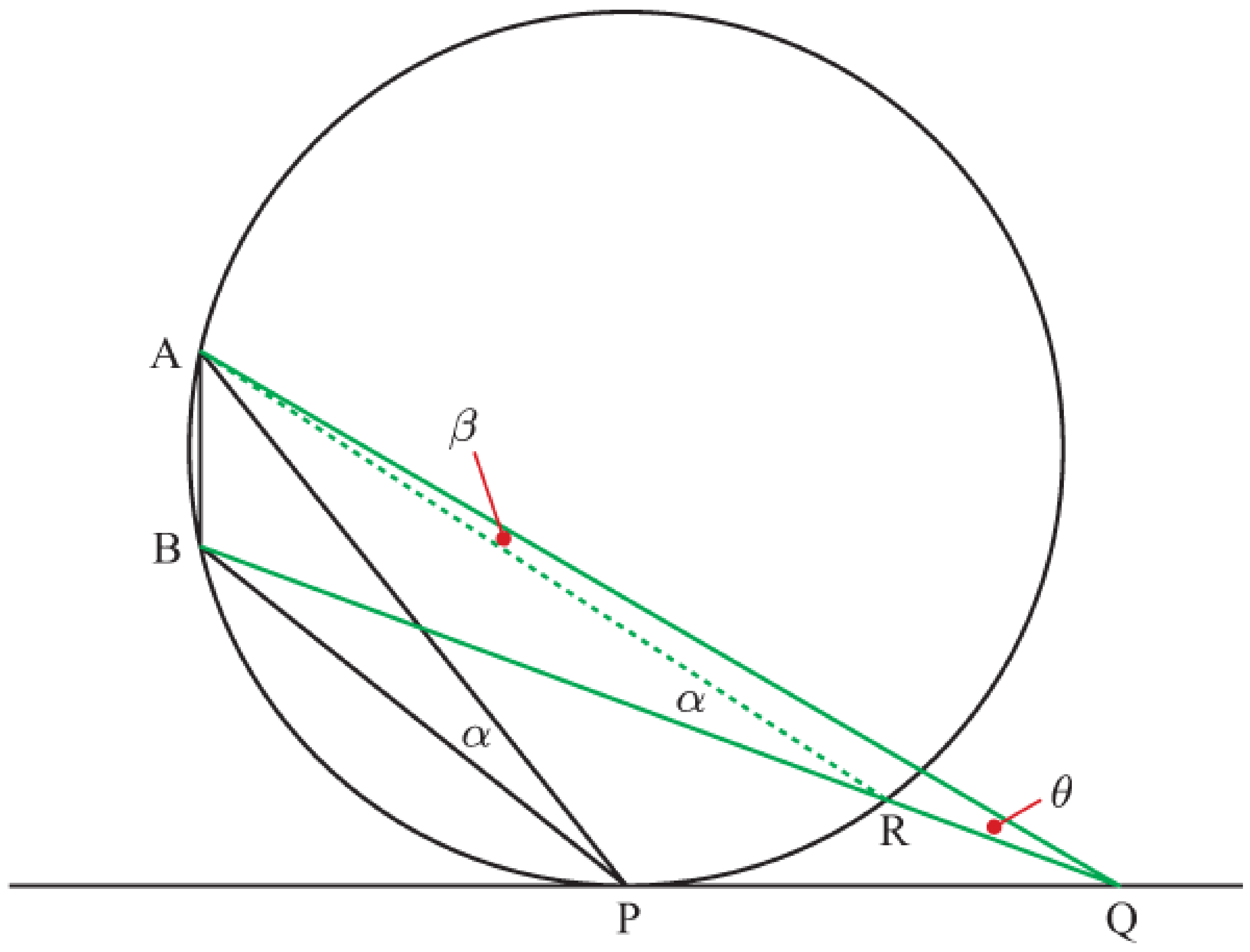
$R\widehat{P}Q + Q\widehat{R}P = P\widehat{Q}S$ {exterior angle of a triangle}

$\therefore$ $R\widehat{P}Q + \alpha = \theta + \alpha$

$\therefore$ $R\widehat{P}Q = \theta$

$\therefore$ [QP] bisects $R\widehat{P}S$.

31



Let P be the point on the boundary line such that the circle through A, B, and P has the boundary line as a tangent.
Let Q be any other point on the boundary line.

Let $A\widehat{P}B = \alpha$, $A\widehat{Q}B = \theta$, and $Q\widehat{A}R = \beta$.

$\therefore$ $A\widehat{R}B = \alpha$ {angles subtended by the same arc}

$\alpha = \theta + \beta$ {exterior angle of a triangle}

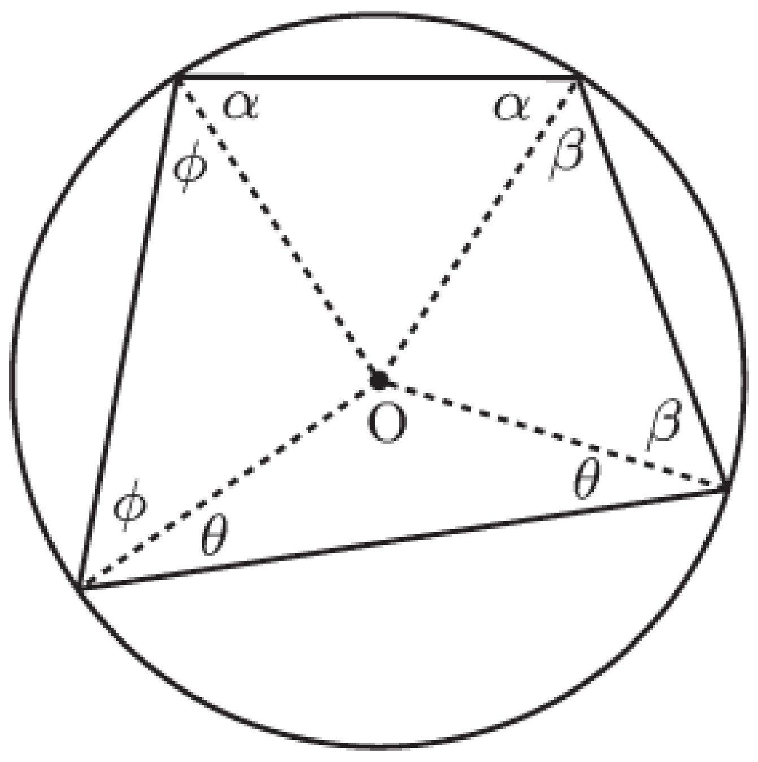
$\therefore$ $\alpha > \theta$

$\therefore$ $A\widehat{P}B$ is always greater than $A\widehat{Q}B$

$\therefore$ the angle of view is maximised when P is chosen such that the boundary line is a tangent to the circle through A, B, and P.

EXERCISE 2E

1



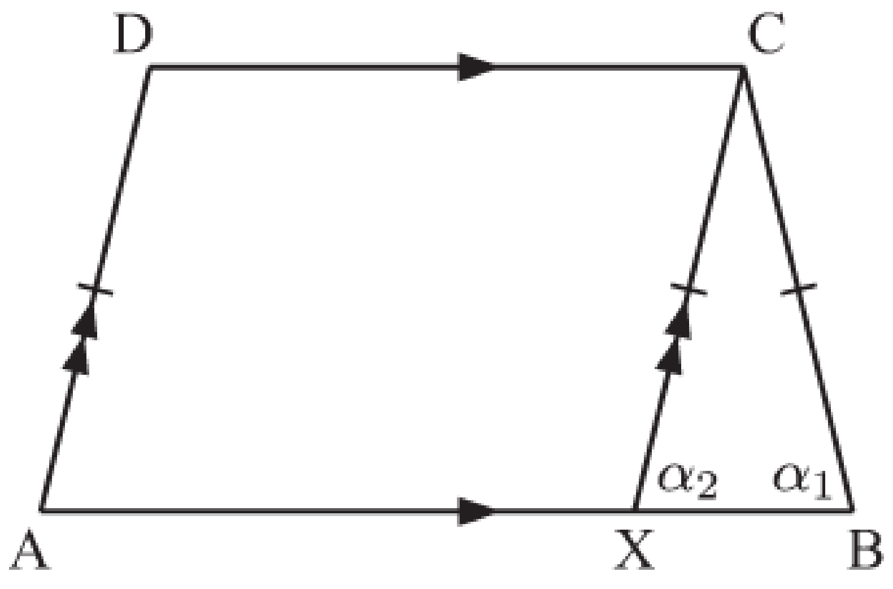
We have four isosceles triangles {equal radii} with equal base angles which we mark α, β, θ, ϕ.

Now $2\alpha + 2\beta + 2\theta + 2\phi = 360°$ {angles of a quadrilateral}

$\therefore$ $\alpha + \beta + \theta + \phi = 180°$

$\therefore$ opposite angles are supplementary.

2



We draw [CX] parallel to [DA], meeting [AB] at X.

We let $A\widehat{B}C = \alpha_1$.

Now ADCX is a parallelogram

$\therefore$ DA = CX {parallelogram theorem}

$\therefore$ CX = CB

$\therefore$ $\triangle$CBX is isosceles

$\therefore$ $\alpha_1 = \alpha_2$ {isosceles triangle theorem}

Now $A\widehat{X}C = 180° - \alpha$ {angles on a line}

$\therefore$ $A\widehat{D}C = 180° - \alpha$ {parallelogram theorem}

$\therefore$ $A\widehat{D}C + A\widehat{B}C = 180° - \alpha + \alpha = 180°$

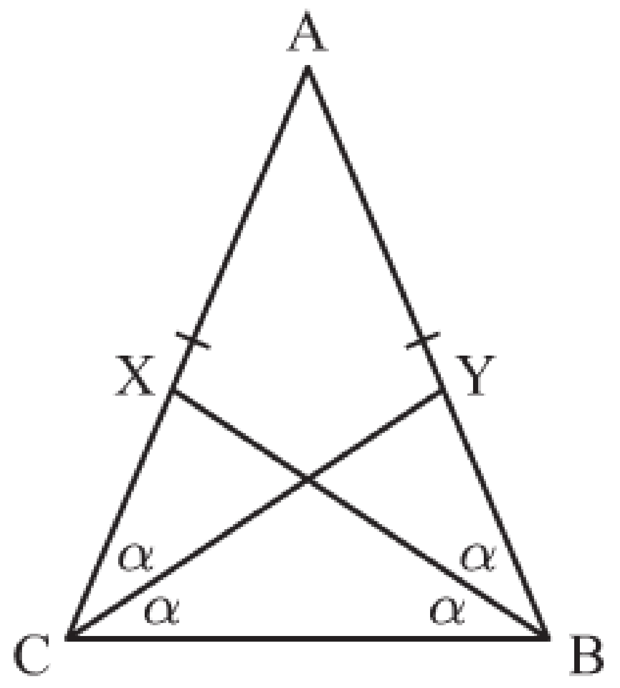
$\therefore$ ADCB is a cyclic quadrilateral
{converse of opposite angles of a cyclic quadrilateral}

3 $2\alpha + 2\beta = 360°$ {angles of a quadrilateral}

$\therefore$ $\alpha + \beta = 180°$

$\therefore$ ABCD is a cyclic quadrilateral.
{converse of opposite angles of a cyclic quadrilateral}

4



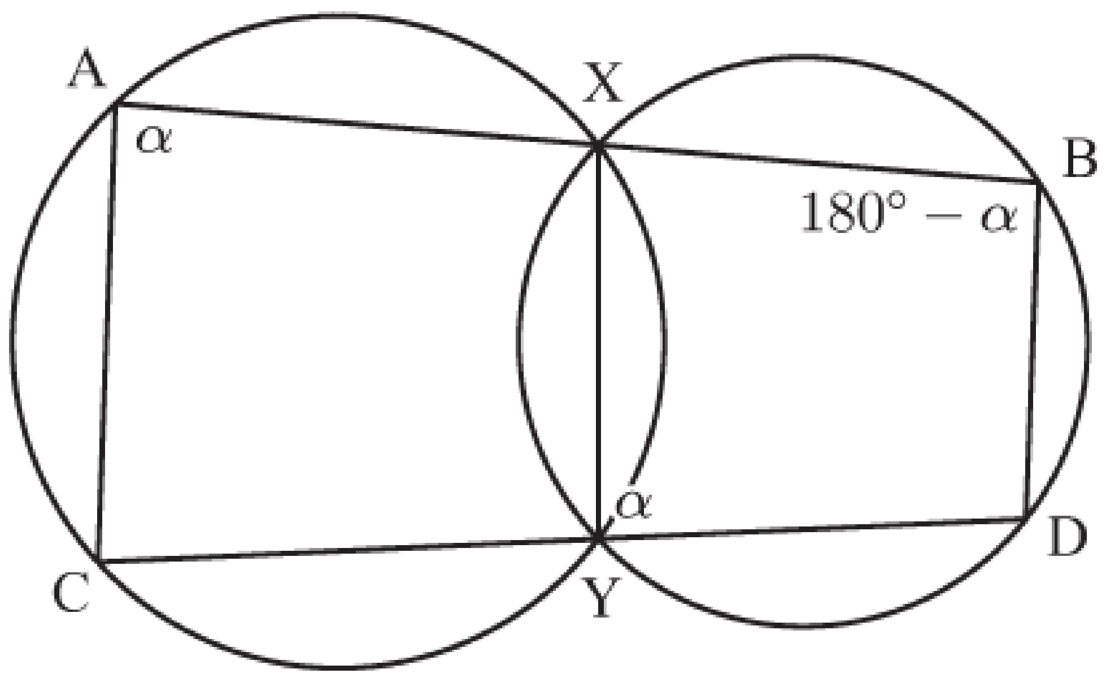
$A\widehat{B}C = A\widehat{C}B$
{isosceles triangle theorem}

$\therefore$ we let angles ABX, CBX, ACY, BCY all be α

$\therefore$ [XY] subtends equal angles of α at B and C.

$\therefore$ BCXY is a cyclic quadrilateral.

5



We join [XY] and let $C\widehat{A}B = \alpha$

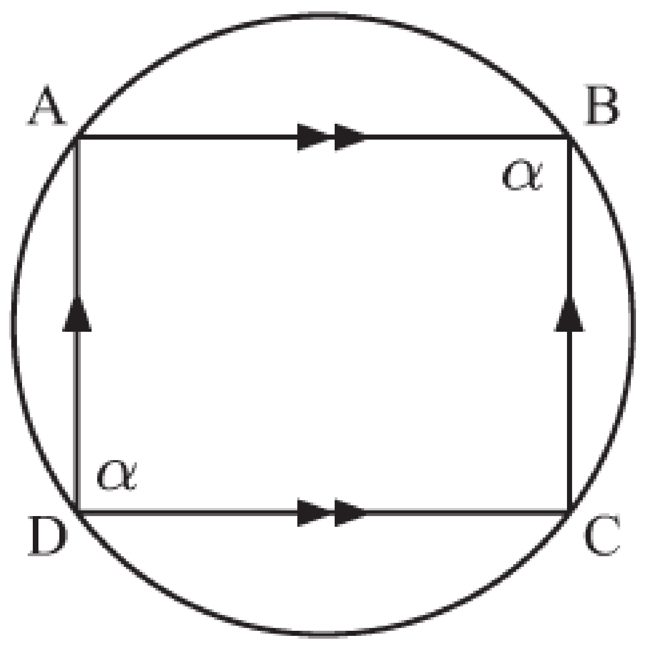
$\therefore$ $X\widehat{Y}D = \alpha$ {exterior angle of a cyclic quadrilateral}

$\therefore$ $A\widehat{B}D = 180° - \alpha$ {opposite angles of a cyclic quadrilateral}

$\therefore$ $C\widehat{A}B + A\widehat{B}D = \alpha + 180° - \alpha$
$= 180°$

$\therefore$ [AC] $\parallel$ [BD] {converse of allied angles}

6



Consider the given figure and let $A\widehat{B}C = \alpha$

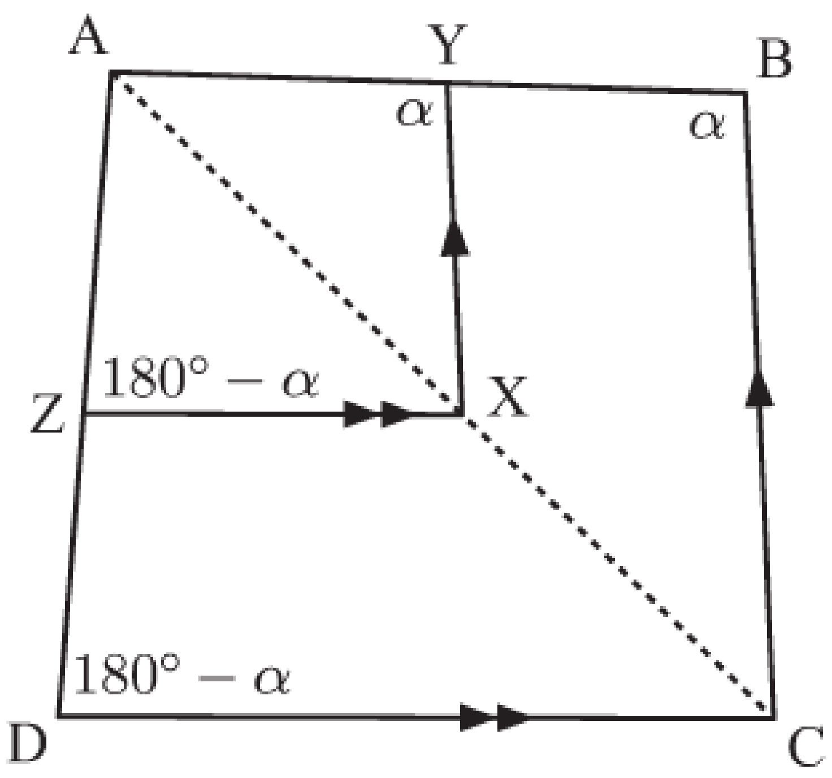
$\therefore$ $A\widehat{D}C = \alpha$
{parallelogram theorem}

$\therefore$ $2\alpha = 180°$
{opposite angles of a cyclic quadrilateral}

$\therefore$ $\alpha = 90°$

$\therefore$ ABCD is a rectangle
{a parallelogram with one angle 90° is a rectangle}

7



Let $A\widehat{B}C = \alpha$

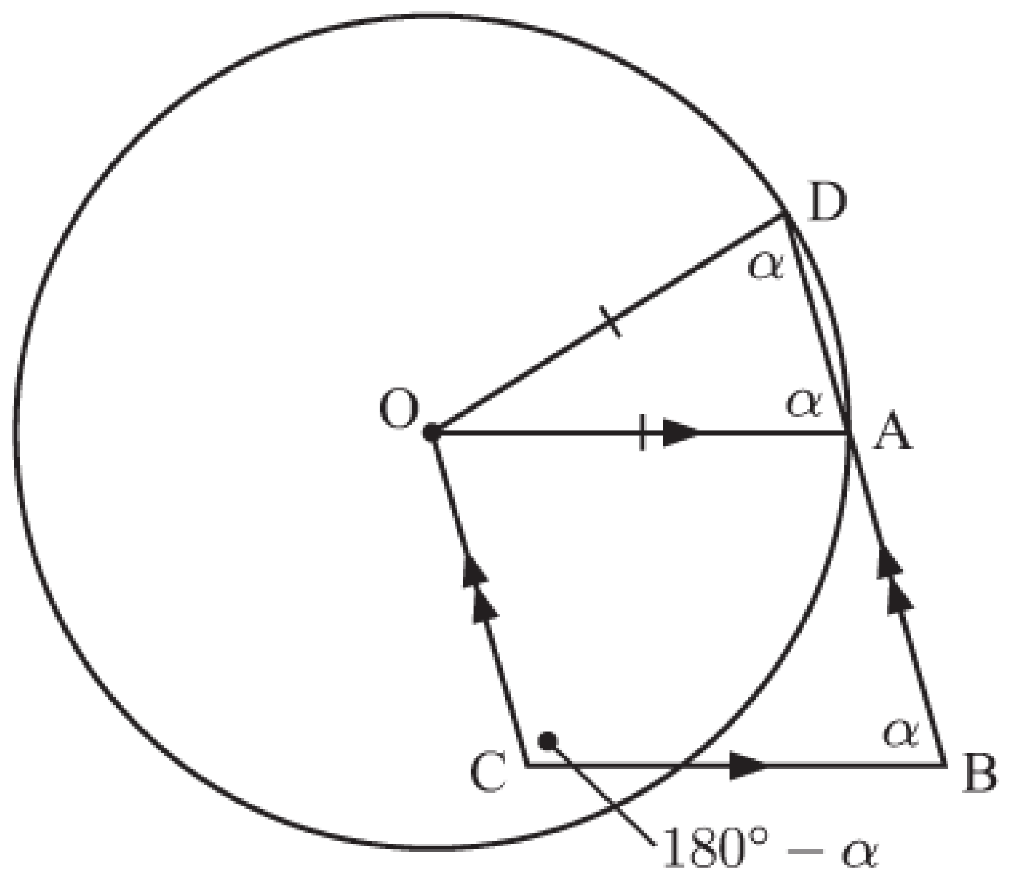
$\therefore$ $A\widehat{D}C = 180° - \alpha$
{opposite angles of a cyclic quadrilateral}

$\therefore$ $A\widehat{Y}X = \alpha$ and
$A\widehat{Z}X = 180° - \alpha$
{equal corresponding angles}

$\therefore$ $A\widehat{Y}X + A\widehat{Z}X = \alpha + 180° - \alpha = 180°$

$\therefore$ XYAZ is a cyclic quadrilateral
{converse of opposite angles of a cyclic quadrilateral}

8



We join [OD] and let $\widehat{ABC} = \alpha.$

$\therefore\ \widehat{OAD} = \alpha$ {corresponding angles}

$OA = OD$ {equal radii}

$\therefore$ $\triangle OAD$ is isosceles

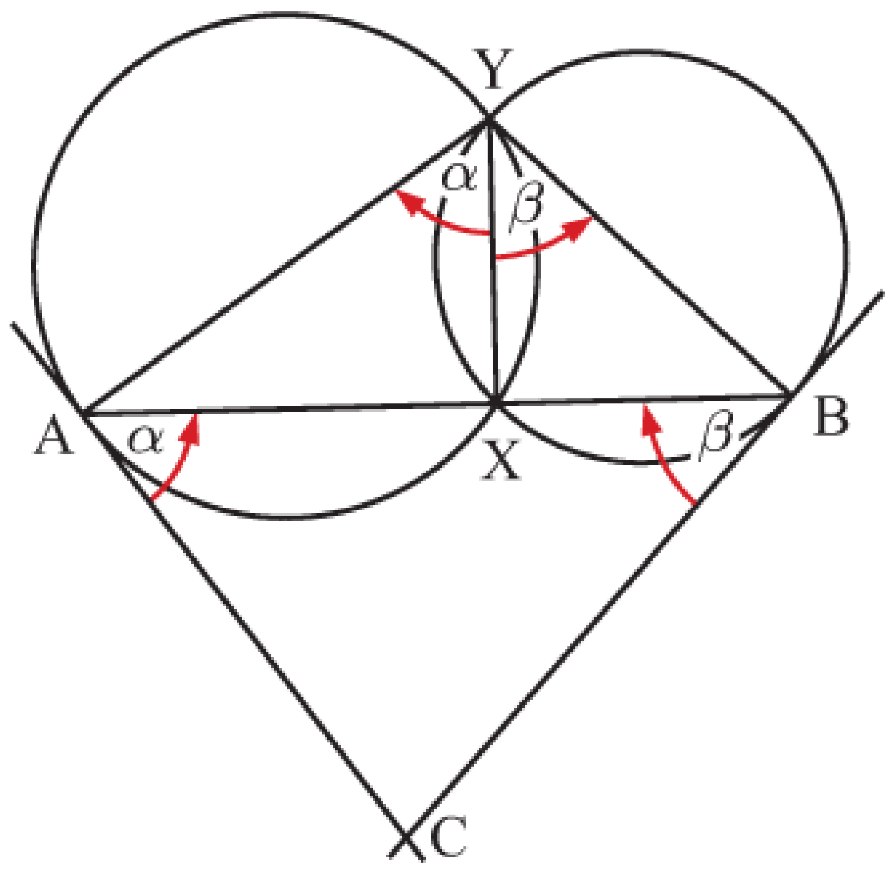
$\therefore\ \widehat{ODA} = \widehat{OAD} = \alpha$ {isosceles triangle theorem}

$\widehat{OCB} = 180° - \alpha$ {supplementary allied angles}

$\therefore\ \widehat{ODB} + \widehat{OCB} = \alpha + 180° - \alpha = 180°$

$\therefore$ DOCB is a cyclic quadrilateral
{converse of opposite angles of a cyclic quadrilateral}

9



We join [XY].

Let $\widehat{CAB} = \alpha$
and $\widehat{CBA} = \beta$

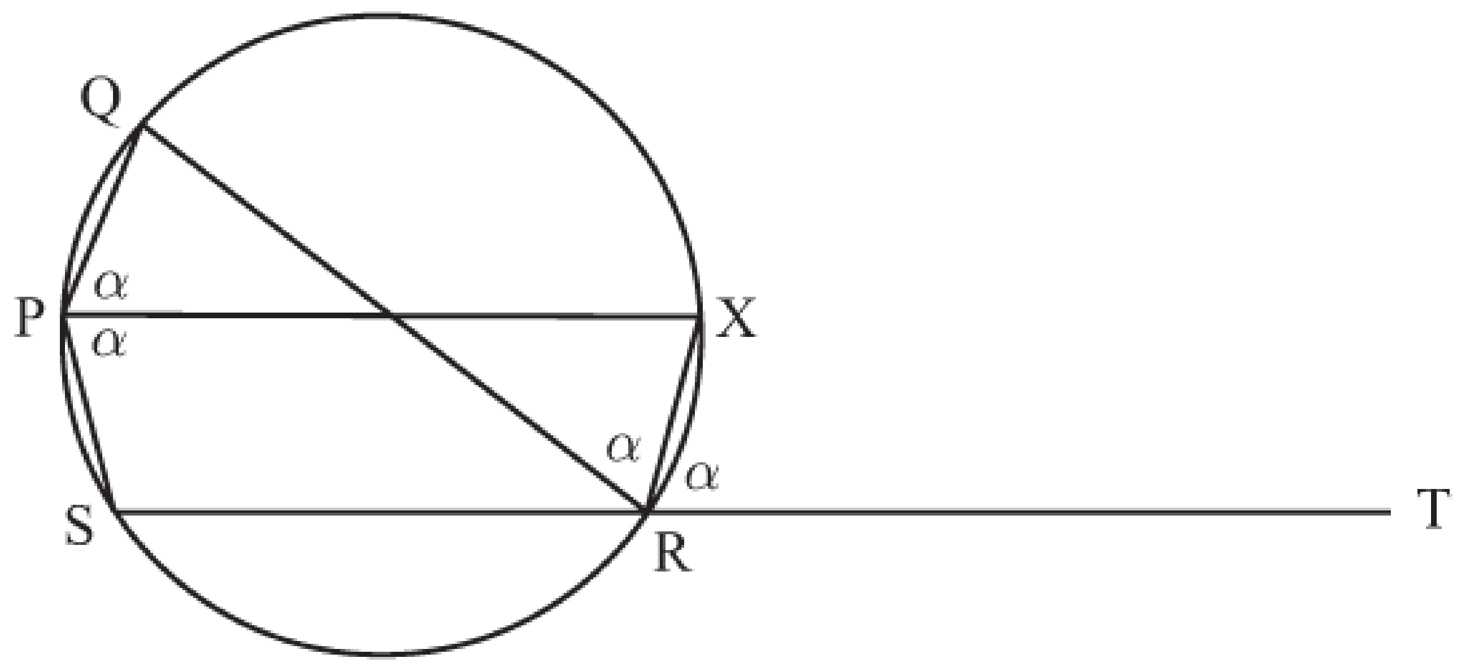
$\therefore\ \widehat{AYX} = \alpha$
and $\widehat{BYX} = \beta$
{angle between tangent and chord}

$\widehat{ACB} = 180° - \alpha - \beta$ {angles of a triangle}

$\therefore\ \widehat{AYB} + \widehat{ACB} = \alpha + \beta + 180° - \alpha - \beta = 180°$

$\therefore$ AYBC is a cyclic quadrilateral
{converse of angles of a cyclic quadrilateral theorem}

10



We join [PX].

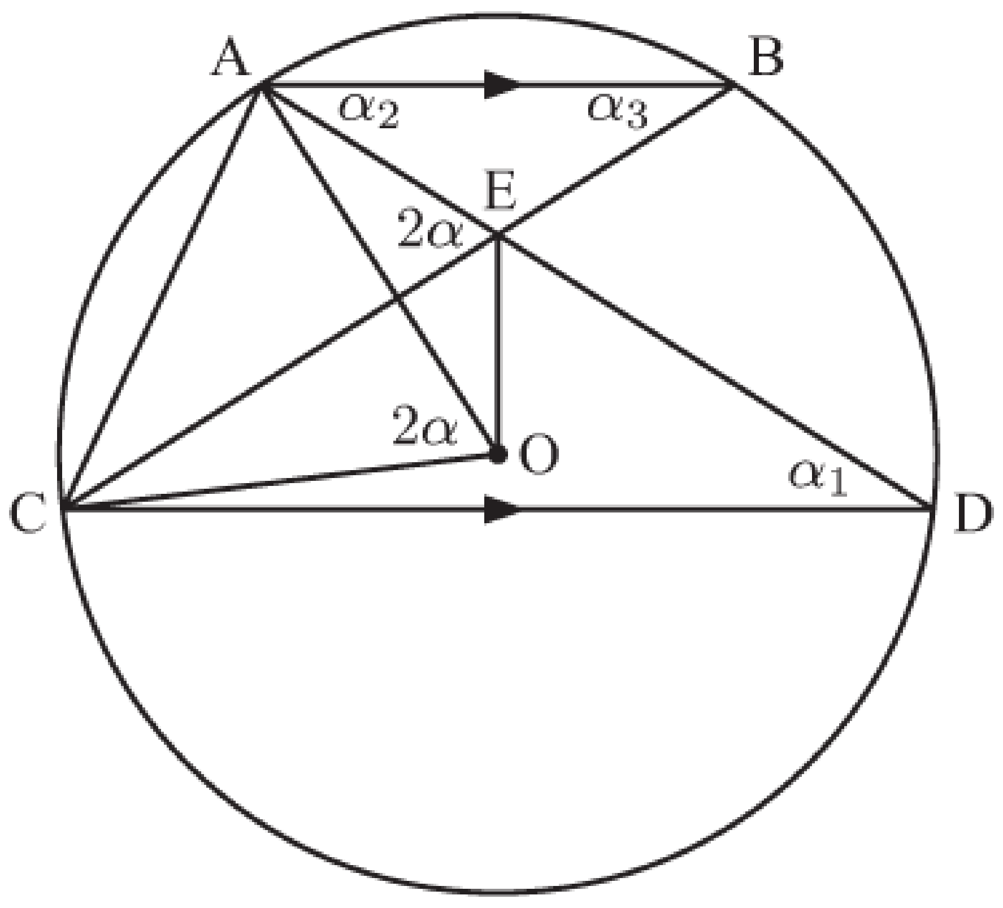
$\widehat{QPX} = \alpha$ {angles subtended by the same arc}

$\widehat{SPX} = \alpha$ {exterior angle of a cyclic quadrilateral}

$\therefore\ \widehat{QPX} = \widehat{SPX}$

$\therefore$ [PX] bisects $\widehat{QPS}$.

11



We join [OE], [OC], [AC], and [OA].

Let $\widehat{ADC} = \alpha_1$

$\therefore\ \alpha_1 = \alpha_2$ {equal alternate angles}

$\therefore\ \alpha_1 = \alpha_3$ {angles subtended by the same arc}

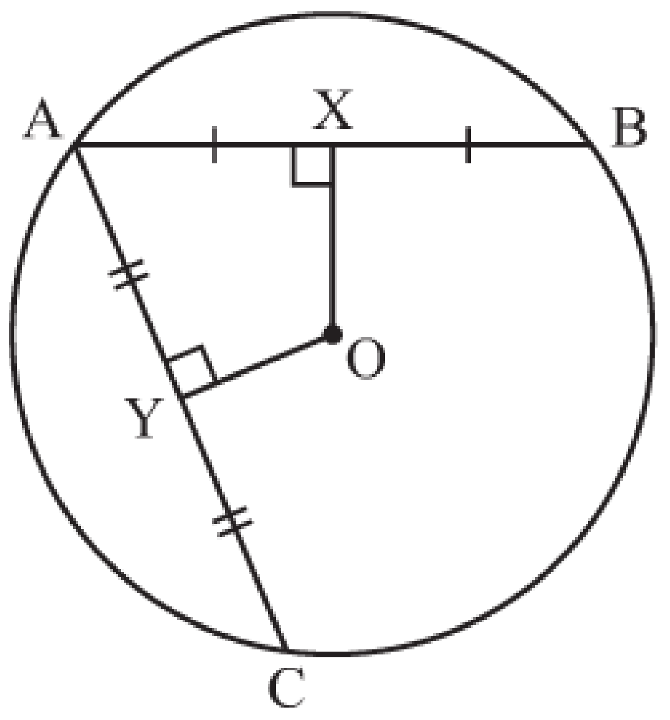
$\therefore\ \widehat{AEC} = \alpha_2 + \alpha_3 = 2\alpha$ {exterior angle of $\triangle ABE$}

$\widehat{AOC} = 2\alpha_1 = 2\alpha$ {angle at the centre}

$\therefore$ [AC] subtends equal angles at E and O

$\therefore$ A, E, O, and C are concyclic {test for concyclic points}

12



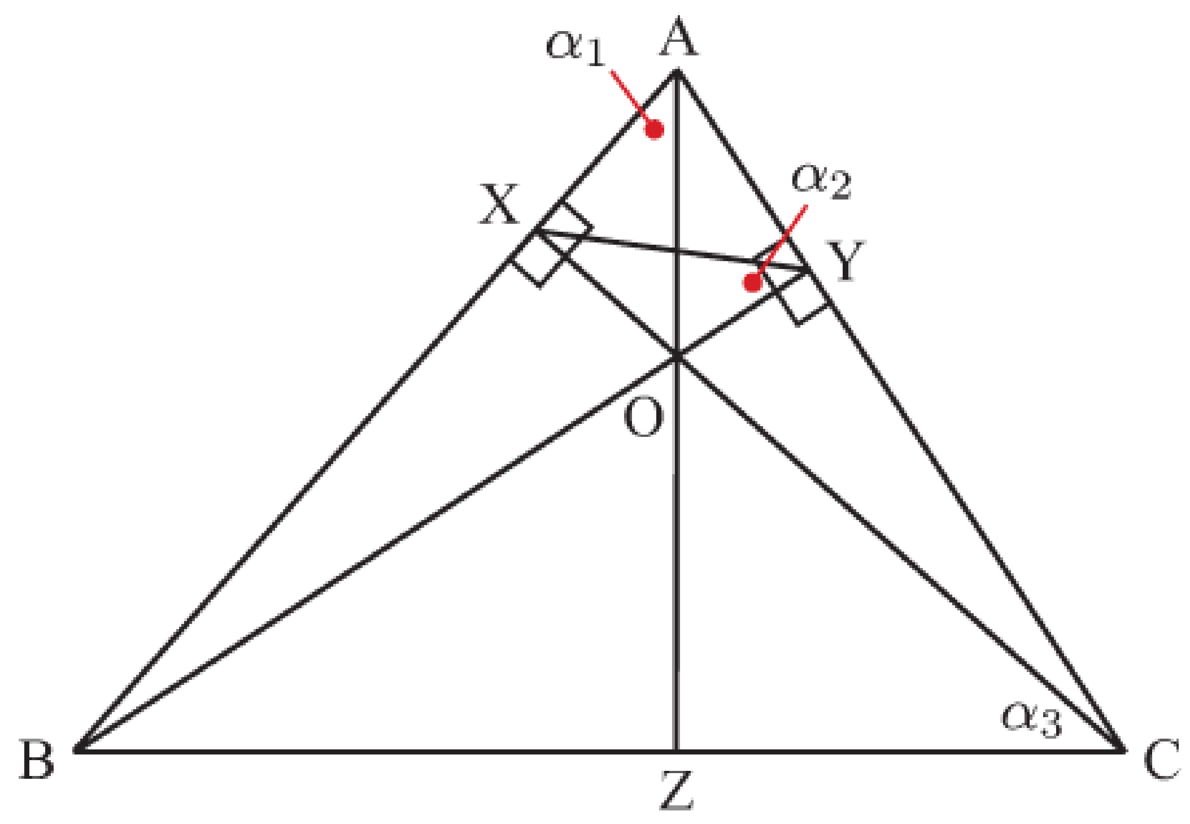
We join [OX] and [OY].

$\widehat{OXA} = \widehat{OYA} = 90°$
{converse 1 of chord of a circle}

$\therefore\ \widehat{OXA} + \widehat{OYA} = 180°$

$\therefore$ O, X, A, and Y are concyclic points {test for concyclic points}

13



a In AXOY, $\widehat{AXO} + \widehat{AYO} = 90° + 90° = 180°$

$\therefore$ AXOY is a cyclic quadrilateral
{converse of opposite angles of a cyclic quadrilateral}

Also, [BC] subtends equal angles at X and Y.

$\therefore$ BXYC is a cyclic quadrilateral
{test for cyclic quadrilaterals}

b Let $\widehat{XAO} = \alpha_1$, $\therefore\ \alpha_1 = \alpha_2$ and $\alpha_2 = \alpha_3$
{angles subtended by the same arc}

$\therefore\ \widehat{XAO} = \widehat{XYO} = \widehat{XCB}.$

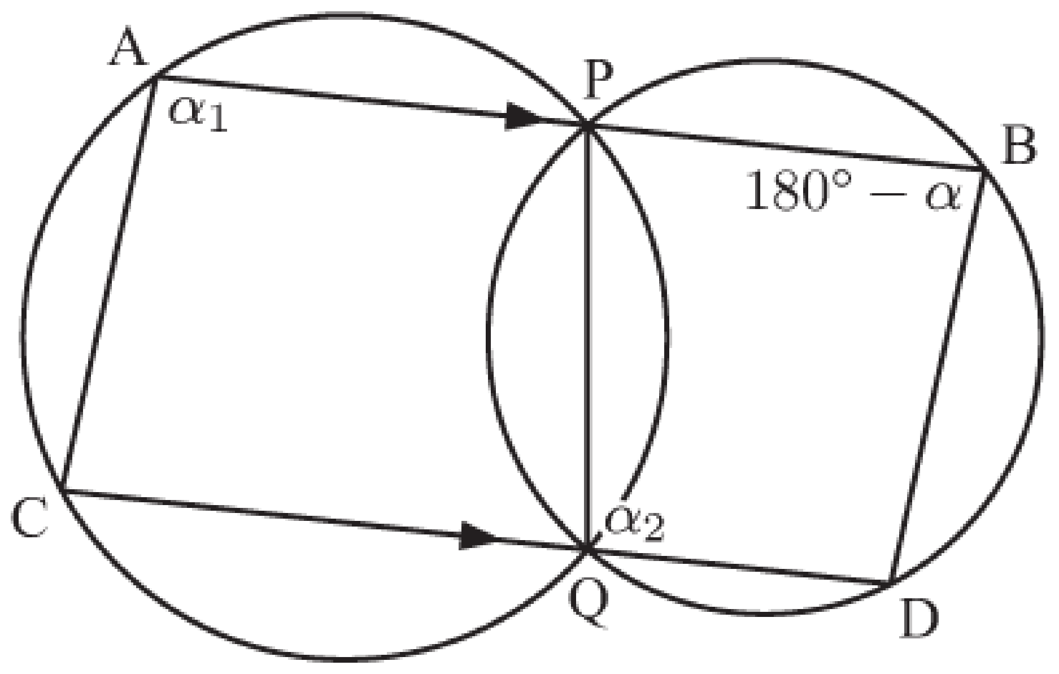
c [XZ] subtends equal angles at A and C $\{\alpha_1 = \alpha_3\}$,

$\therefore$ XZCA is a cyclic quadrilateral
{test for cyclic quadrilaterals}

$\therefore\ \widehat{CZA} = \widehat{CXA} = 90°$
{angles subtended by the same arc}

$\therefore$ [AZ] $\perp$ [BC].

14



We join [PQ], [AC], and [BD], and let $\widehat{PAC} = \alpha_1$.

$\alpha_1 = \alpha_2$ {exterior angle of a cyclic quadrilateral}

$\therefore\ \widehat{PBD} = 180° - \alpha$ {opposite angles of a cyclic quadrilateral}

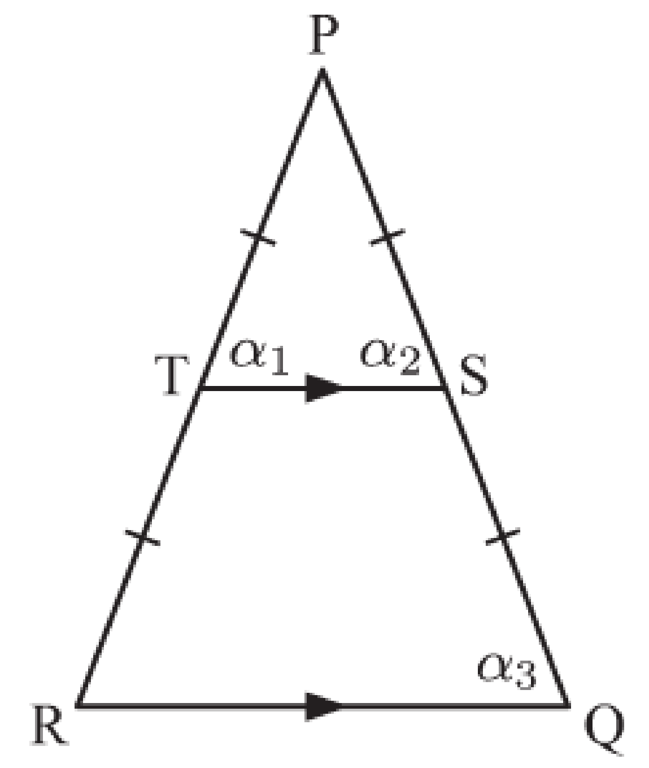
$\therefore\ \widehat{CAB} + \widehat{DBA} = \alpha + 180° - \alpha = 180°$

$\therefore$ [CA] $\parallel$ [DB] {converse of supplementary allied angles}

$\therefore$ ABDC is a parallelogram

$\therefore$ AB = CD {parallelogram theorem}

15



S and T are the midpoints of [PQ] and [PR]

$\therefore$ [ST] $\parallel$ [QR] {midpoint theorem}

Let $P\widehat{T}S = \alpha_1$.

$\alpha_1 = \alpha_2$ {isosceles triangle theorem}

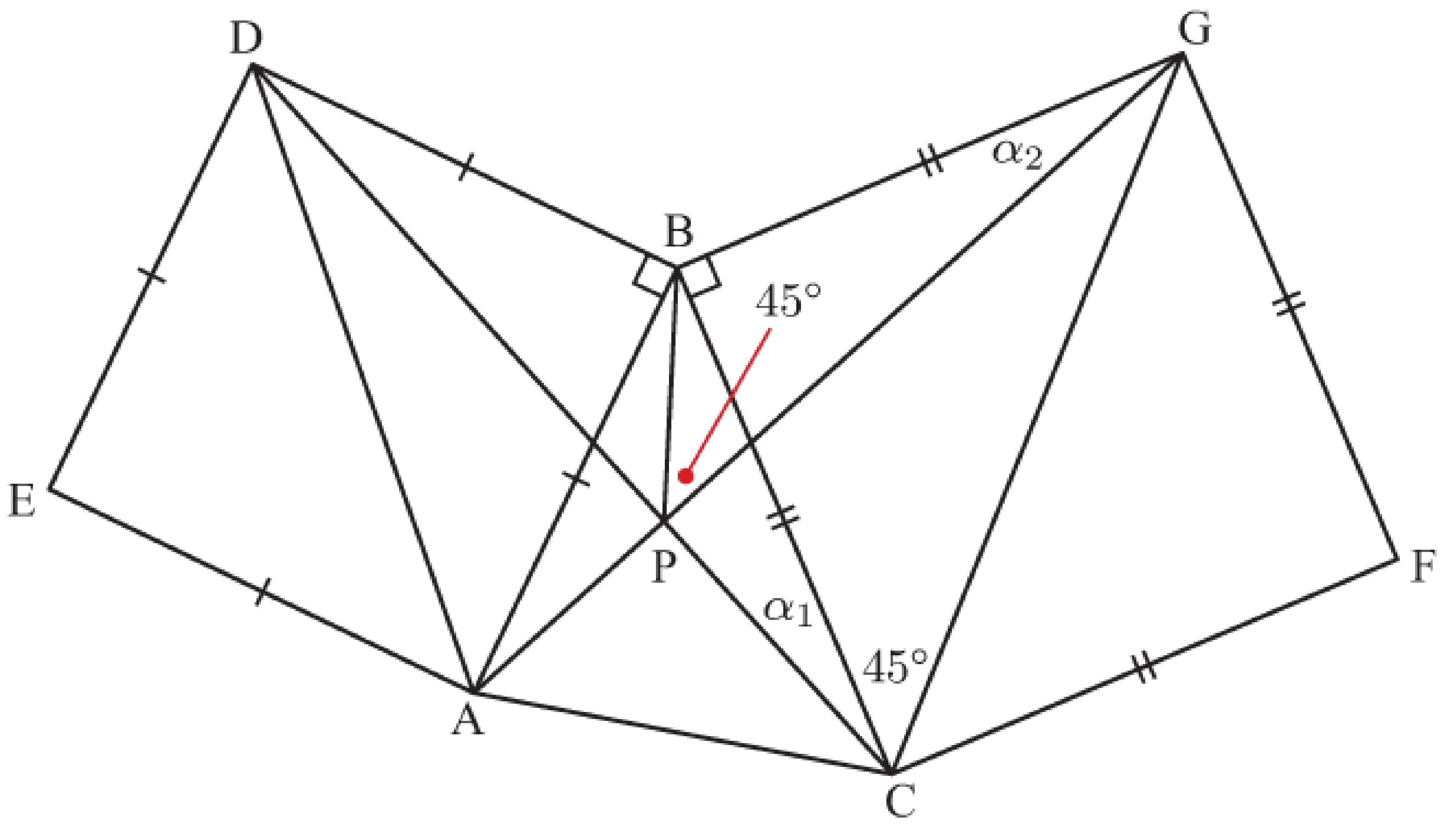
$\alpha_2 = \alpha_3$ {equal corresponding angles}

$\therefore$ $\alpha_1 = \alpha_3$

$\therefore$ STRQ is a cyclic quadrilateral
{converse of exterior angle of a cyclic quadrilateral}

$\therefore$ S, Q, R, and T are concyclic.

16



a In $\triangle$s BDC and ABG, we observe that:

- DB = AB {sides of square ABDE}
- BC = BG {sides of square CBGF}
- $D\widehat{B}C = A\widehat{B}G = 90° + A\widehat{B}C$

$\therefore$ $\triangle$s DBC and ABG are congruent {SAS}

Consequently, $\alpha_1 = \alpha_2$

$\therefore$ BP subtends equal angles at G and C

$\therefore$ B, G, C, and P are concyclic {test for concyclic points}

b Consequently, GC subtends equal angles at B and P.

$\therefore$ $G\widehat{P}C = 90°$

$\therefore$ [DC] $\perp$ [AG]

c Join [AD], [CG], and [BP].

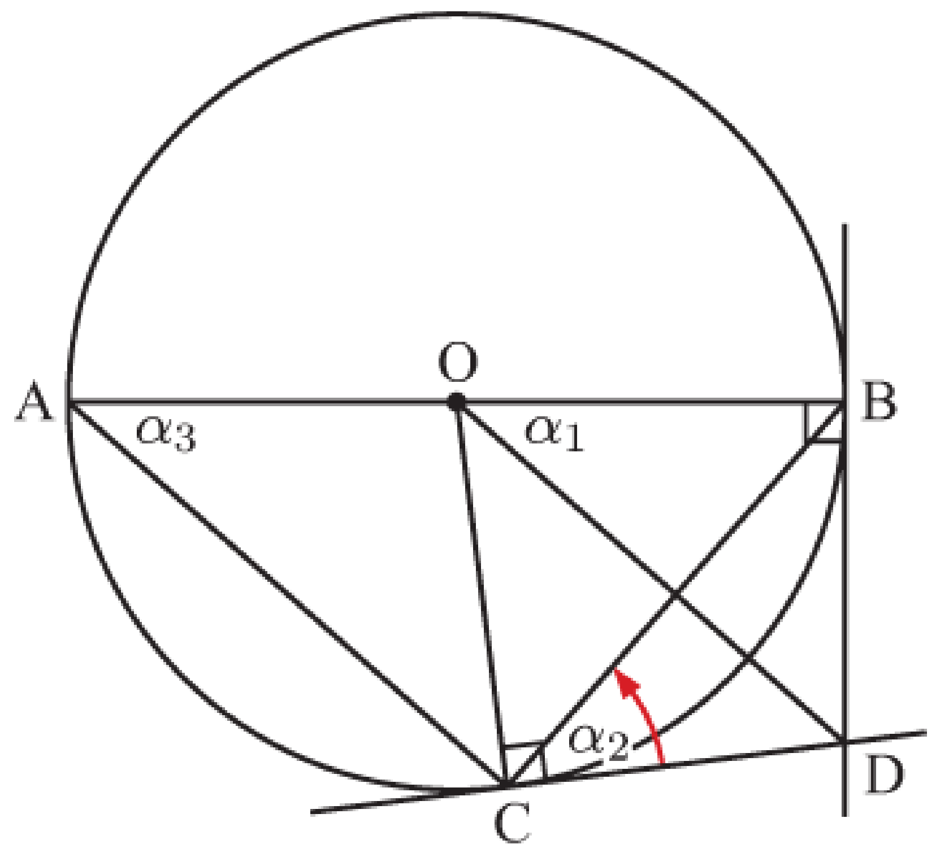
$G\widehat{C}B = 45°$ {CG is a diagonal of square}

and so $B\widehat{P}G = 45°$ {angles subtended by the same arc}

By similar reasoning as in **a**, APBD is a cyclic quadrilateral and BPD = 45°

$\therefore$ BP bisects $D\widehat{P}G$.

17



We join [OC] and [BC], and let $B\widehat{O}D = \alpha_1$.

$O\widehat{B}D = OCD = 90°$ {radius-tangent}

$\therefore$ OBDC is a cyclic quadrilateral
{converse of opposite angles of a cyclic quadrilateral}

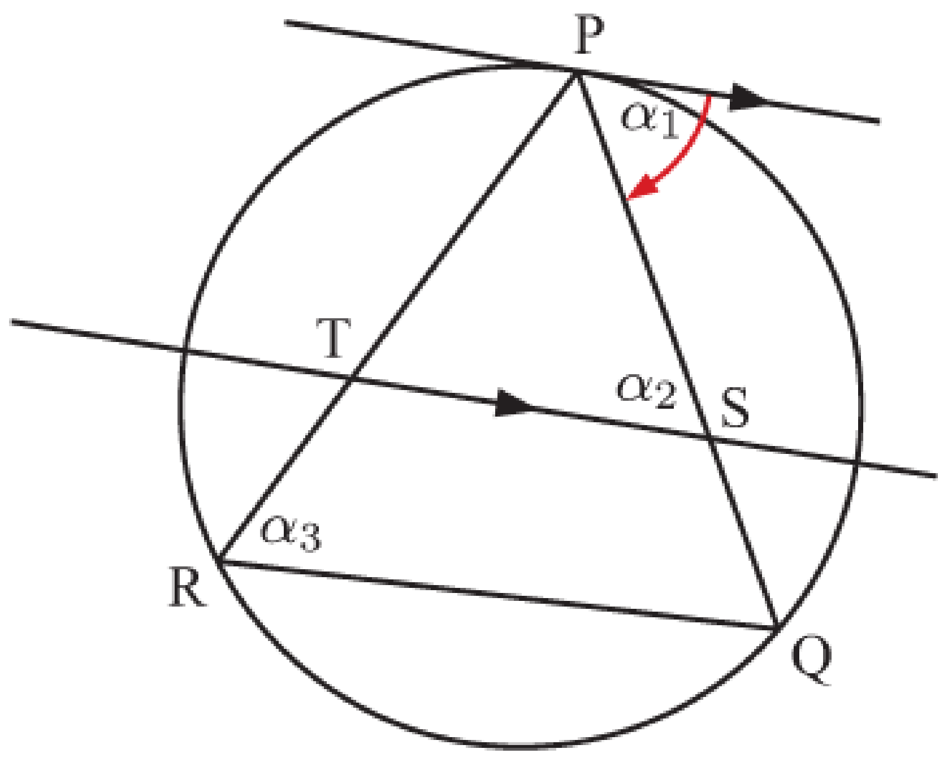
$\therefore$ $\alpha_1 = \alpha_2$ {angles subtended by the same arc}

$\alpha_2 = \alpha_3$ {angle between tangent and chord}

$\therefore$ $\alpha_1 = \alpha_3$

$\therefore$ [OD] $\parallel$ [AC] {converse of equal corresponding angles}

18



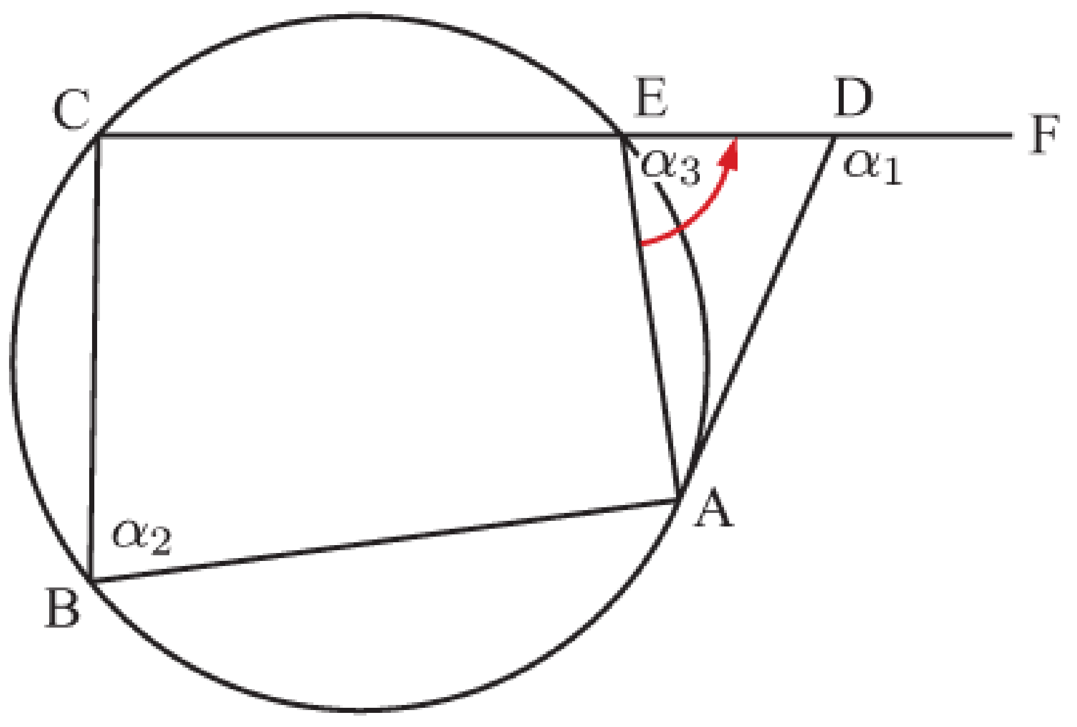
$\alpha_1 = \alpha_2$ {equal alternate angles}

and $\alpha_1 = \alpha_3$ {angle between tangent and chord}

$\therefore$ $\alpha_2 = \alpha_3$

$\therefore$ SQRT is a cyclic quadrilateral
{converse of exterior angle of a cyclic quadrilateral}

19



Let ABCD be a quadrilateral where an exterior angle is equal to the interior opposite angle.

$\therefore$ $\alpha_1 = \alpha_2$

We now draw a circle through A, B, C, and E where E is on [CD] or [CD] produced.

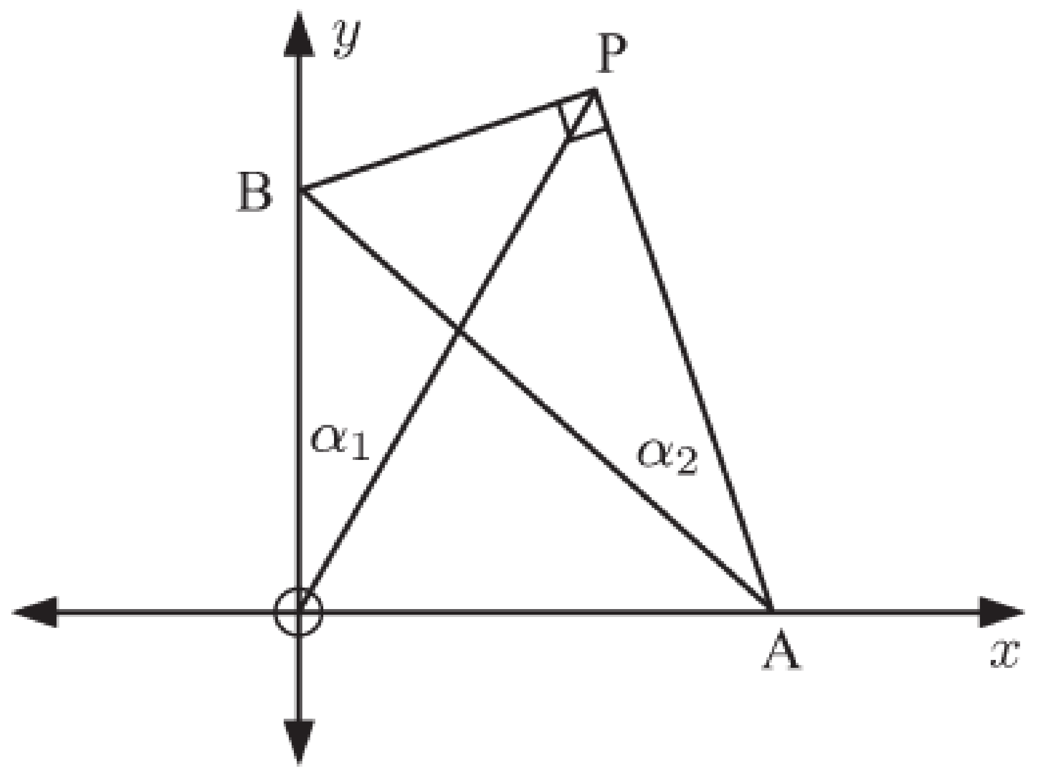
However $\alpha_2 = \alpha_3$ {exterior angle of a cyclic quadrilateral}

$\therefore$ $\alpha_1 = \alpha_3$

$\therefore$ [AE] $\parallel$ [AD] {converse of equal corresponding angles}
which is possible only if E and D coincide.

$\therefore$ ABCD is a cyclic quadrilateral.

20



As the axes are at right angles, OAPB is a cyclic quadrilateral no matter where the set square moves. {converse of opposite angles of a cyclic quadrilateral}

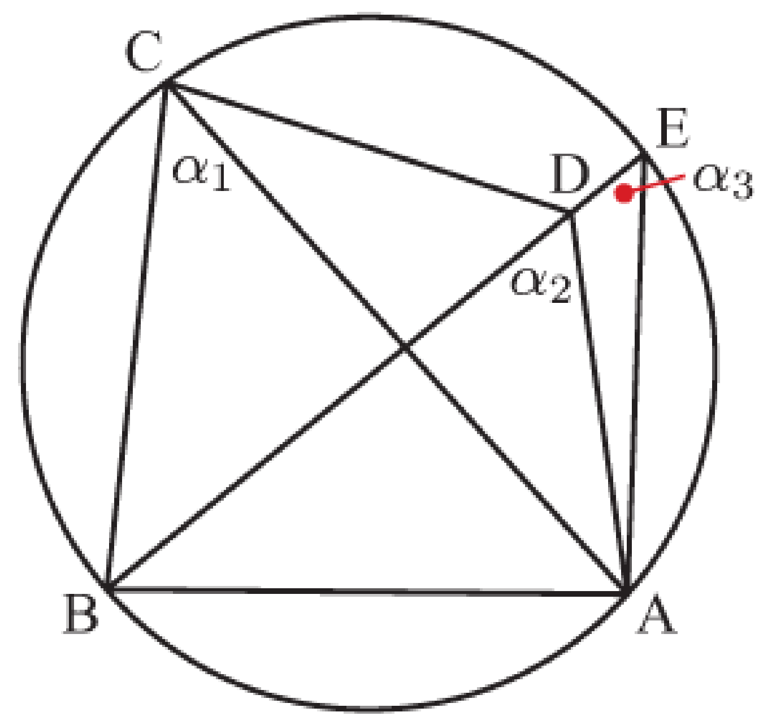
We join OP.

Now $\alpha_1 = \alpha_2$ {angles subtended by the same arc}

As α_2 is fixed, α_1 must also be fixed.

$\therefore$ P lies on a straight line segment through O.

21



For the quadrilateral ABCD where [AB] subtends equal angles at C and D we draw a circle through A, B, and C. Let E be on [BD] or [BD] produced. (The latter case is shown.)

Let $\widehat{ACB} = \alpha_1$.

$\alpha_1 = \alpha_2$ {given}

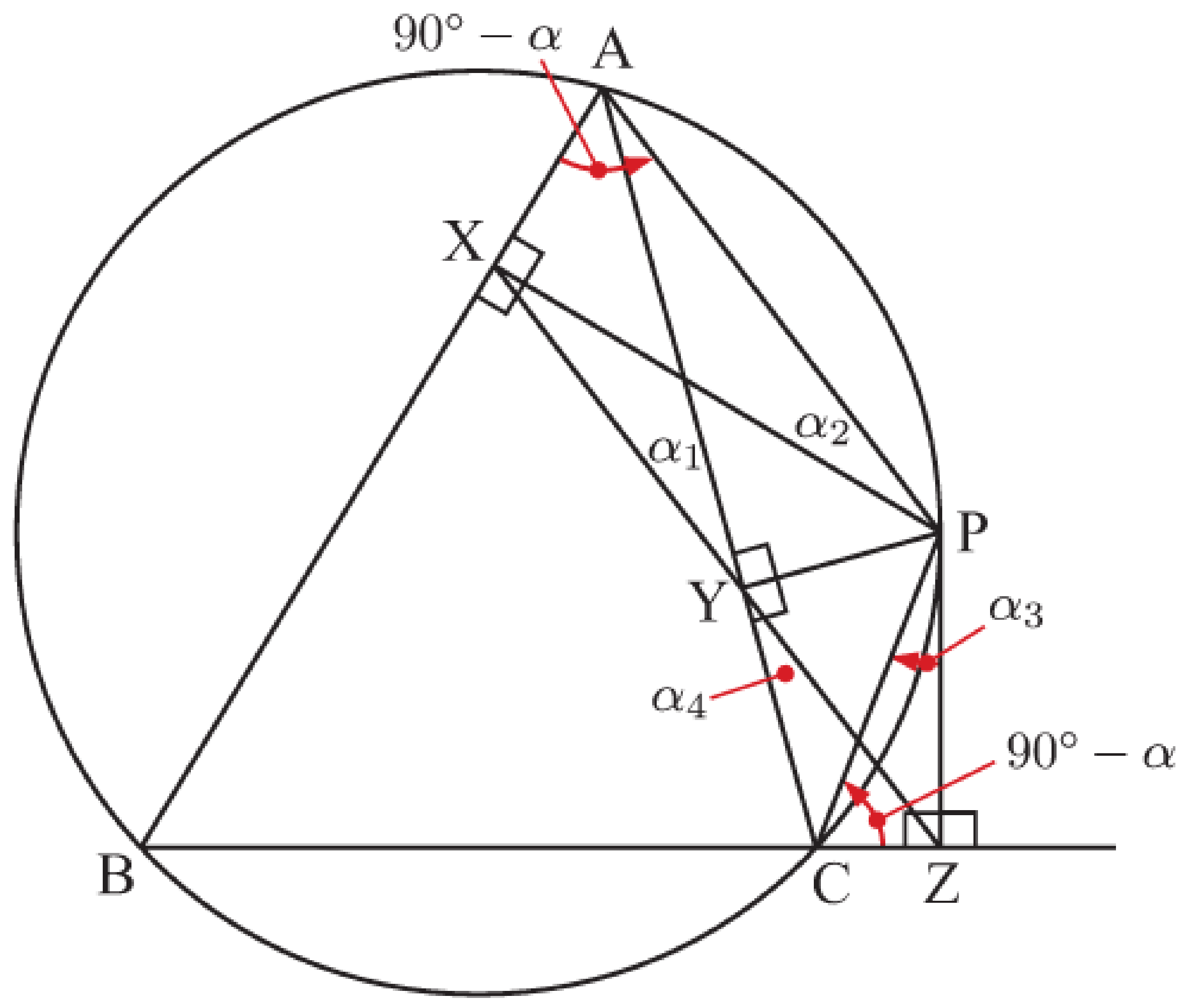
But $\alpha_1 = \alpha_3$ {angles subtended by the same arc}

$\therefore\ \alpha_2 = \alpha_3$

$\therefore$ [DA] $\parallel$ [EA] {converse of equal corresponding angles} and this is only possible if D and E coincide.

Thus, A, B, C, and D are concyclic.

22



We join [XY], [YZ], [AP], and [PC], and let $\widehat{XYA} = \alpha_1$.

As [AP] subtends equal angles at X and Y, APYX is a cyclic quadrilateral. {test for cyclic quadrilaterals}

$\therefore\ \alpha_1 = \alpha_2$ {angles subtended by the same arc}

In $\triangle$PAX, $\widehat{XAP} = 90° - \alpha$ {angles of a triangle}

But APCB is a cyclic quadrilateral

$\therefore\ \widehat{ZCP} = 90° - \alpha$ {exterior angle of a cyclic quadrilateral}

$\therefore\ \widehat{ZPC} = \alpha_3$ {angles of a triangle}

But ZPYC is a cyclic quadrilateral

{converse of opposite angles of a cyclic quadrilateral}

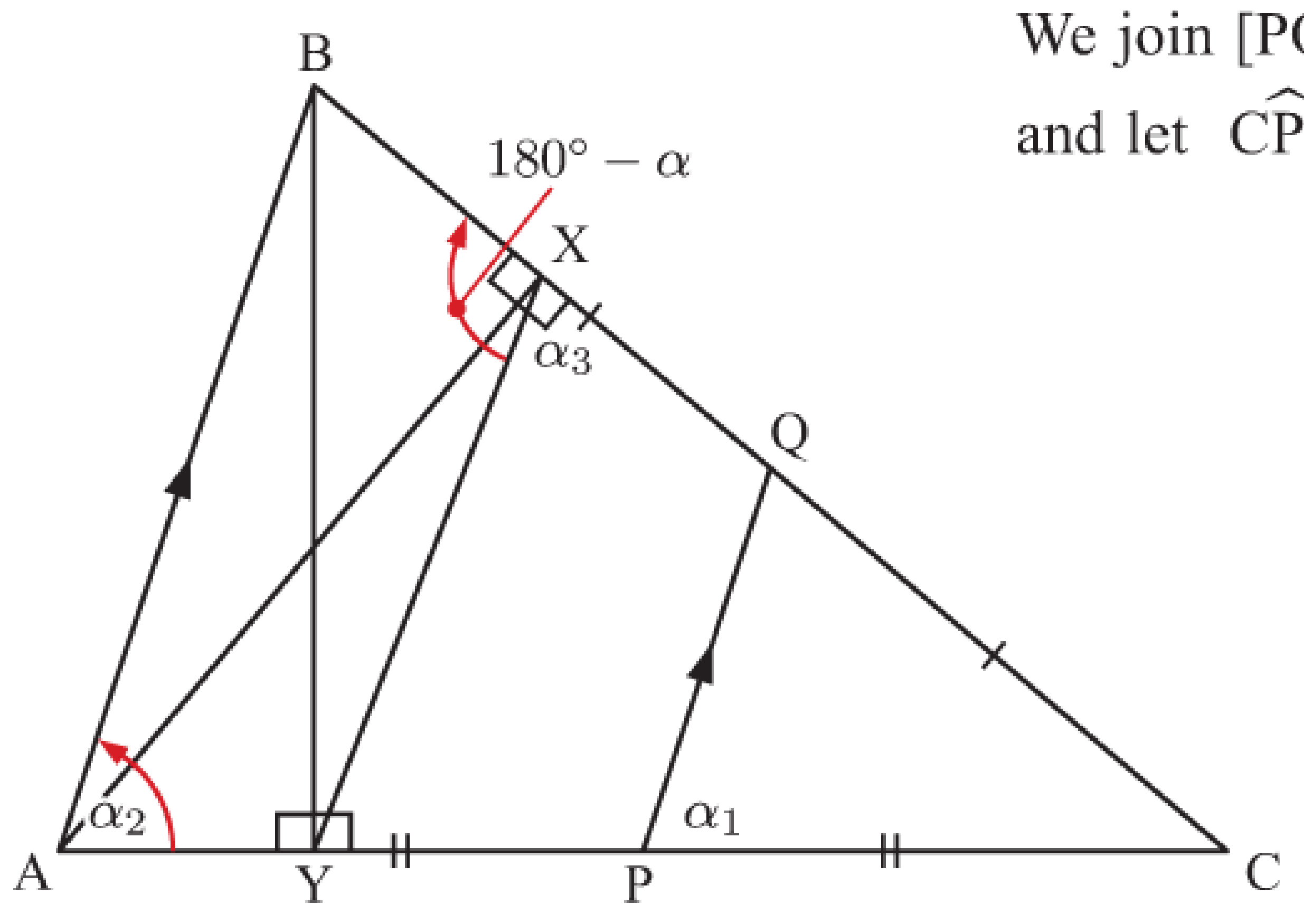
$\therefore\ \alpha_3 = \alpha_4$ {angles subtended by the same arc}

$\therefore\ \alpha_1 = \alpha_4$

But AYC is a fixed straight line.

Hence X, Y, and Z are collinear.

23



We join [PQ] and [XY], and let $\widehat{CPQ} = \alpha_1$.

[PQ] $\parallel$ [AB] {midpoint theorem}

$\therefore\ \alpha_1 = \alpha_2$ {equal corresponding angles}

Now [AB] subtends equal angles at X and Y (90°)

$\therefore$ ABXY is a cyclic quadrilateral
{test for cyclic quadrilaterals}

$\therefore\ \widehat{YXB} = 180° - \alpha$ {opposite angles of a cyclic quadrilateral}

$\therefore\ \widehat{YXQ} = \alpha_3$ {angles on a line}

In XQPY, $\alpha_1 = \alpha_3$

$\therefore$ XQPY is a cyclic quadrilateral.
{converse of exterior angle of a cyclic quadrilateral}

EXERCISE 2F

1 **a** By the intersecting chords theorem,

$$x \times 6 = 3 \times 7$$
$$\therefore\ 6x = 21$$
$$\therefore\ x = 3.5$$

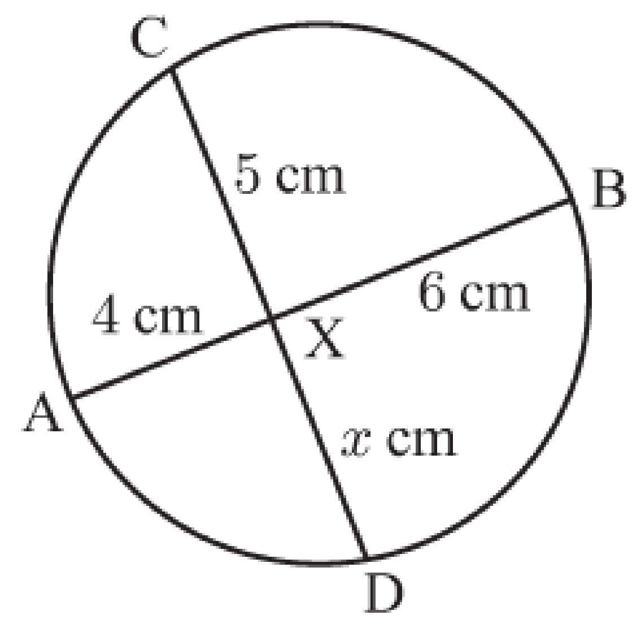
b By the secant-secant theorem,

$$x(x+5) = 2 \times (2+8)$$
$$\therefore\ x^2 + 5x = 2 \times 10$$
$$\therefore\ x^2 + 5x - 20 = 0$$
$$\therefore\ x = \frac{-5 \pm \sqrt{25+80}}{2}$$
$$\therefore\ x = \frac{-5 + \sqrt{105}}{2} \quad \{\text{as } x > 0\}$$
$$\therefore\ x \approx 2.62$$

c By the secant-tangent theorem,

$$x(x+8) = 5^2$$
$$\therefore\ x^2 + 8x - 25 = 0$$
$$\therefore\ x = \frac{-8 \pm \sqrt{64+100}}{2}$$
$$\therefore\ x = \frac{-8 \pm 2\sqrt{41}}{2}$$
$$\therefore\ x = -4 + \sqrt{41} \quad \{\text{as } x > 0\}$$
$$\therefore\ x \approx 2.40$$

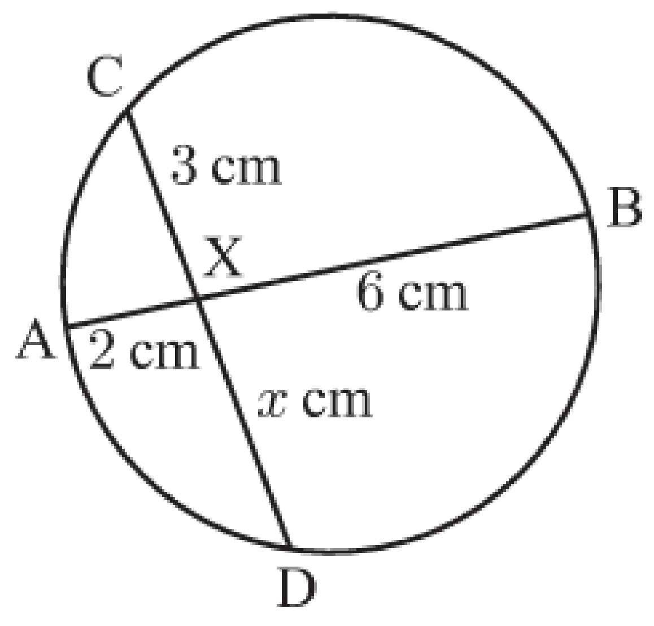
2 **a**



By the intersecting chords theorem,

$$5 \times x = 4 \times 6$$
$$\therefore\ 5x = 24$$
$$\therefore\ x = 4.8$$
$$\therefore\ \text{DX} = 4.8 \text{ cm}$$

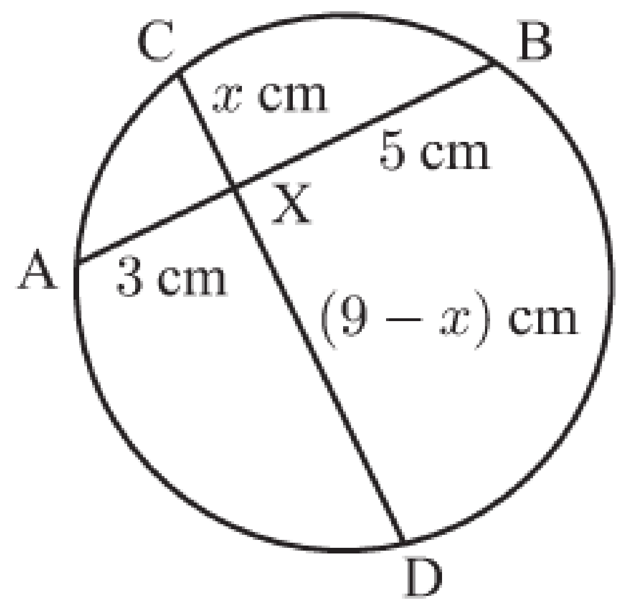
b



By the intersecting chords theorem,

$$3 \times x = 2 \times 6$$
$$\therefore\ 3x = 12$$
$$\therefore\ x = 4$$
$$\therefore\ \text{CD} = 7 \text{ cm}$$

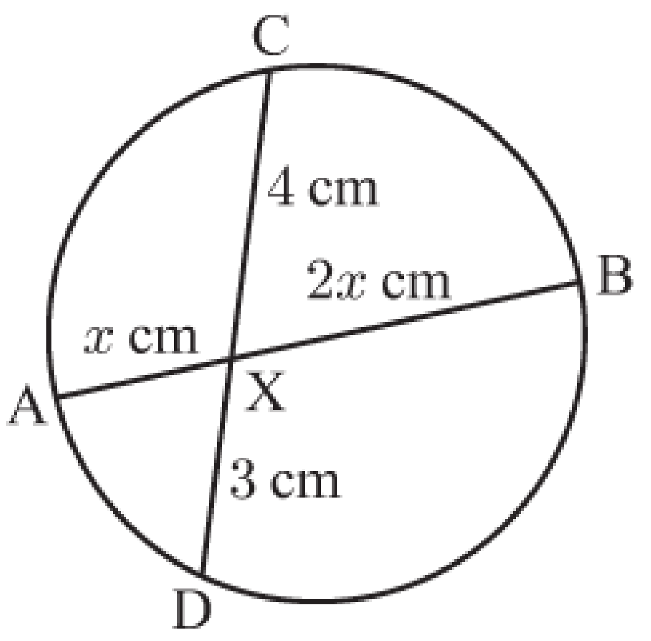
c



By the intersecting chords theorem,

$$x(9-x) = 3 \times 5$$
$$\therefore\ 9x - x^2 = 15$$
$$\therefore\ x^2 - 9x + 15 = 0$$
$$\therefore\ x = \frac{9 \pm \sqrt{81-60}}{2}$$
$$\therefore\ x = \frac{9 \pm \sqrt{21}}{2}$$
$$\therefore\ \text{CX} = \frac{9 \pm \sqrt{21}}{2} \text{ cm}$$
$$\approx 6.79 \text{ cm or } 2.21 \text{ cm}$$

d



By the intersecting chords theorem,

$x \times 2x = 3 \times 4$

$\therefore\ 2x^2 = 12$

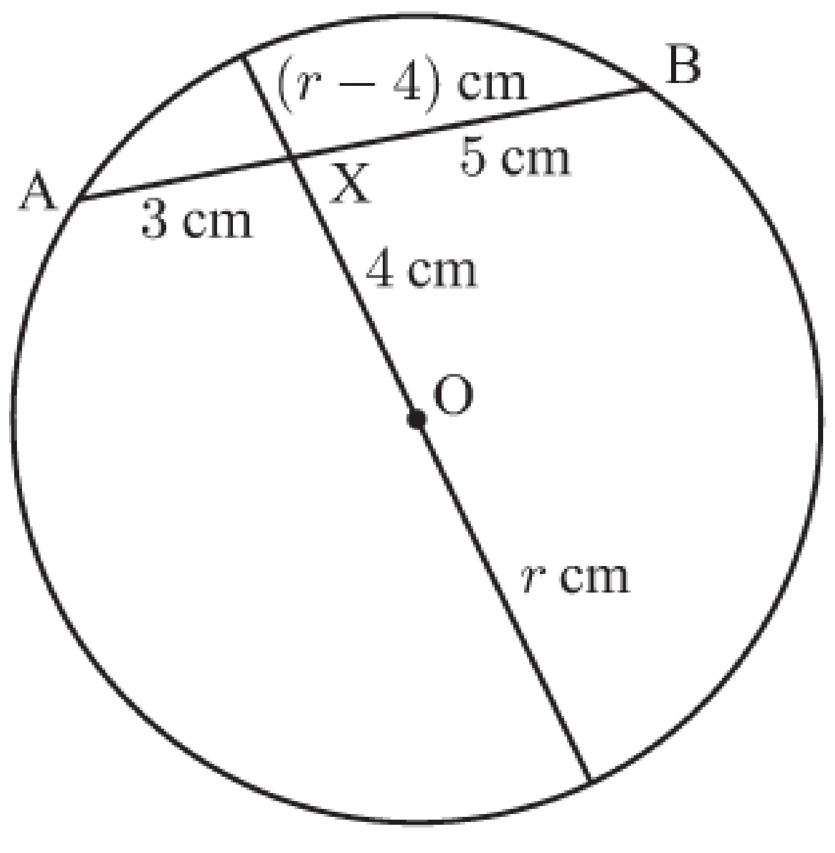
$\therefore\ x^2 = 6$

$\therefore\ x = \sqrt{6}$ {as $x > 0$}

$\therefore\ \text{AB} = 3\sqrt{6}$ cm

≈ 7.35 cm

3



Let the radius be r cm.
By the intersecting chords theorem,

$(r+4)(r-4) = 3 \times 5$

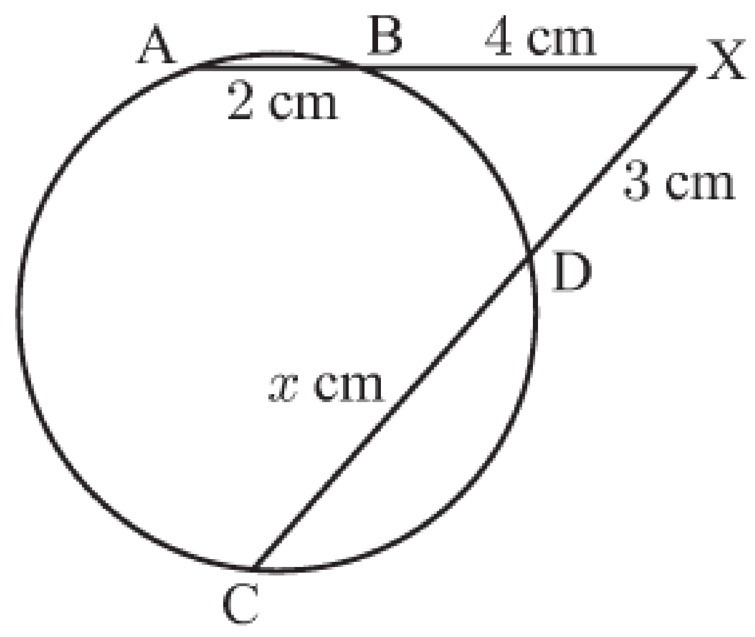
$\therefore\ r^2 - 16 = 15$

$\therefore\ r^2 = 31$

$\therefore\ r = \sqrt{31}$ {as $r > 0$}

$\therefore$ the radius is $\sqrt{31} \approx 5.57$ cm

4 **a**



By the secant-secant theorem,

$3(x+3) = 4 \times 6$

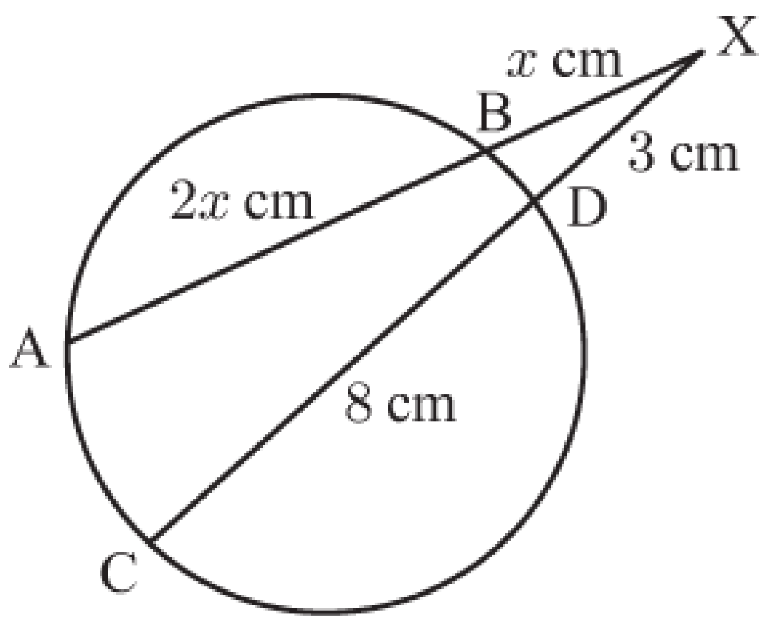
$\therefore\ 3(x+3) = 24$

$\therefore\ x+3 = 8$

$\therefore\ x = 5$

$\therefore\ \text{CD} = 5$ cm

b



Let $\text{BX} = x$ cm

$\therefore\ \text{AX} = 3x$ cm

$\therefore\ \text{AB} = 2x$ cm

By the secant-secant theorem,

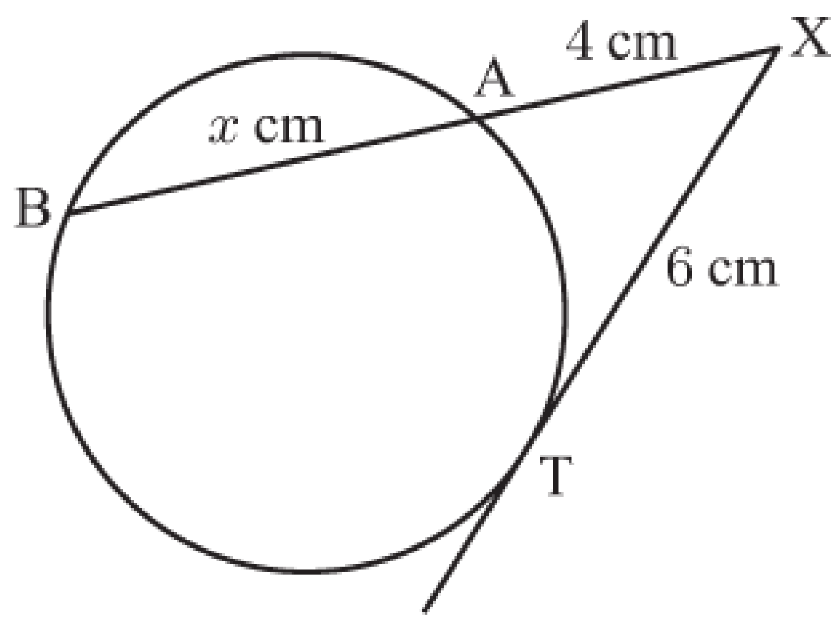
$x \times 3x = 3 \times 11$

$\therefore\ x^2 = 11$

$\therefore\ x = \sqrt{11}$ {as $x > 0$}

$\therefore\ \text{AB} = 2\sqrt{11} \approx 6.63$ cm

5 **a**



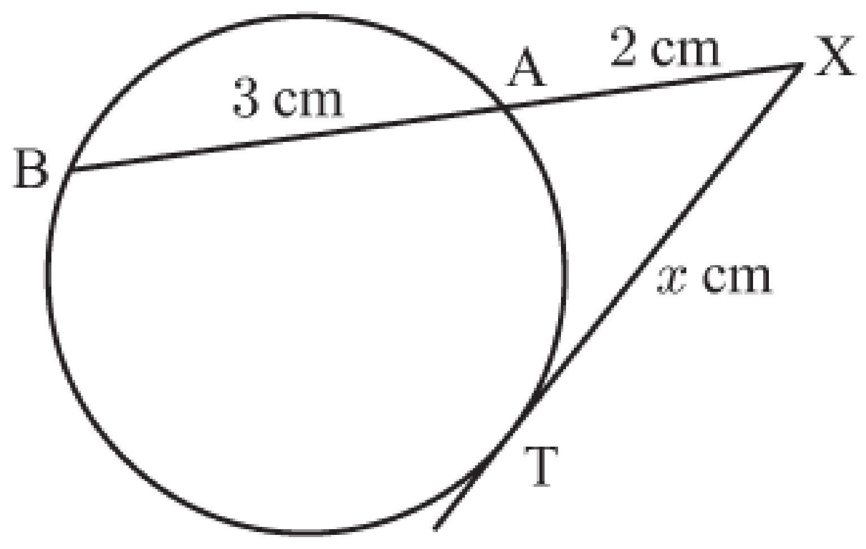
If $\text{AB} = x$ cm, by the secant-tangent theorem,

$6^2 = 4(4+x)$

$\therefore\ 4+x = 9$

$\therefore\ \text{BX} = 9$ cm

b



Let $\text{XT} = x$ cm.
By the secant-tangent theorem,

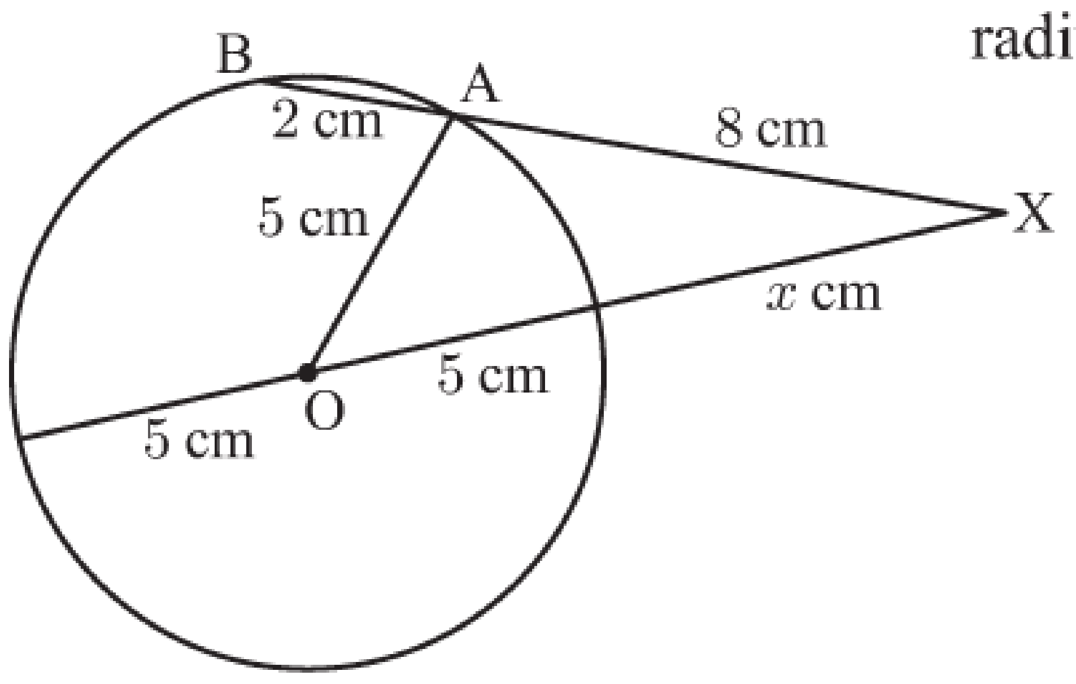
$x^2 = 2 \times 5$

$\therefore\ x = \sqrt{10}$

$\therefore\ \text{XT} = \sqrt{10}$

≈ 3.16 cm

c



radius = 5 cm

By the secant-secant theorem,

$x(x+10) = 8 \times 10$

$\therefore\ x^2 + 10x - 80 = 0$

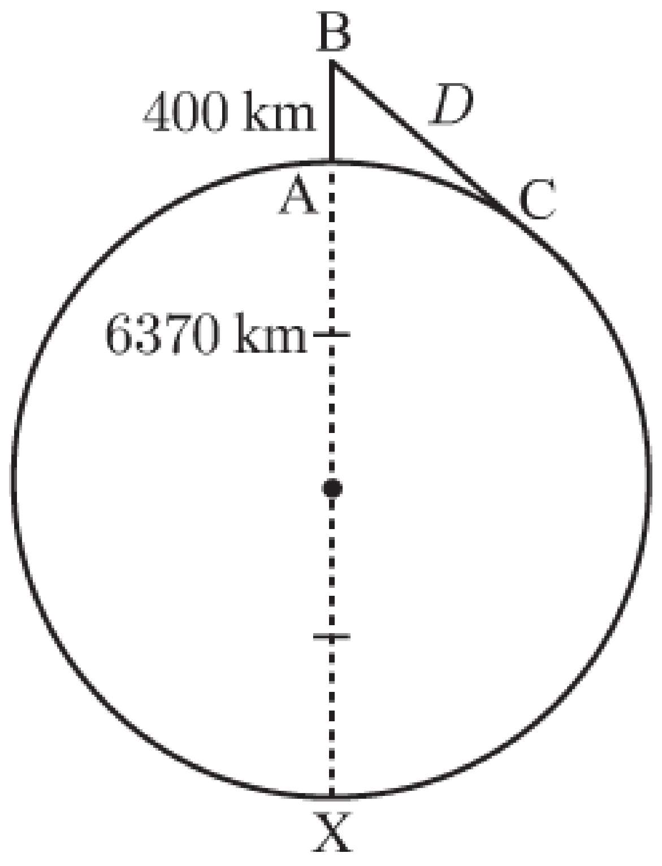
$\therefore\ x = \dfrac{-10 \pm \sqrt{100 - 4(1)(-80)}}{2}$

$\therefore\ x = \dfrac{-10 \pm \sqrt{420}}{2}$

$\therefore\ x = -5 + \sqrt{105}$ {as $x > 0$}

$\therefore\ \text{OX} = x + 5 = \sqrt{105} \approx 10.2$ cm

6 **a**



$\text{AB} = 400$ km
By the secant-tangent theorem,

$\text{BA.BX} = \text{BC}^2$

$\therefore\ \text{BC}^2 \approx 400 \times 13\,140$

$\therefore\ \text{BC}^2 \approx 5\,256\,000$

$\therefore\ \text{BC} \approx 2290$ km

$\therefore$ distance to visible horizon ≈ 2290 km.

b Let $\text{BC} = D$ km and $\text{AB} = h$ km.
By the secant-tangent theorem,

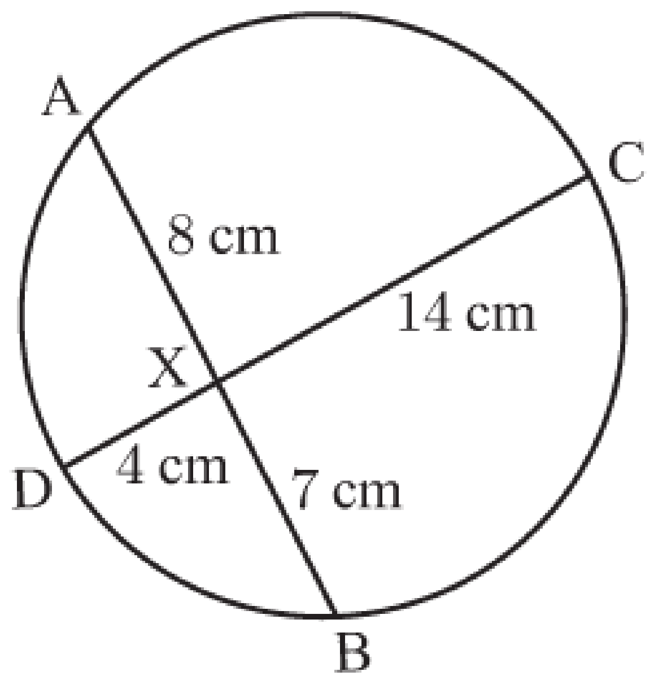
$D^2 \approx h(h + 12\,740)$

$\therefore\ D \approx \sqrt{h^2 + 12\,740h}$ km

7 By the secant-tangent theorem, $\text{CT}^2 = \text{CB.CA} = \text{CS}^2$

$\therefore\ \text{CT} = \text{CS}$

8 **a**



$\text{AX.BX} = 8 \times 7 = 56$

$\text{CX.DX} = 14 \times 4 = 56$

$\therefore\ \text{AX.BX} = \text{CX.DX}$

$\therefore$ A, B, C, and D are concyclic.
{converse of intersecting chords}

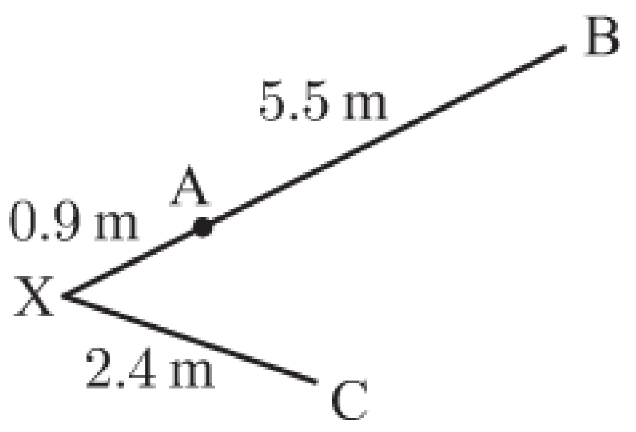
b $\text{AX.BX} = 5 \times 3.2 = 16$

$\text{CX.DX} = 8 \times 2 = 16$

$\therefore\ \text{AX.BX} = \text{CX.DX}$

$\therefore$ A, B, C, and D are concyclic.
{converse of intersecting chords}

9



$\text{XA.XB} = 0.9 \times 6.4$

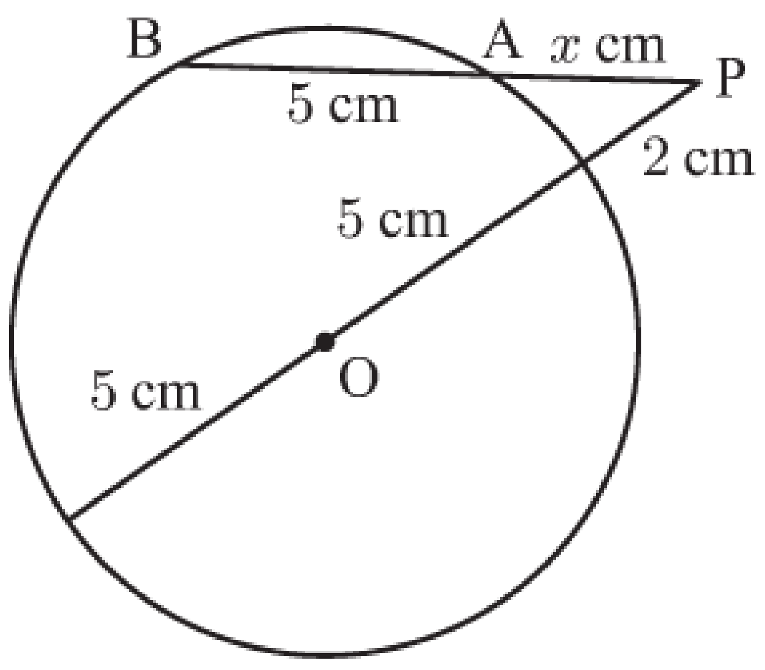
$= 5.76$

and $\text{XC}^2 = 2.4^2 = 5.76$

$\therefore\ \text{XC}^2 = \text{XA.XB}$

$\therefore$ [CX] is a tangent to the circle through A, B, C.
{converse of secant-tangent theorem}

10

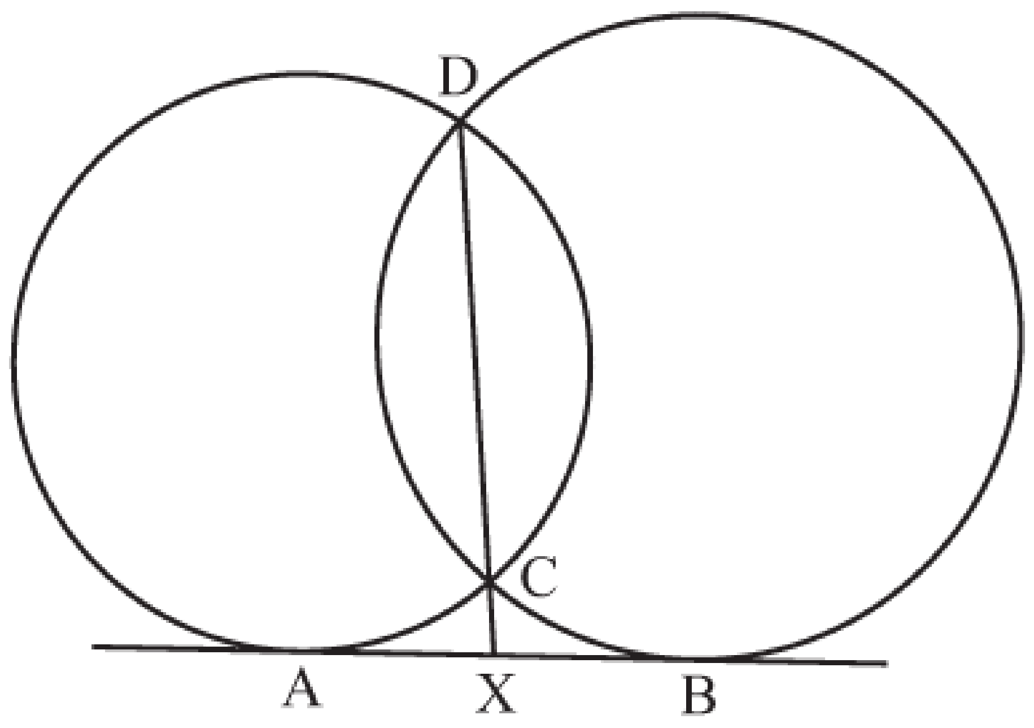


Let AP $= x$ cm.

By the secant-secant theorem,

$$x(x+5) = 2 \times 12$$
$$\therefore\ x^2 + 5x - 24 = 0$$
$$\therefore\ (x+8)(x-3) = 0$$
$$\therefore\ x = 3 \quad \{\text{as } x > 0\}$$
$$\therefore\ \text{AP} = 3 \text{ cm}$$

11

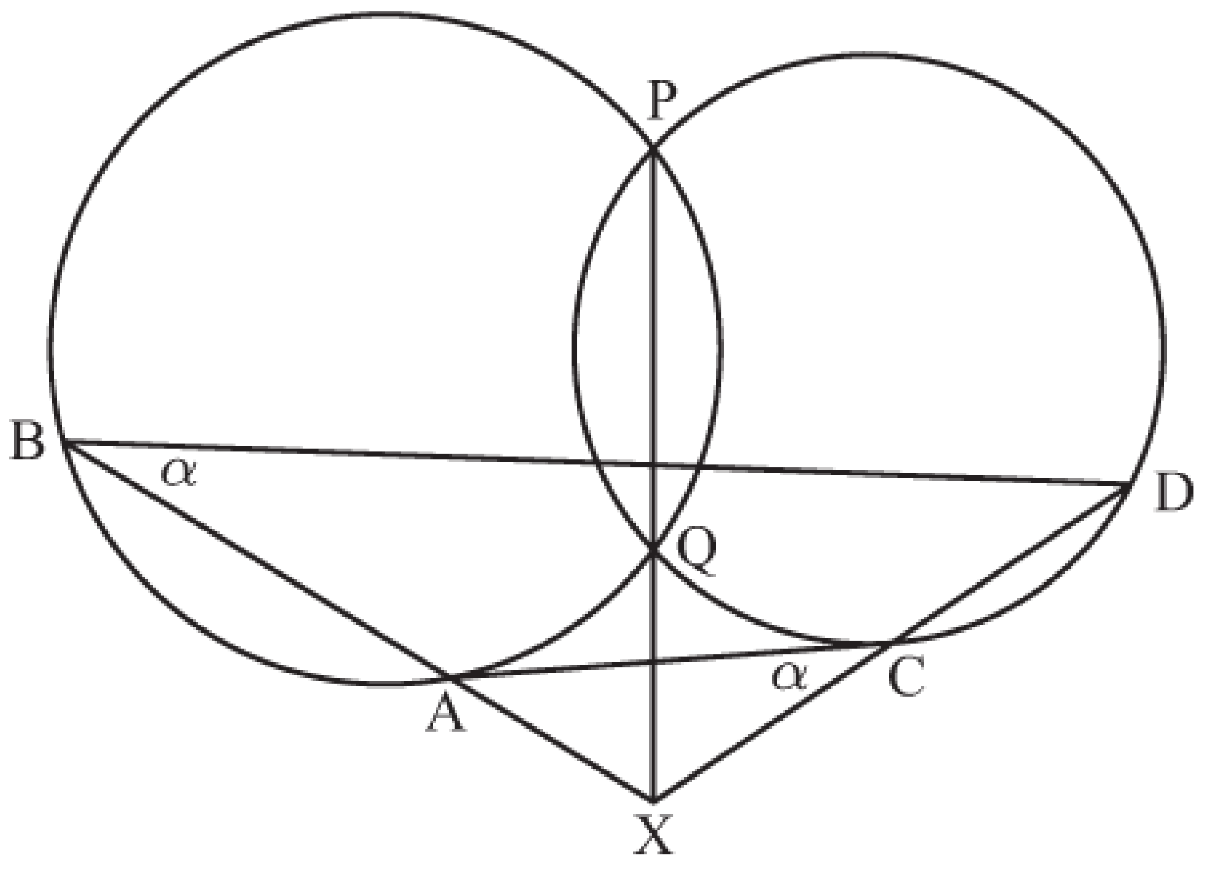


By the secant-tangent theorem, $\text{XA}^2 = \text{XC.XD} = \text{XB}^2$

$\therefore$ XA = XB

$\therefore$ X bisects [AB].

12



By the secant-secant theorem,

XA.XB = XQ.XP in one circle

and XC.XD = XQ.XP in the other

$\therefore$ XA.XB = XC.XD .... ($*$)

$\therefore$ in $\triangle$s AXC, DXB, $\dfrac{\text{XA}}{\text{XD}} = \dfrac{\text{XC}}{\text{XB}}$ {from $*$ }

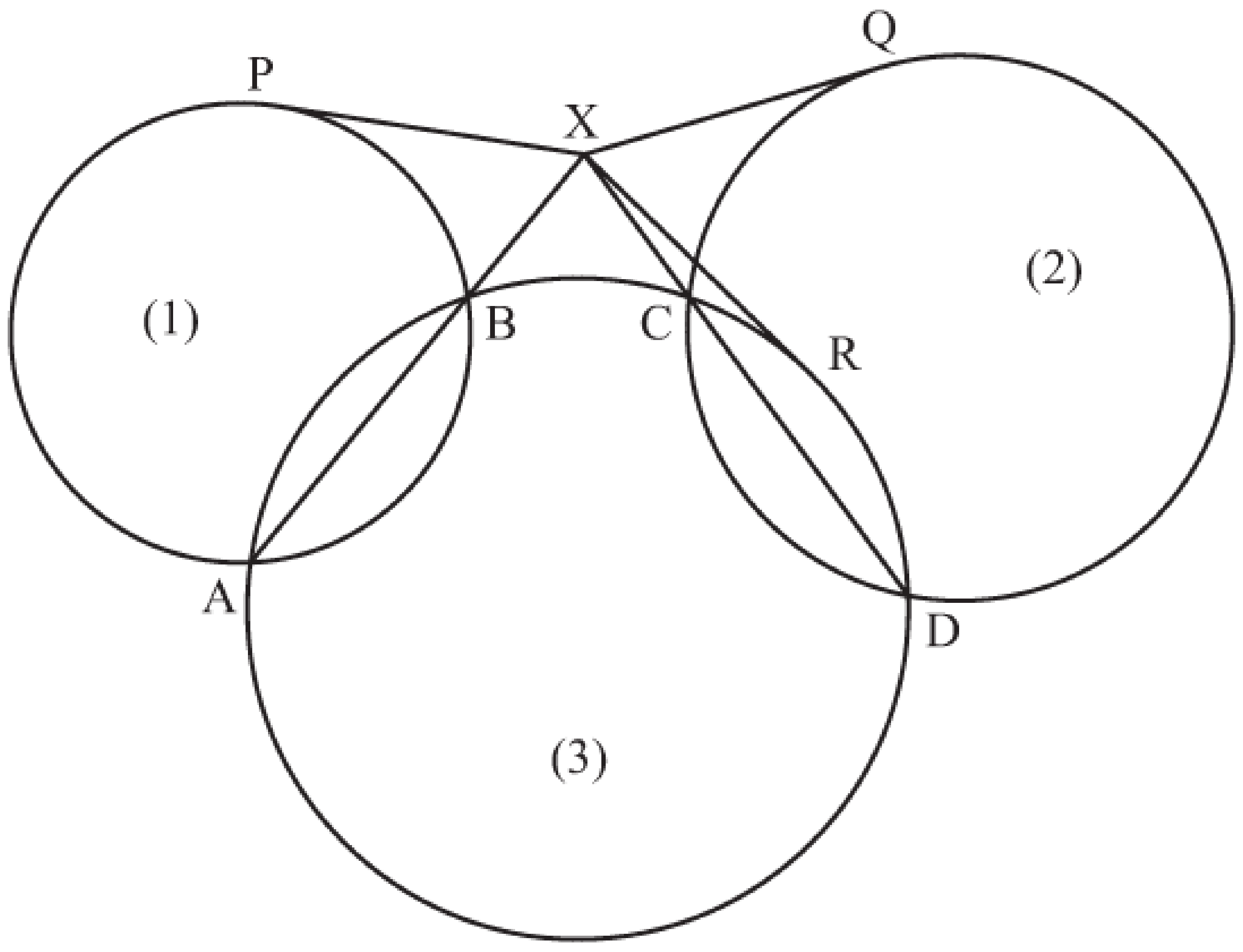
and $\widehat{\text{AXC}} = \widehat{\text{DXB}}$

$\therefore$ $\triangle$s AXC and DXB are similar {two sides of each triangle are in the same ratio and the included angles are equal}.

Consequently, $\widehat{\text{ACX}} = \widehat{\text{DBX}} = \alpha$, say

$\therefore$ ACDB is a cyclic quadrilateral.
{converse of exterior angle of a cyclic quadrilateral}

13



Let [XP], [XQ], and [XR] be the three tangents from X to the three circles.

By the secant-tangent theorem,

$\text{XP}^2 = \text{XB.XA}$ (in circle (1))

and $\text{XR}^2 = \text{XB.XA}$ (in circle (3))

$\therefore$ $\text{XP}^2 = \text{XR}^2$

$\therefore$ XP = XR

By the secant-tangent theorem,

$\text{XQ}^2 = \text{XC.XD}$ (in circle (2))

and $\text{XR}^2 = \text{XC.XD}$ (in circle (3))

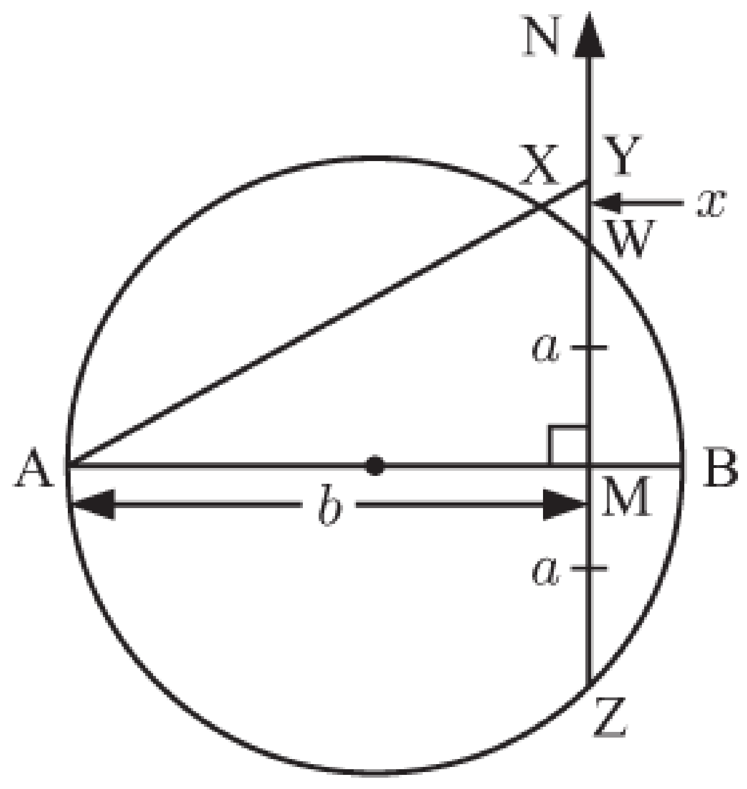
$\therefore$ $\text{XQ}^2 = \text{XR}^2$

$\therefore$ XQ = XR

$\therefore$ XP = XR = XQ

$\therefore$ the tangents from X to all three circles are equal in length.

14

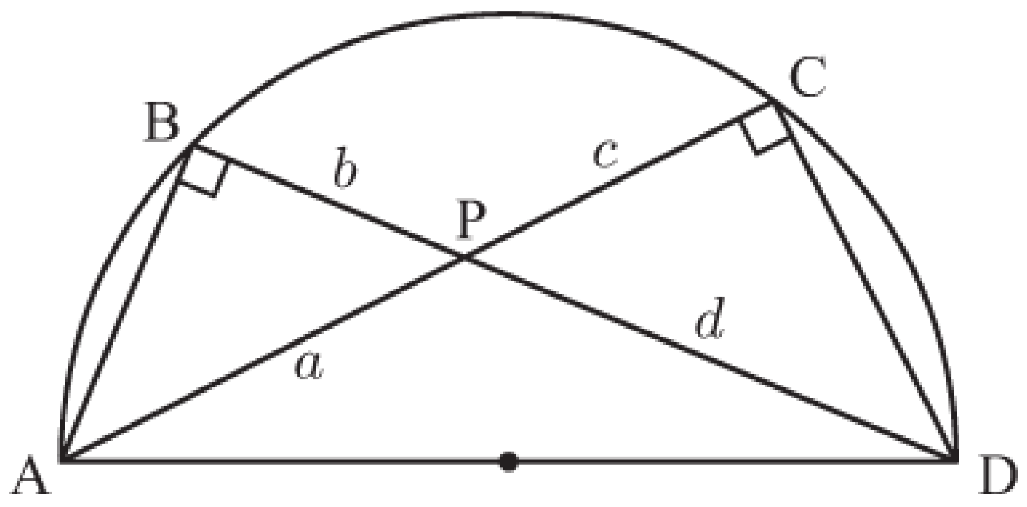


Let [MY] meet the circle at W.
W is a fixed point.
Let WM = MZ $= a$
{chord of a circle}
and AM $= b$ and YW $= x$,
where x varies.

By the secant-secant theorem,

$$\text{YX.YA} = \text{YW.YZ}$$
$$\therefore\ (\text{YA} - \text{AX}).\text{YA} = x(x+2a)$$
$$\therefore\ \text{YA}^2 - \text{AX.AY} = x^2 + 2ax$$
$$\therefore\ b^2 + (a+x)^2 = \text{AX.AY} + x^2 + 2ax \quad \{\text{Pythagoras}\}$$
$$\therefore\ b^2 + a^2 + 2ax + x^2 = \text{AX.AY} + x^2 + 2ax$$
$$\therefore\ \text{AX.AY} = a^2 + b^2 \quad \text{which is constant}$$

15



We join [AB] and [CD].
$\widehat{\text{ABD}} = \widehat{\text{ACD}} = 90°$
{angle in a semi-circle}

By the intersecting chords theorem, PA.PC = PB.PD

If we let PA $= a$, PB $= b$, PC $= c$, and PD $= d$, then $ac = bd$ .... (1)

Now AP.AC + DP.DB

$= a(a+c) + d(b+d)$

$= a^2 + ac + bc + d^2$

$= a^2 + 2ac + d^2$ {using (1)}

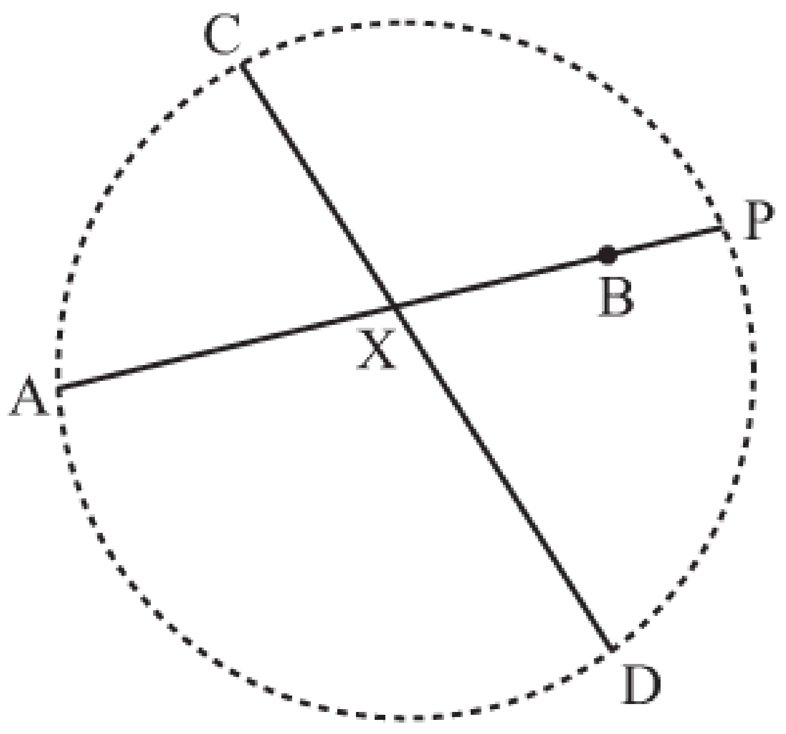
$= a^2 + 2ac + c^2 + d^2 - c^2$

$= (a+c)^2 + d^2 - c^2$

$= \text{AC}^2 + \text{CD}^2$ {Pythagoras in $\triangle$PCD}

$= \text{AD}^2$ {Pythagoras in $\triangle$ACD}

16



Let P be on [AB] or [AB] produced such that ACPD is a cyclic quadrilateral.

$\therefore$ XA.XP = XC.XD

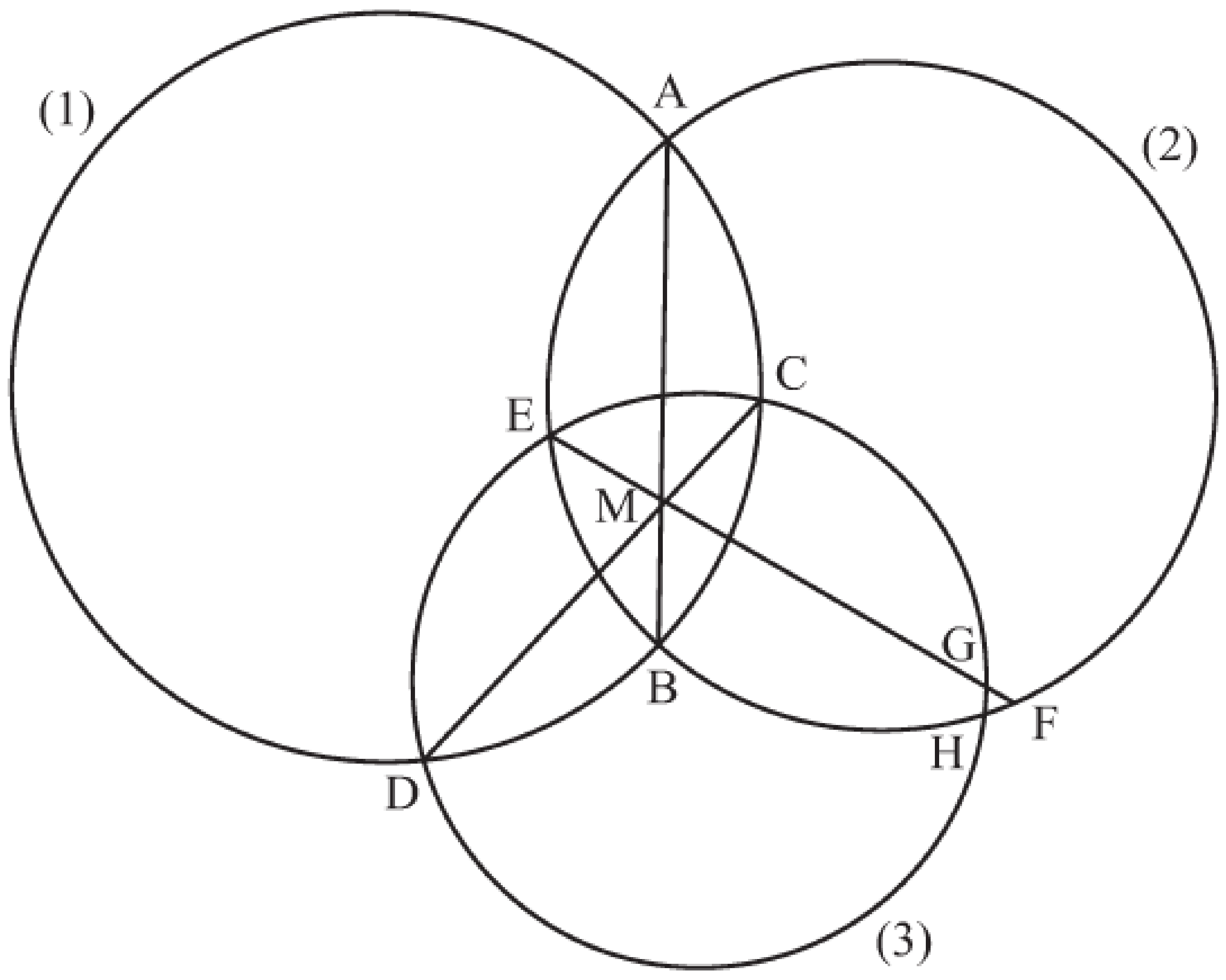
But XA.XB = XC.XD {given}

$\therefore$ XP = XB

$\therefore$ P is B

$\therefore$ A, B, C, and D are concyclic.

17



Let [EMF] and [EMG] be chords of circles (2) and (3) respectively.

EM.MF = AM.MB {in (2)}
= CM.MD {in (1)}
= EM.MG {in (3)}

∴ F and G coincide
∴ F and G are on both circles (2) and (3)
∴ F and G are really H.

So, we have indirectly shown that all three chords are concurrent.

EXERCISE 2G

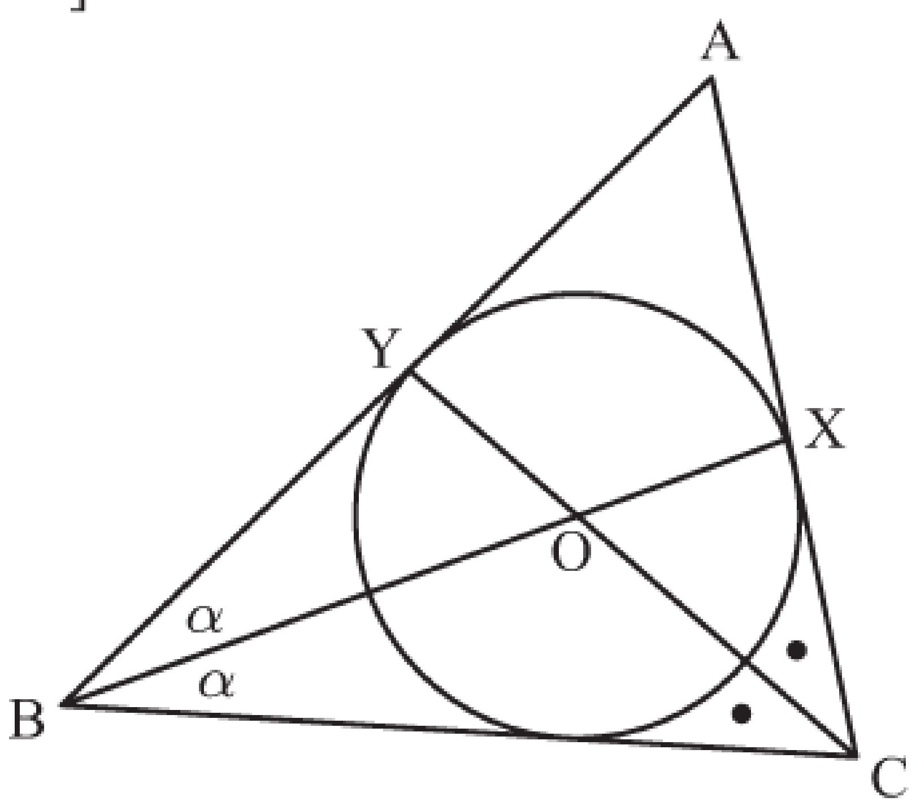
1 **a** Infinitely many circles can be drawn.

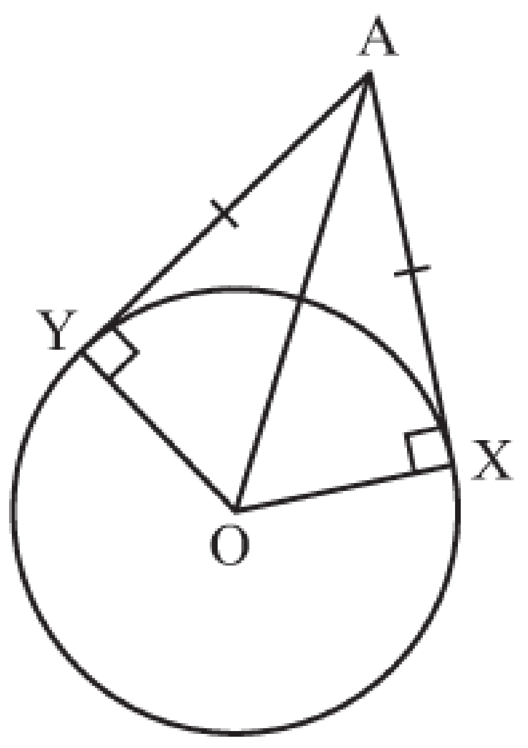
b Infinitely many circles can be drawn.

c Where [BX] and [CY] intersect we have a circle which touches [AB], [AC], and [BC].

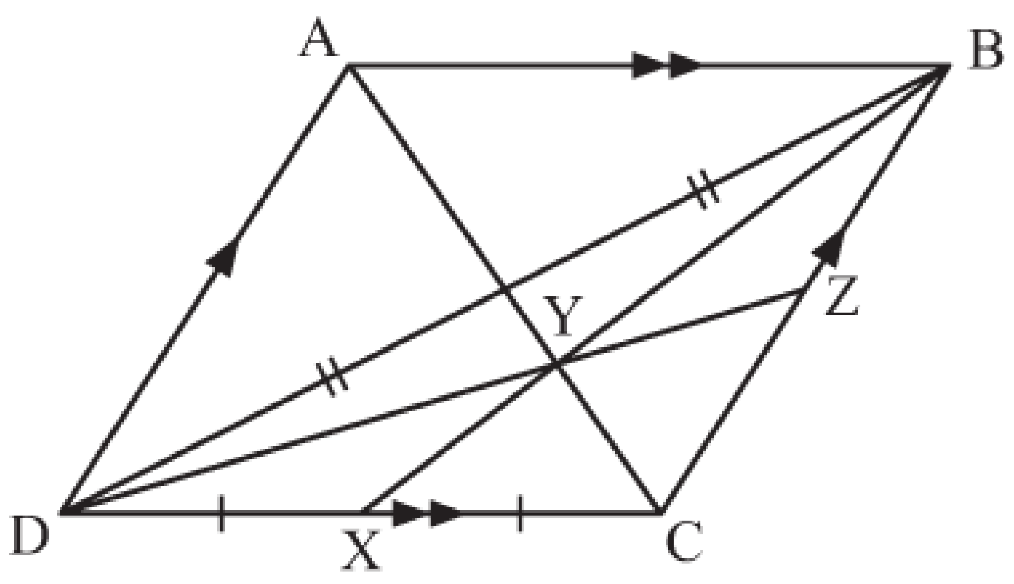
d We join [AO].
Consider △AXO, △AYO:
AX = AY {tangents from an external point}
AO is common
$A\widehat{X}O = A\widehat{Y}O = 90°$
{radius-tangent}
∴ △s AXO, AYO are congruent {RHS}.

Consequently
$X\widehat{A}O = Y\widehat{A}O$.





2



Let [DY] produced meet [BC] at Z.
Join [DB].
Diagonal [BD] is bisected by diagonal [AC]
{parallelogram theorem}

∴ Y is the centroid of △BCD.
∴ [DZ] is a median to the third side.
∴ Z is the midpoint of [BC].
∴ [DY] bisects [BC].

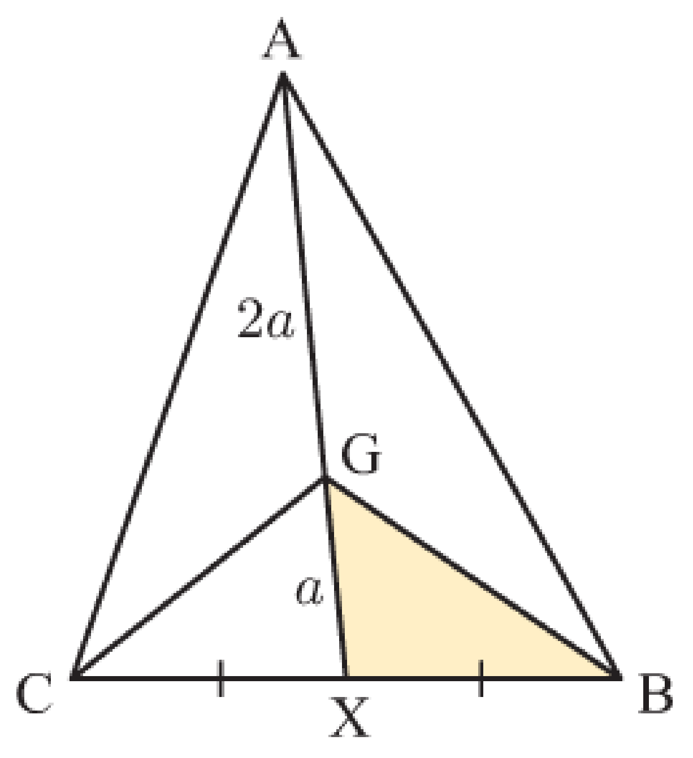
3 **a** X lies on the perpendicular bisector of [AB].
∴ △ABX is isosceles
{converse of isosceles triangle theorem}

b Likewise, △ACX is isosceles.

c Now as △s ABX and ACX are isosceles,
AX = BX and AX = CX
∴ BX = CX
∴ △BXC is isosceles
△s BXN and CXN are congruent. {RHS}

d Consequently, BN = CN
We have indirectly shown that all three perpendicular bisectors of the sides of a triangle are concurrent.

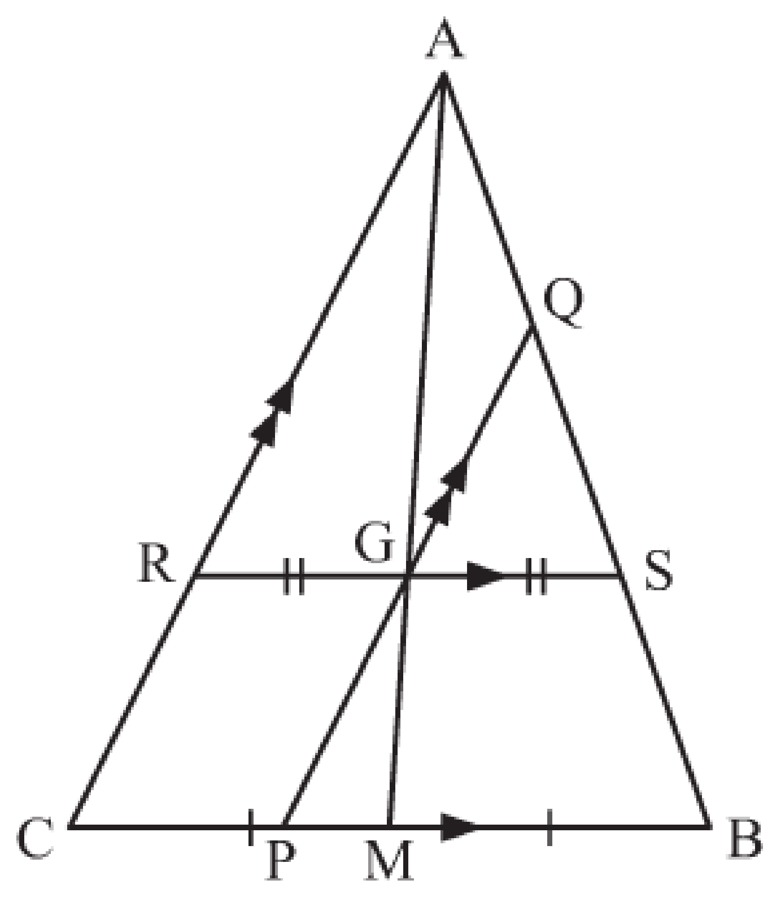
4



We join [GC].
BX = CX {given}
$GX = \frac{1}{3}AX$ {medians of a triangle}

∴ area of $\triangle GBX = \frac{1}{3} \times$ area of △ABX {equal bases}
and area of $\triangle ABX = \frac{1}{2} \times$ area of △ABC {equal altitudes}
∴ area of $\triangle GBX = \frac{1}{3} \times \frac{1}{2} \times$ area of △ABC
$= \frac{1}{6} \times$ area of △ABC

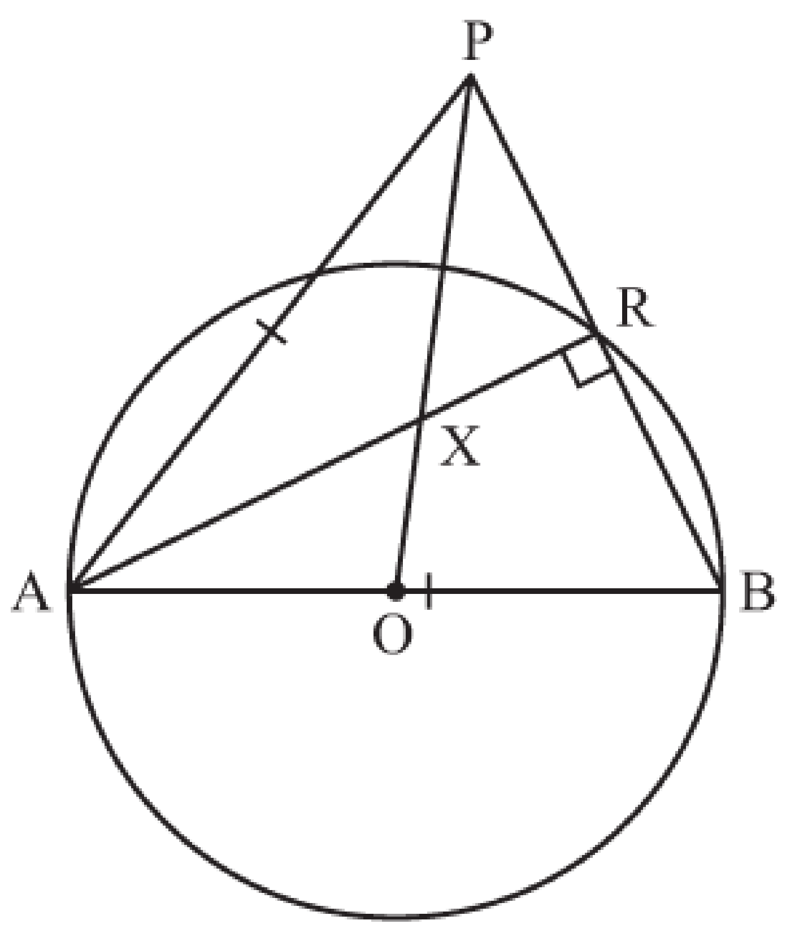
5



As G is the centroid of △ABC,
$\frac{AG}{GM} = \frac{2}{1}$
But △AGS is similar to △AMB
{equal corresponding angles}
∴ $\frac{AG}{AM} = \frac{AS}{AB} = \frac{2}{3}$
∴ $AS = \frac{2}{3}AB$
∴ S is a point of trisection of [AB].

Similarly, $\frac{RG}{CM} = \frac{2}{3}$ and $\frac{GS}{MB} = \frac{2}{3}$ {similar triangles}
∴ $RG = \frac{2}{3}CM$ and $GS = \frac{2}{3}MB$
∴ RG = GS {as BM = MC, given}
In △s SQG, SAR, [QG] ∥ [AR]
∴ Q bisects [AS] {converse of midpoint theorem}
∴ AQ = QS = SB
∴ Q and S trisect [AB].

6



Consider triangles APR and ABR:
$A\widehat{R}B = 90°$
{angle in semi-circle}
∴ $A\widehat{R}P = A\widehat{R}B = 90°$
{angles on a line}
AP = AB {given}
AR is common

∴ △s APR and ABR are congruent {RHS}

Consequently, PR = BR

∴ [AR] is a median of △APB.

OA = OB {equal radii}

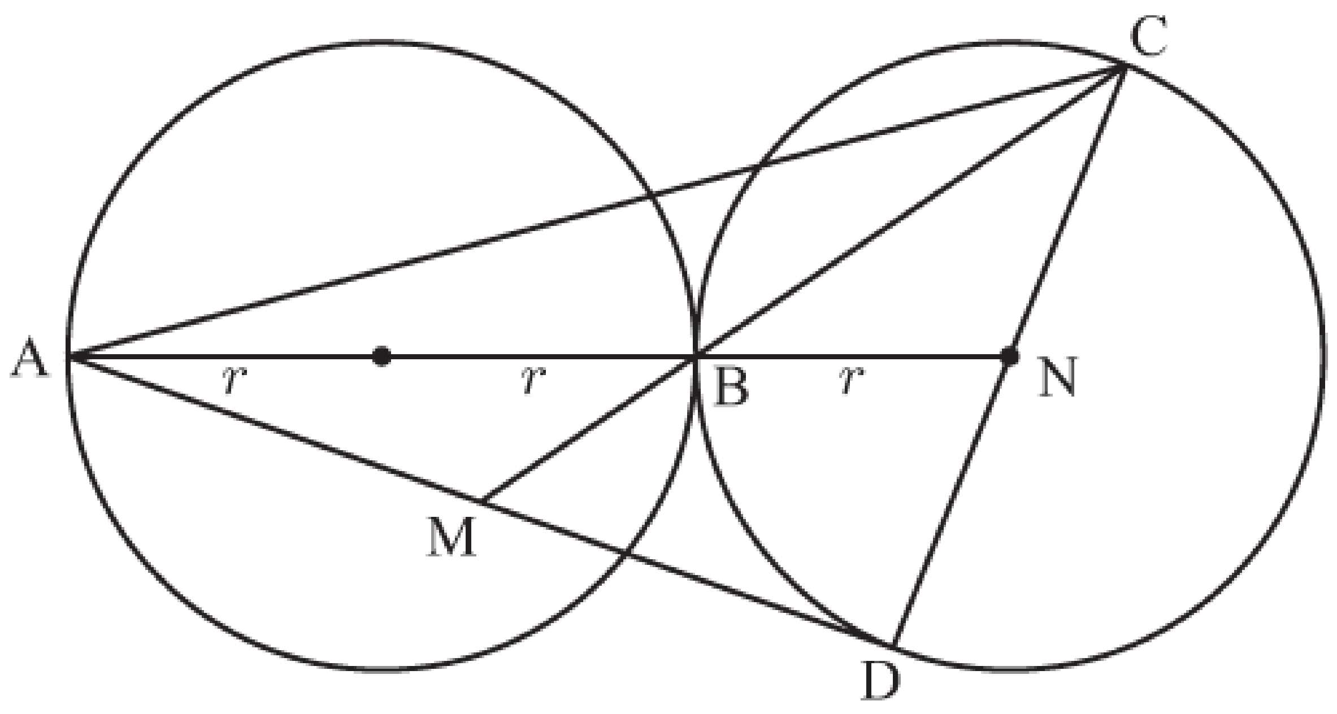
∴ [PO] is also a median.

∴ X is the centroid.

∴ X divides [PO] in the ratio 2 : 1 {medians of a triangle}

∴ [XP] is twice as long as [OX].

7



Let [BC] produced meet [AD] at M, let N be the second circle's centre, join [AC], and join [CD].

[CD] is a diameter.

∴ N is the midpoint of [CD].

∴ [AN] is a median of △ACD.

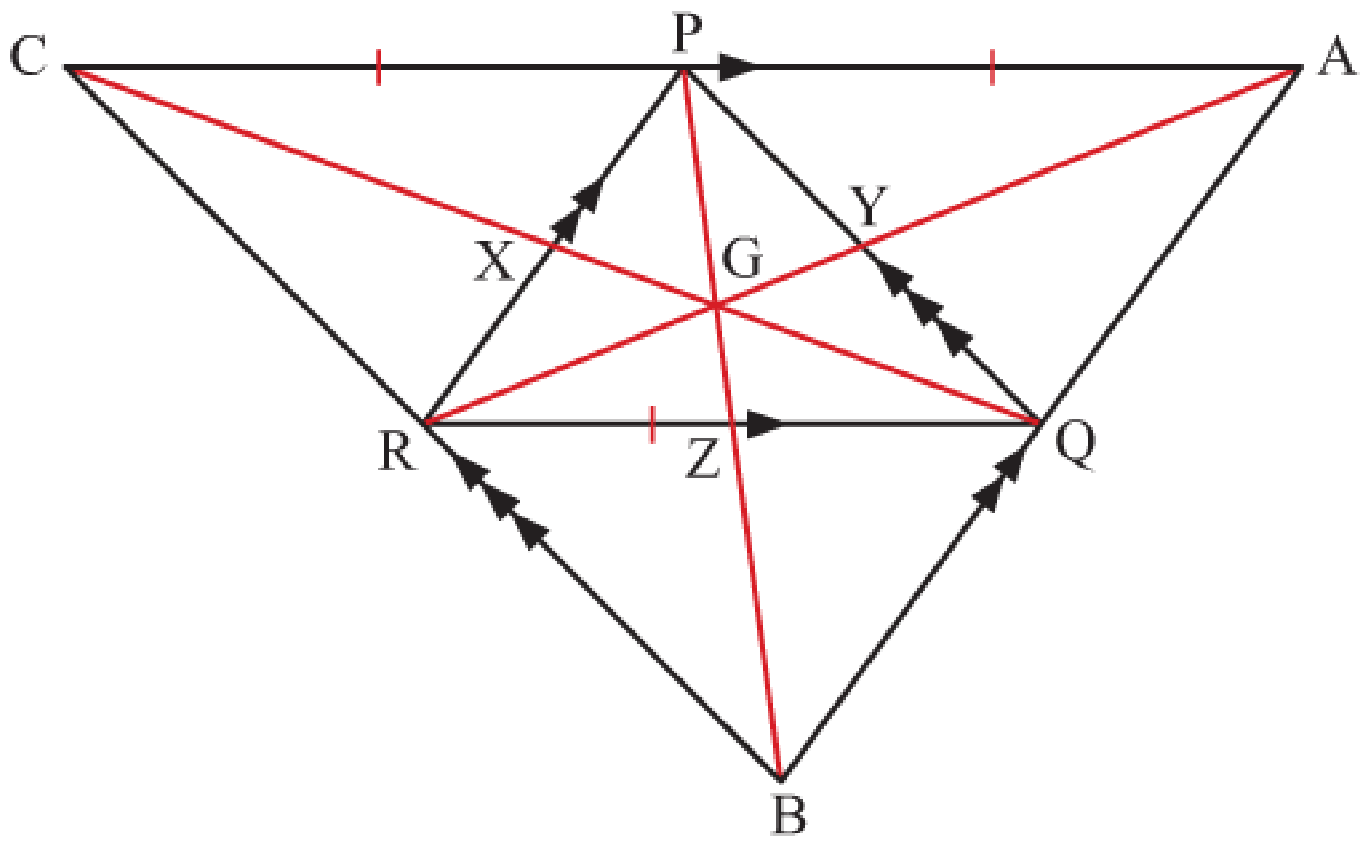
As AB : BN = $2r : r = 2 : 1$,

B is the centroid of △ACD.

∴ CM is another median

∴ M bisects [AD].

8



As CPQR is a parallelogram, CP = RQ .... (1)
{parallelogram theorem}

Likewise as PAQR is a parallelogram, PA = RQ .... (2)

From (1) and (2), CP = PA

∴ P is the midpoint of [CA].

By similar reasoning, Q is the midpoint of [AB] and
R is the midpoint of [BC].

We now join [PB], [RA], and [QC].

Let G be the centroid of △ABC and let
[CQ] meet [PR] at X, [AR] meet [PQ] at Y, and
[BP] meet [RQ] at Z.

Now △s BRZ, BCP are equiangular and therefore similar.

$$\therefore \quad \frac{\text{RZ}}{\text{CP}} = \frac{\text{BR}}{\text{BC}} = \frac{1}{2}$$

∴ RZ = $\frac{1}{2}$CP

Likewise in △s BZQ, BPA, ZQ = $\frac{1}{2}$PA

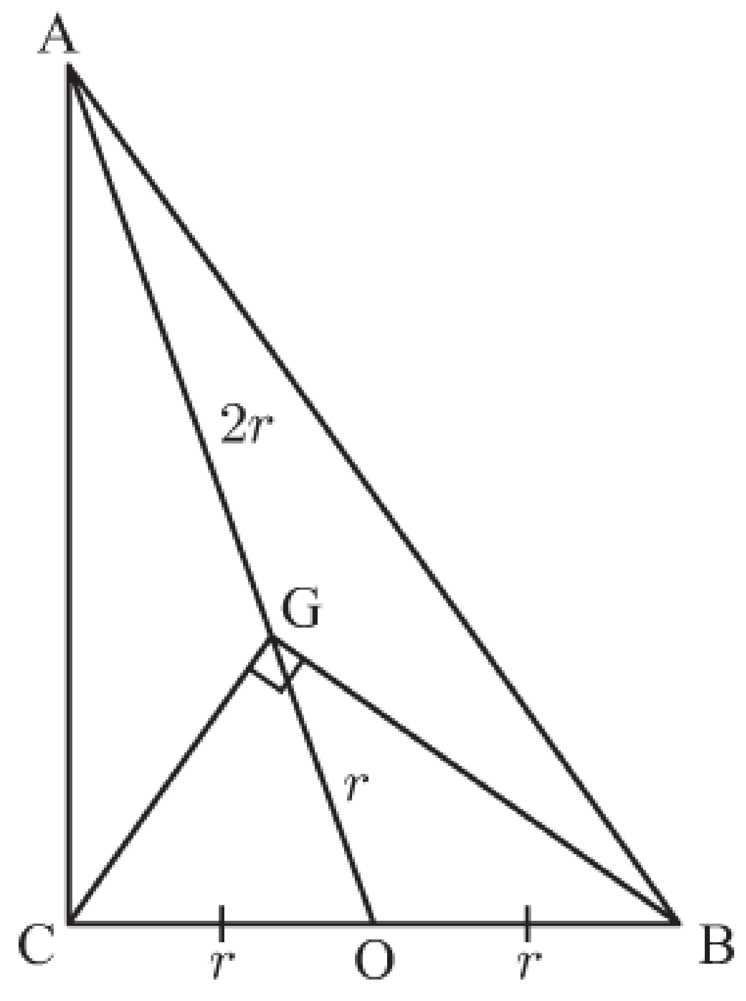
Thus RZ = ZQ {as CP = PA}

∴ PZ is a median of △PQR.

By similar reasoning using different similar △s, [QX] and [RY] are also medians of △PQR.

These 3 medians meet at G, the centroid of △ABC.

∴ △s PQR, ABC have a common centroid.



Let

and OB = BC = r, where r is constant.

∴ G lies on a semi-circle with centre O and radius r {converse of angle in a semi-circle}.

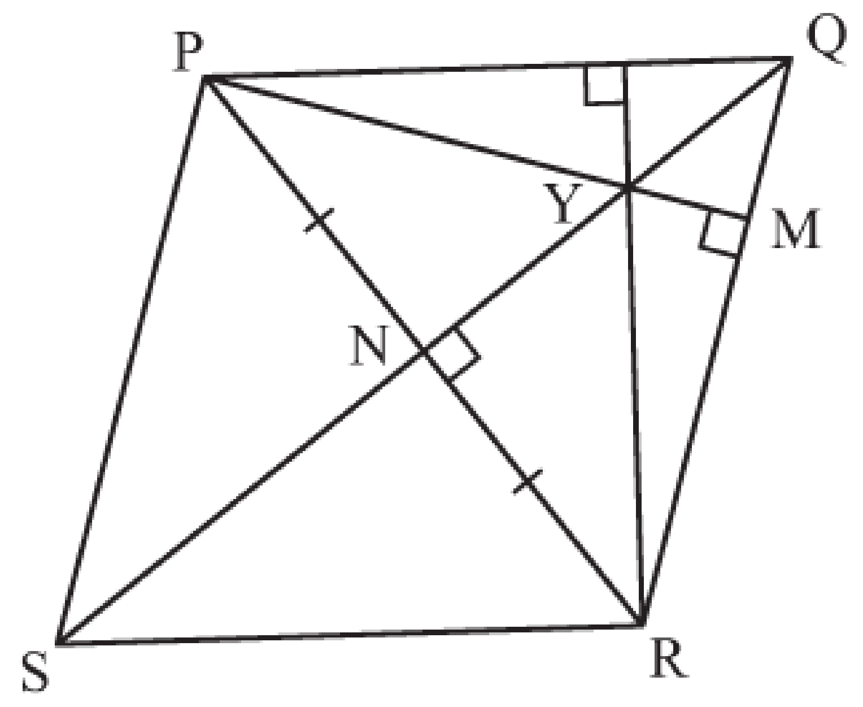
∴ OG = r and AG = $2r$ {medians of a triangle}

∴ OA = $3r$

∴ OA is constant for all positions of A.

∴ the locus of A is a circle with centre the midpoint of [BC] and radius $\frac{3}{2}$ × BC.

10



Join [PR] and let [PR] and [OS] meet at N.

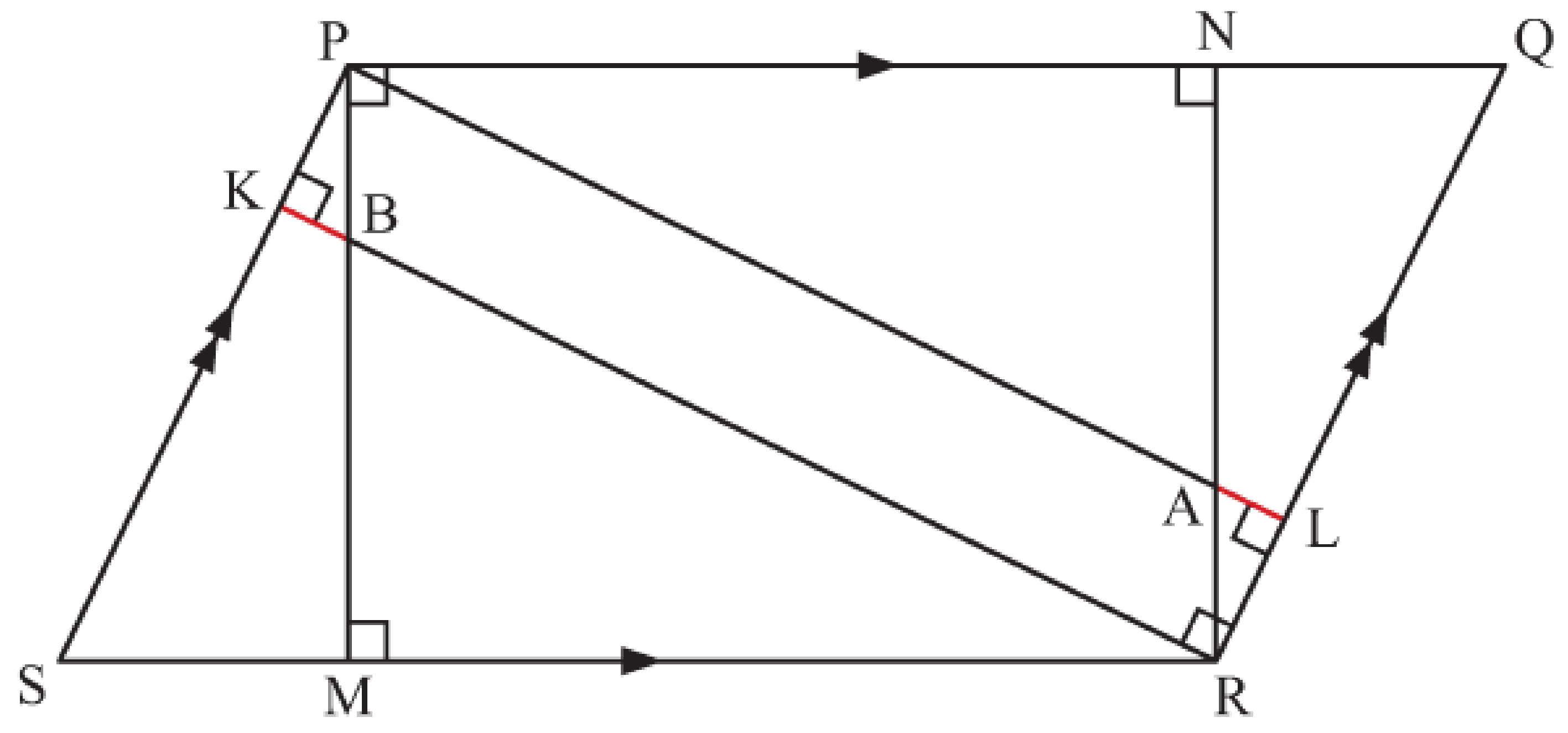
As PQRS is a rhombus, its diagonals bisect each other at right angles.

Thus [PM] and [QN] are altitudes of △PQR and as they meet at Y, Y is the orthocentre of △PQR.

∴ [RY] produced is also an altitude.

Thus [RY] ⊥ [PQ].

11



Let [PB] produced meet [SR] at M and [RA] produced meet [PQ] at N.

[PM] and [RN] are altitudes. {A, B are orthocentres}

∴ $\text{R}\widehat{\text{M}}\text{P} = \text{P}\widehat{\text{N}}\text{R} = 90^\circ$

∴ $\text{M}\widehat{\text{P}}\text{N} = 90^\circ$ {supplementary allied angles}

∴ [PM] ∥ [NR] {converse of allied angles}

Let [RB] produced meet [PS] at K and [PA] produced meet [QR] at L.

[RK] and [PL] are altitudes. {A, B are orthocentres}

∴ $\text{P}\widehat{\text{K}}\text{R} = \text{R}\widehat{\text{L}}\text{P} = 90^\circ$

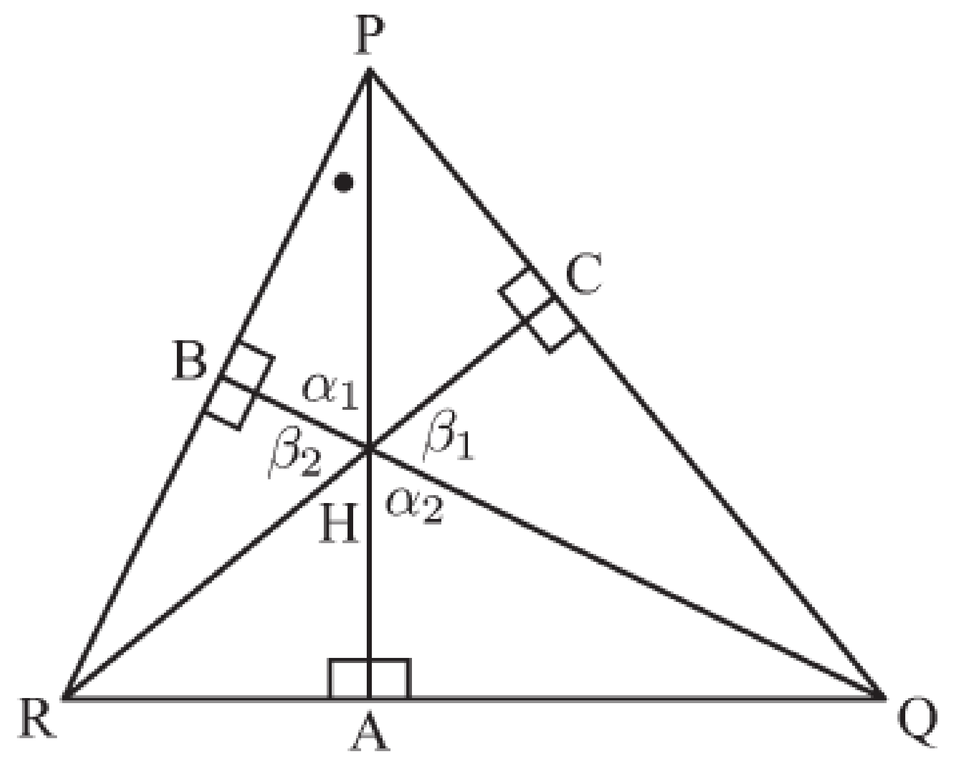
∴ $\text{K}\widehat{\text{R}}\text{L} = 90^\circ$ {supplementary allied angles}

∴ [KR] ∥ [PL] {converse of allied angles}

So, [PB] ∥ [AR] and [BR] ∥ [PA]

∴ PARB is a parallelogram.

12



a Consider △s PBH, PAR:

$\widehat{\text{P}}$ is common {•} and

$\text{P}\widehat{\text{B}}\text{H} = \text{P}\widehat{\text{A}}\text{R} = 90^\circ$

∴ △s PBH and PAR are equiangular and therefore similar.

$$\therefore \quad \frac{\text{PB}}{\text{PA}} = \frac{\text{PH}}{\text{PR}}$$

∴ PH.PA = PB.PR

b Consider $\triangle$s PHB, QHA:

$\alpha_1 = \alpha_2$ {vertically opposite angles}

and $\widehat{PBH} = \widehat{QAH} = 90°$

$\therefore$ $\triangle$s PHB and QHA are equiangular and therefore similar.

$$\therefore \quad \frac{PH}{QH} = \frac{HB}{HA}$$

$\therefore$ PH.HA = QH.HB .... (1)

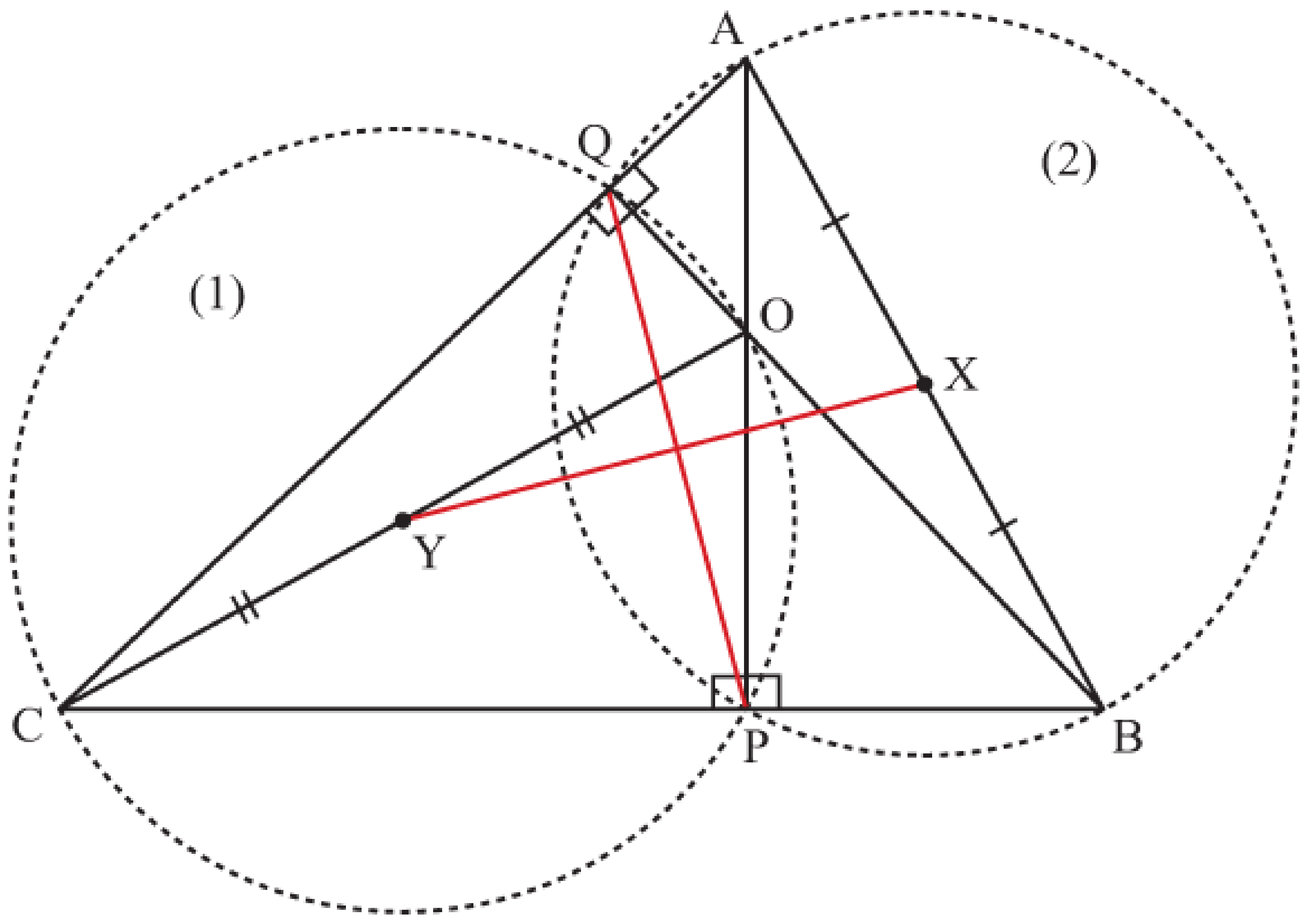
Likewise, $\triangle$s CHQ and BHR are similar.

$$\therefore \quad \frac{HC}{HB} = \frac{QH}{RH}$$

$\therefore$ QH.HB = RH.HC .... (2)

From (1) and (2), PH.HA = QH.HB = RH.HC

13



In OPCQ, $\widehat{OPC} + \widehat{OQC} = 90° + 90° = 180°$

$\therefore$ OPCQ is a cyclic quadrilateral
{converse of opposite angles of a cyclic quadrilateral}

[OC] is a diameter of circle (1) as it subtends right angles at P and Q {converse of angle in a semi-circle}

$\therefore$ Y is the centre of circle (1).

As [AB] subtends equal angles of 90° at P and Q, ABPQ is a cyclic quadrilateral {test for cyclic quadrilaterals}

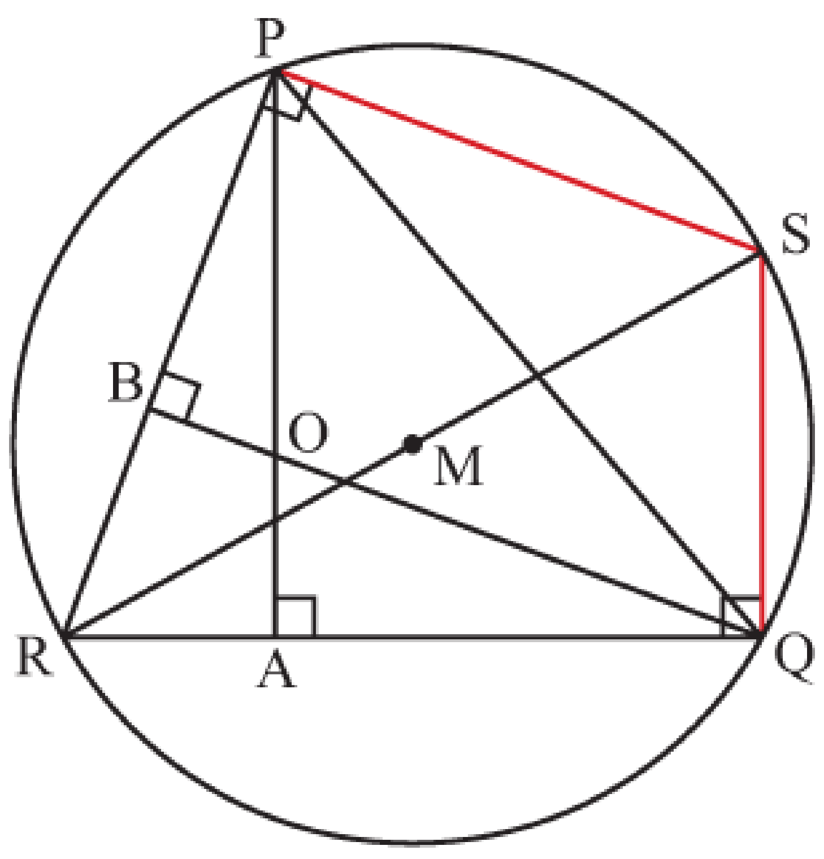
[AB] is a diameter of circle (2) as it subtends right angles at P and Q {converse of angle in a semi-circle}

$\therefore$ X is the centre of circle (2).

So, circles (1) and (2) have a common chord [PQ] and the line [XY] connects the circles' centres.

$\therefore$ [XY] bisects [PQ] at right angles {intersecting circles}

14



We let the circle's centre be M and locate the orthocentre O at the intersection of perpendiculars [PA] to [QR] and [QB] to [PR].

Now as [RS] is a diameter, it subtends right angles at P and Q. {angle in a semi-circle}

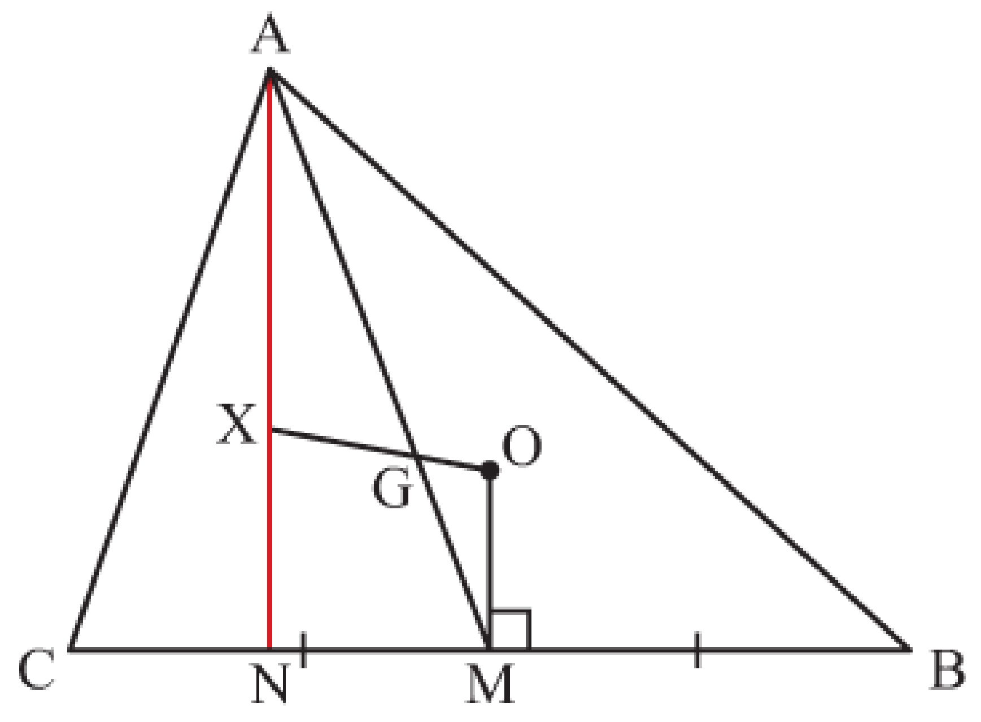
As $\widehat{SPR} + \widehat{QBP} = 90° + 90° = 180°$,

[SP] $\parallel$ [QB] {converse of supplementary allied angles}

Likewise, as $\widehat{SQR} + \widehat{QAP} = 180°$, [SQ] $\parallel$ [AP].

Thus SQOP is a parallelogram.

15



Using the given hint, we now have to prove that X is the orthocentre of $\triangle$ABC.

We join [AX] and produce it to [BC] meeting [BC] at N.

Let M be the midpoint of [BC].

[OM] $\perp$ [BC] {O is the circumcentre}

AGM is a median of $\triangle$ABC

where AG : GM = 2 : 1 {medians of a triangle}

Now GX : OG = 2 : 1 {construction}

and $\widehat{AGX} = \widehat{MGO}$ {vertically opposite}

$\therefore$ $\triangle$s AGX and MGO are similar
{Two pairs of sides in proportion and included angles equal}

Consequently, $\widehat{XAG} = \widehat{OMG}$

$\therefore$ [AX] $\parallel$ [OM] {converse of equal alternate angles}

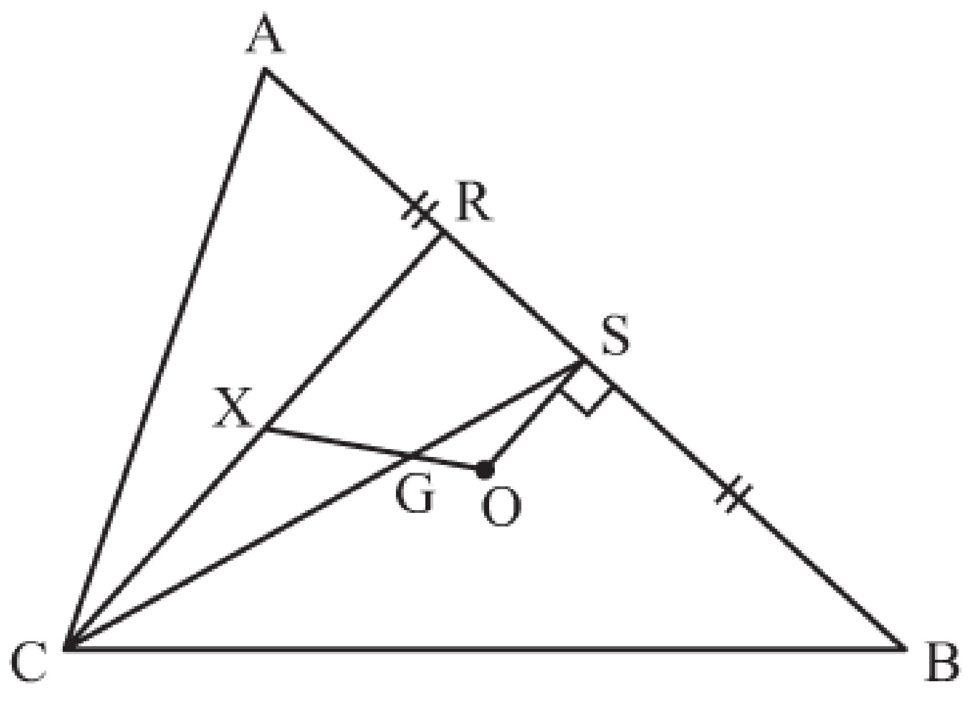
But $\widehat{OMB}$ is a right angle {O is the circumcentre}

$\therefore$ $\widehat{ANB}$ is a right angle

$\therefore$ [AN] is an altitude of $\triangle$ABC .... (1)

By similar reasoning we can show that [CX] produced to R on [AB] is also an altitude of $\triangle$ABC.

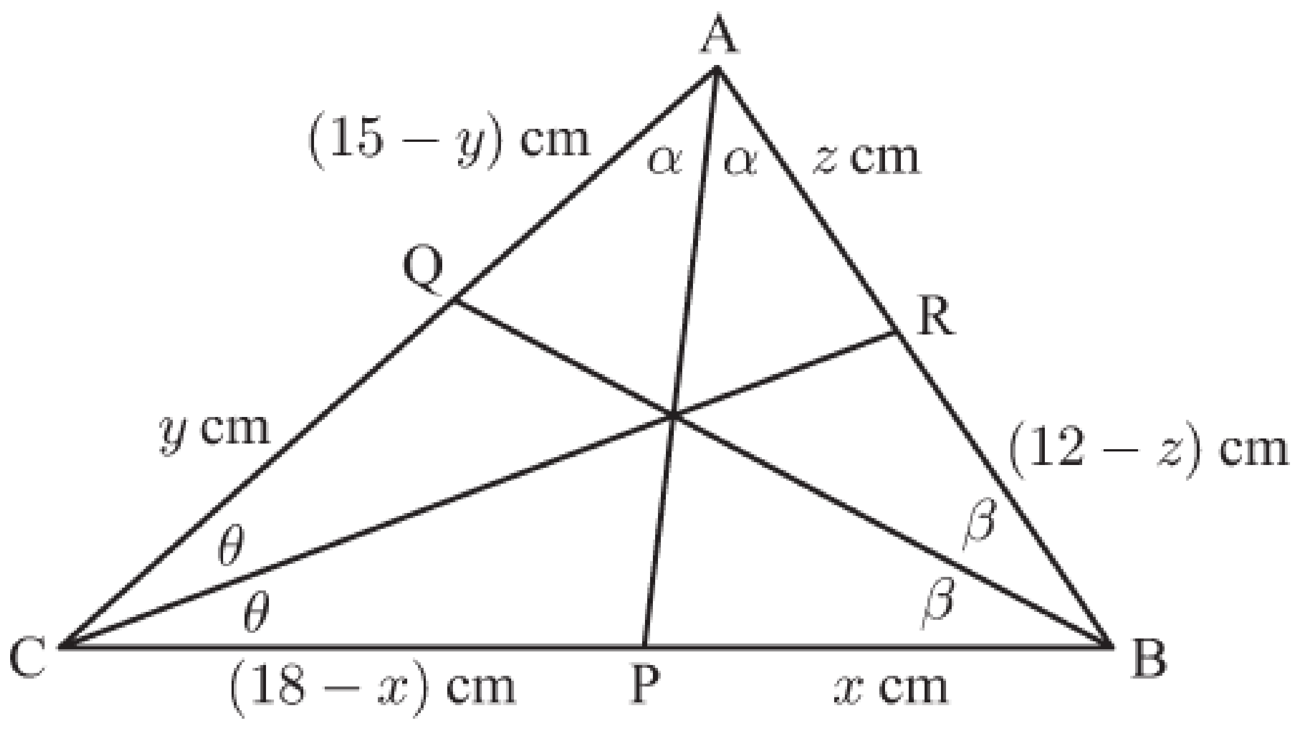
Use:



Thus as [CR] and [AN] are altitudes which meet at X, then X is the orthocentre.

EXERCISE 2H

1



We draw the angle bisectors of the triangle:

[AP] meets [BC] at P, [BQ] meets [AC] at Q, and [CR] meets [AB] at R.

We let BP = x cm, CQ = y cm, and AZ = r cm.

By the angle bisector theorem, $\dfrac{12}{15} = \dfrac{4}{5} = \dfrac{x}{18 - x}$

$$\therefore \quad 72 - 4x = 5x$$

$$\therefore \quad 9x = 72$$

$$\therefore \quad x = 8$$

$$\text{and} \quad 18 - x = 10$$

$\therefore$ BP = 8 cm, PC = 10 cm

Likewise $\frac{18}{12} = \frac{3}{2} = \frac{y}{15-y}$ and $\frac{18}{15} = \frac{6}{5} = \frac{12-z}{z}$

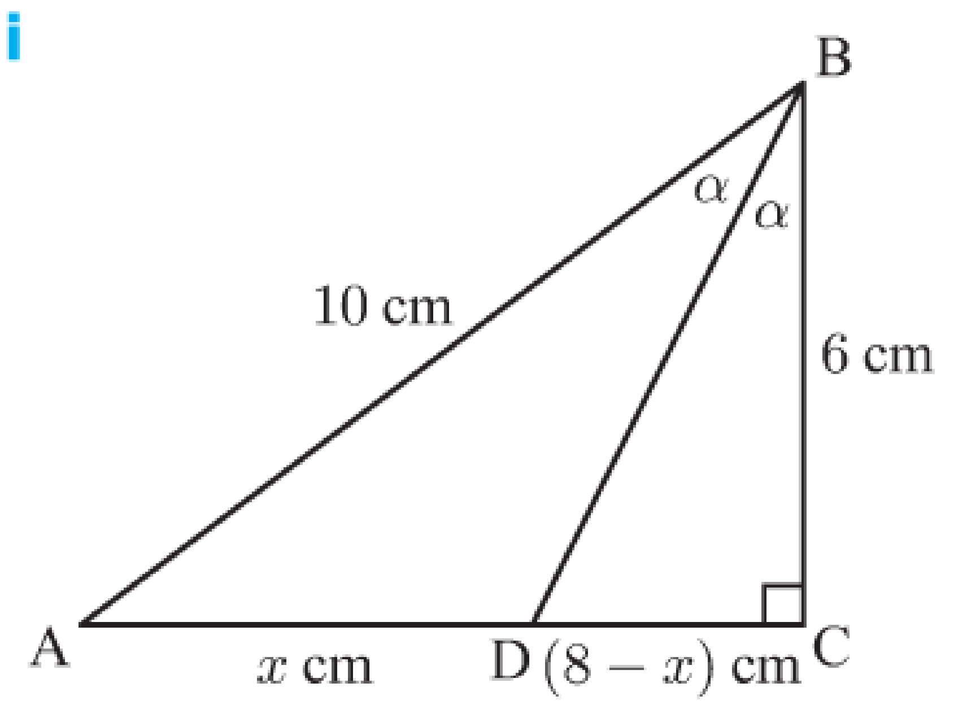
$\therefore\ 45 - 3y = 2y$ $\qquad \therefore\ 6z = 60 - 5z$

$\therefore\ 5y = 45$ $\qquad \therefore\ 11z = 60$

$\therefore\ y = 9$ $\qquad \therefore\ z = \frac{60}{11}$

$\therefore$ CQ = 9 cm, QA = 6 cm, AR ≈ 5.45 cm, and RB ≈ 6.55 cm.

2 a i



Let AD = x cm.
By the angle bisector theorem,

$$\frac{6}{10} = \frac{8-x}{x}$$

$\therefore\ 6x = 80 - 10x$

$\therefore\ 16x = 80$

$\therefore\ x = 5$

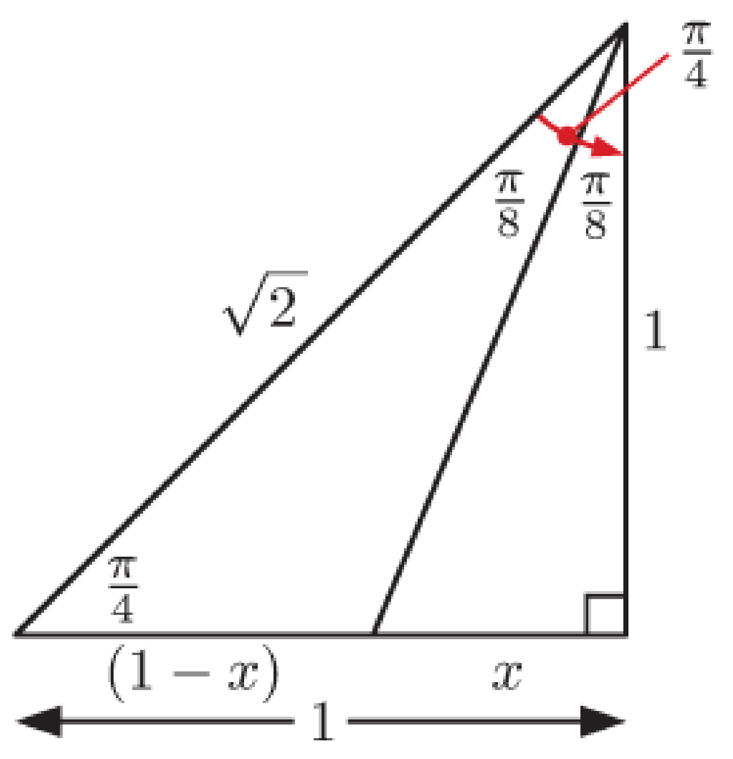
$\therefore$ AD = 5 cm,
CD = 3 cm

ii △ABC is right angled at C {ratio of sides 3 : 4 : 5}

$$\therefore\ \tan\left(\frac{\widehat{ABC}}{2}\right) = \frac{CD}{6} = \frac{3}{6}$$

$$\therefore\ \tan\left(\frac{\widehat{ABC}}{2}\right) = \frac{1}{2}$$

b



Consider the right-angled isosceles triangle shown. The two equal sides have length 1 unit.
By Pythagoras, the hypotenuse is $\sqrt{2}$ units.

By the angle bisector theorem,

$$\frac{1}{\sqrt{2}} = \frac{x}{1-x}$$

$\therefore\ 1 - x = x\sqrt{2}$

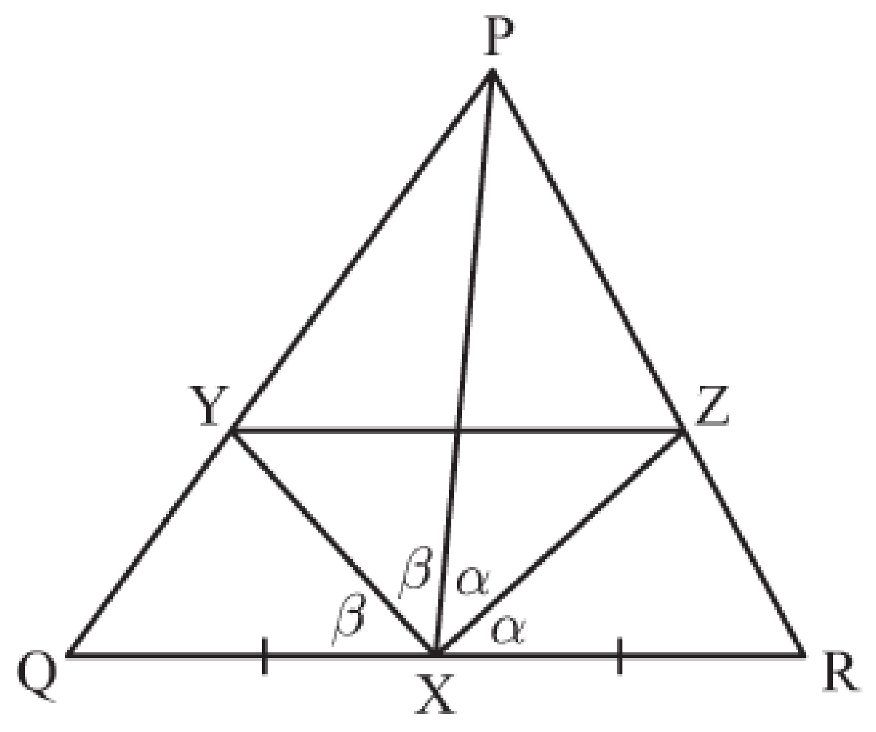
$\therefore\ 1 = x(\sqrt{2}+1)$

$$\therefore\ x = \frac{1}{\sqrt{2}+1}\left(\frac{\sqrt{2}-1}{\sqrt{2}-1}\right)$$

$\therefore\ x = \sqrt{2} - 1$

Consequently, $\tan\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}-1}{1} = \sqrt{2} - 1.$

3



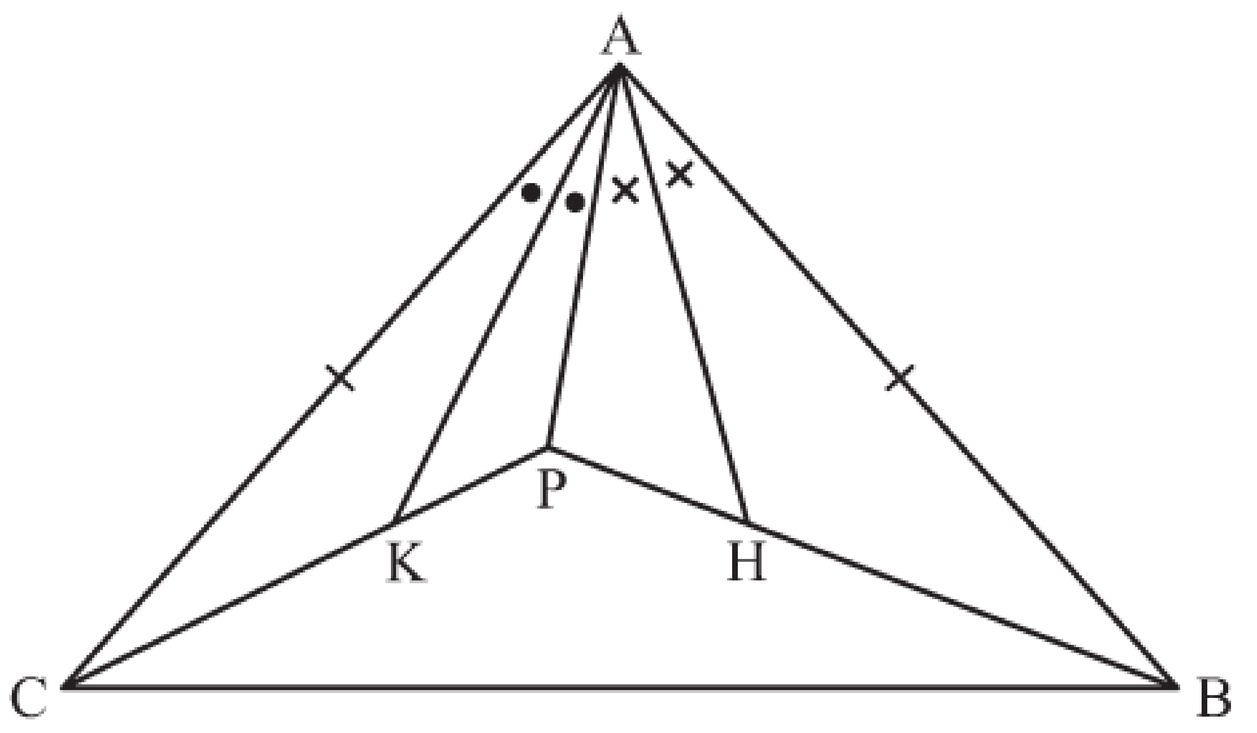
Join [YZ].
By the angle bisector theorem, $\frac{PX}{QX} = \frac{PY}{YQ}$ and $\frac{PX}{XR} = \frac{PZ}{ZR}$.

$\therefore$ as QX = XR, LHS of each fraction are equal.

$$\therefore\ \frac{PY}{YQ} = \frac{PZ}{ZR}$$

$\therefore$ [YZ] ∥ [QR] {converse to parallel lines within a triangle}

4



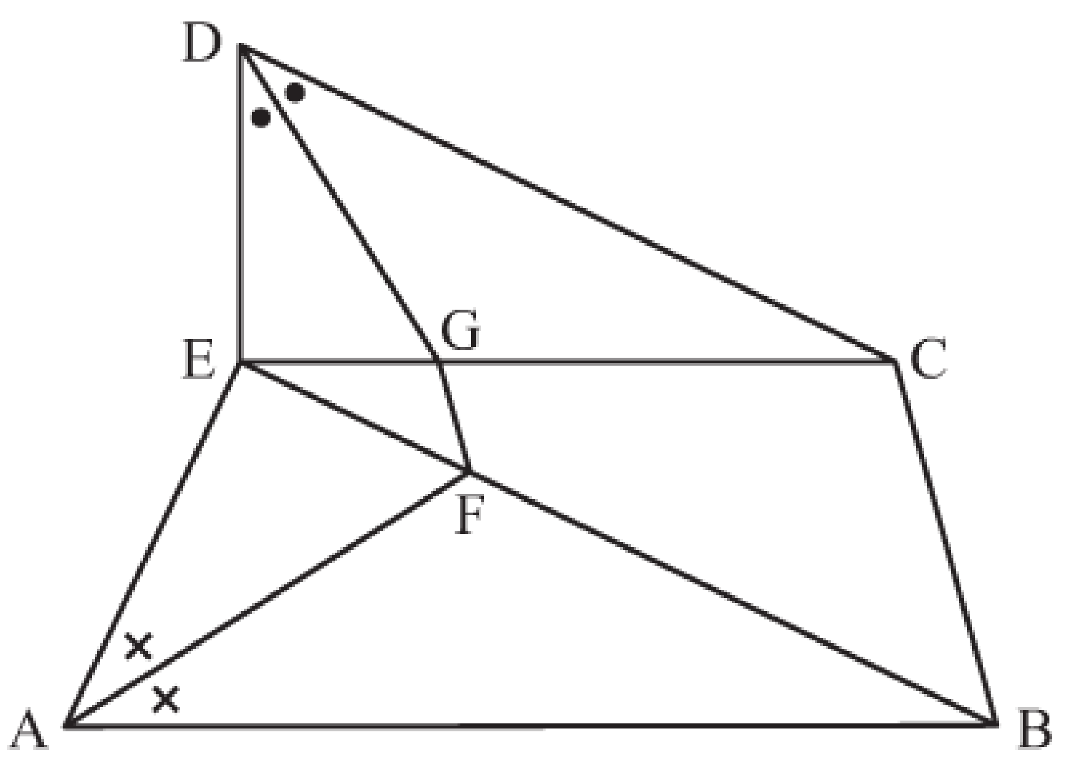
By the angle bisector theorem, $\frac{AP}{AB} = \frac{PH}{HB}$ and $\frac{AP}{AC} = \frac{PK}{KC}$

But AB = AC {given}

$$\therefore\ \frac{PH}{HB} = \frac{PK}{KC}$$

$\therefore$ [HK] ∥ [BC] {converse to parallel lines within a triangle}

5



By the angle bisector theorem,

$\frac{DC}{DE} = \frac{GC}{EG}$ .... (1) and

$\frac{AB}{AE} = \frac{BF}{FE}$ .... (2)

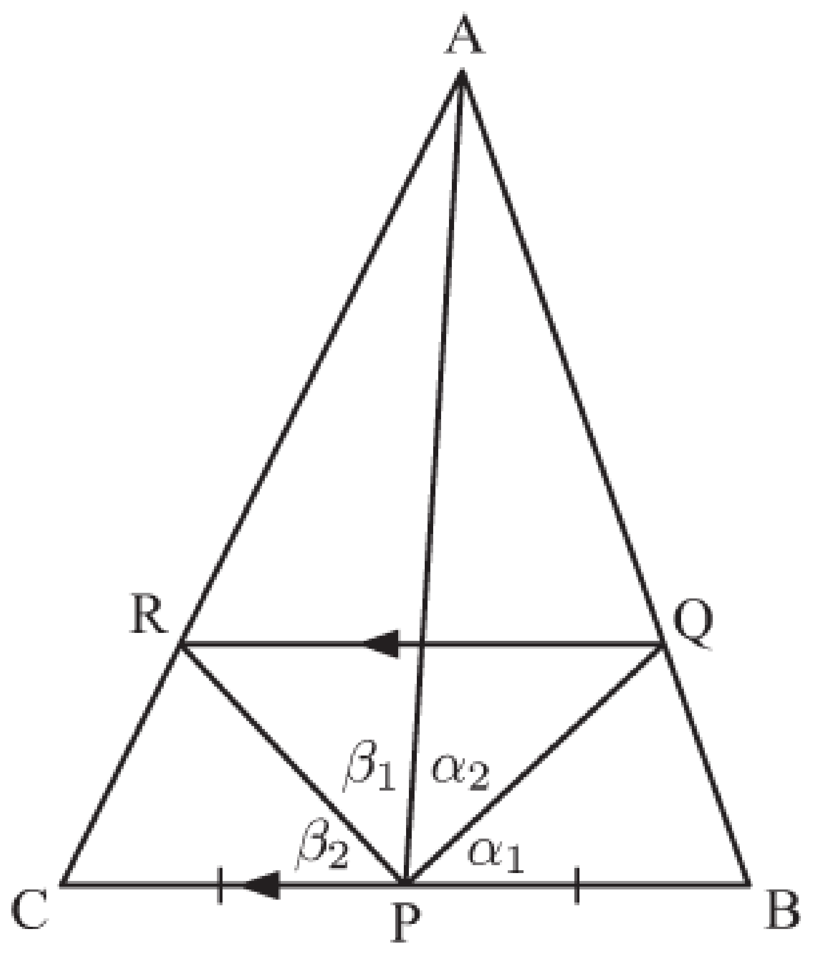
△s EAB and EDG are similar {given}

$$\therefore\ \frac{AB}{AE} = \frac{DC}{DE}$$

$\therefore\ \frac{BF}{FE} = \frac{GC}{EG}$ {using (1) and (2)}

$\therefore$ [GF] ∥ [CB] {converse to parallel lines within a triangle}

6



We join [PR].

$\alpha_1 = \alpha_2$ {given}

$\therefore\ \frac{AP}{BP} = \frac{AQ}{BQ}$ .... (1)
{angle bisector theorem}

But $\frac{AQ}{BQ} = \frac{AR}{RC}$ .... (2)
{parallel lines within a triangle}

$\therefore\ \frac{AP}{BP} = \frac{AR}{RC}$
{from (1), (2)}

However BP = PC {given}

$$\therefore\ \frac{AP}{PC} = \frac{AR}{RC}$$

$\therefore$ [PR] bisects $\widehat{APC}$ {converse of angle bisector theorem}

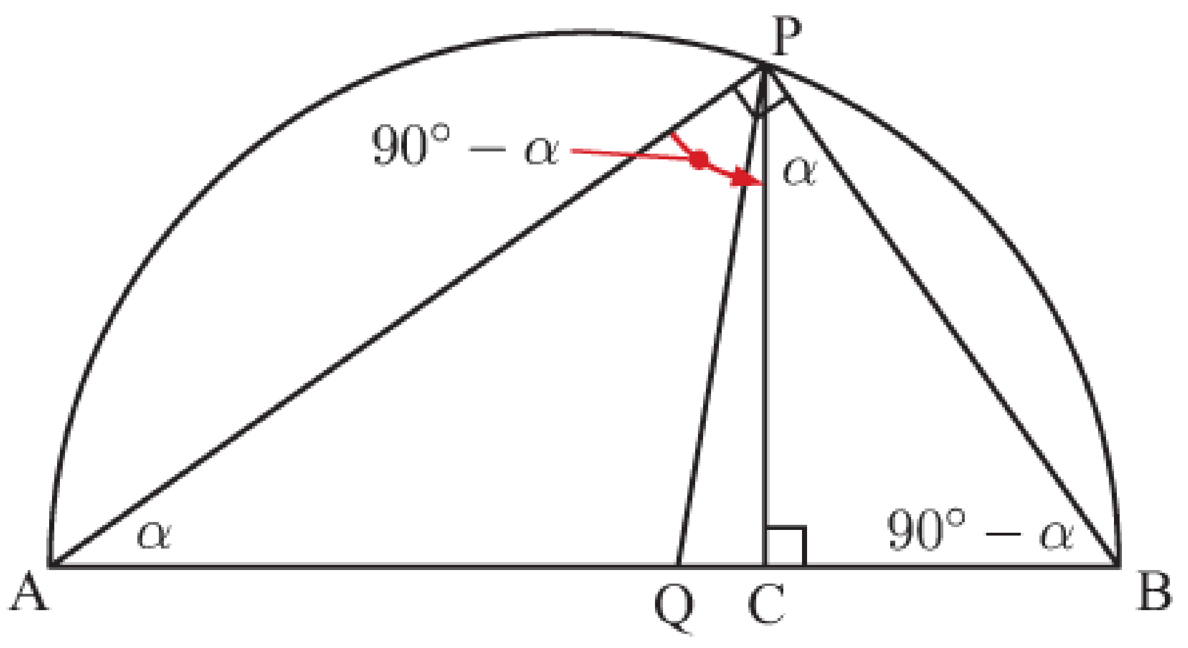
$\beta_1 = \beta_2$

$\therefore\ 2\alpha + 2\beta = 180°$ {angles on a line}

$\therefore\ \alpha + \beta = 90°$

That is, $\widehat{QPR}$ is a right angle.

7



As QP bisects $\widehat{APB}$,

$$\frac{PA}{PB}=\frac{AQ}{QB} \quad .... (1) \quad \{\text{angle bisector theorem}\}$$

But $\triangle$s APC and ABP are equiangular and therefore similar.

$$\therefore \quad \frac{AC}{AP}=\frac{PC}{BP}$$

On rearranging, $\dfrac{PA}{PB}=\dfrac{AC}{CP}$ .... (2)

Thus from (1) and (2), $\dfrac{AQ}{QB}=\dfrac{AC}{PC}$.

8 **a** $\dfrac{QR}{QP}=\dfrac{RM}{PM}$ {angle bisector theorem}

But RM = PM

$\therefore$ QR = QP

$\therefore$ $\triangle$PQR is isosceles.

b None of the 4 tests for congruence apply.

9 By the secant-secant theorem in $\triangle$XBD,

XA.XB = XC.XD

$\therefore \quad \dfrac{XA}{XC}=\dfrac{XD}{XB}$ .... (1)

By the angle bisector theorem,

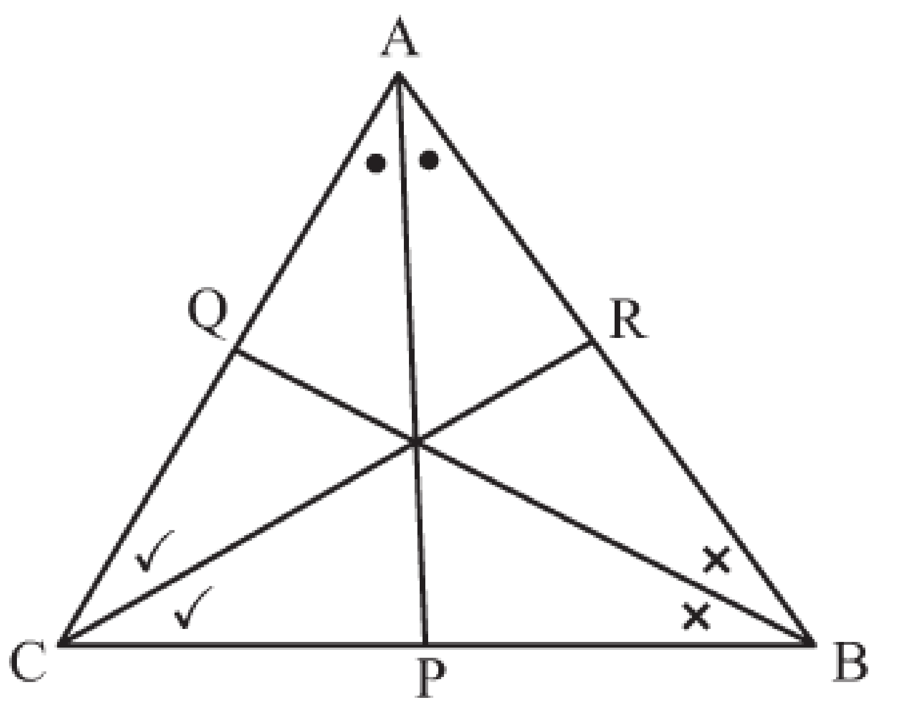
in $\triangle$XAC, $\dfrac{XA}{XC}=\dfrac{MA}{CM}$ .... (2) and

in $\triangle$XBD, $\dfrac{XD}{XB}=\dfrac{ND}{BN}$ .... (3)

$\therefore \quad \dfrac{MA}{CM}=\dfrac{ND}{BN}$ {from (1), (2), (3)}

$\therefore$ MA.BN = ND.CM

10



Using the angle bisector theorem 3 times:

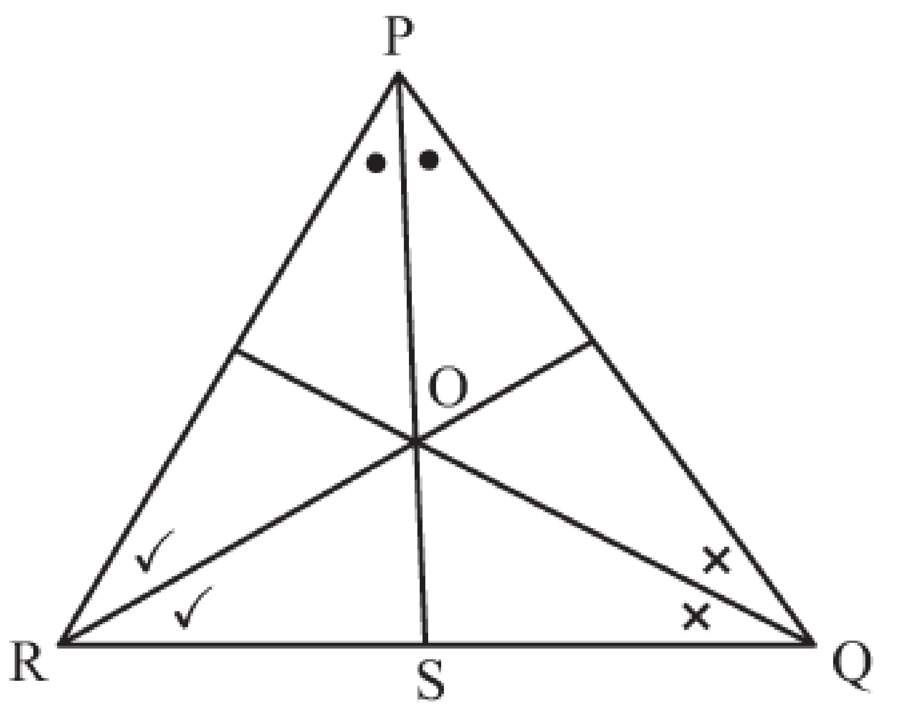
$$\frac{AB}{AC}=\frac{BP}{CP},\quad \frac{AC}{BC}=\frac{AR}{BR},$$

and $\dfrac{AB}{BC}=\dfrac{AQ}{CQ}$

Thus $\quad AR.BP.CQ=\dfrac{\cancel{AC}.BR}{\cancel{BC}}.\dfrac{\cancel{AB}.CP}{\cancel{AC}}.\dfrac{AQ.\cancel{BC}}{\cancel{AB}}$

$= AQ.BR.CP$ {after cancellation}

11



By the angle bisector theorem in $\triangle$PQS,

$\dfrac{PQ}{QS}=\dfrac{PO}{OS}$ .... (1)

and in $\triangle$PRS,

$\dfrac{PR}{SR}=\dfrac{PO}{OS}$ .... (2)

Now QR = QS + SR

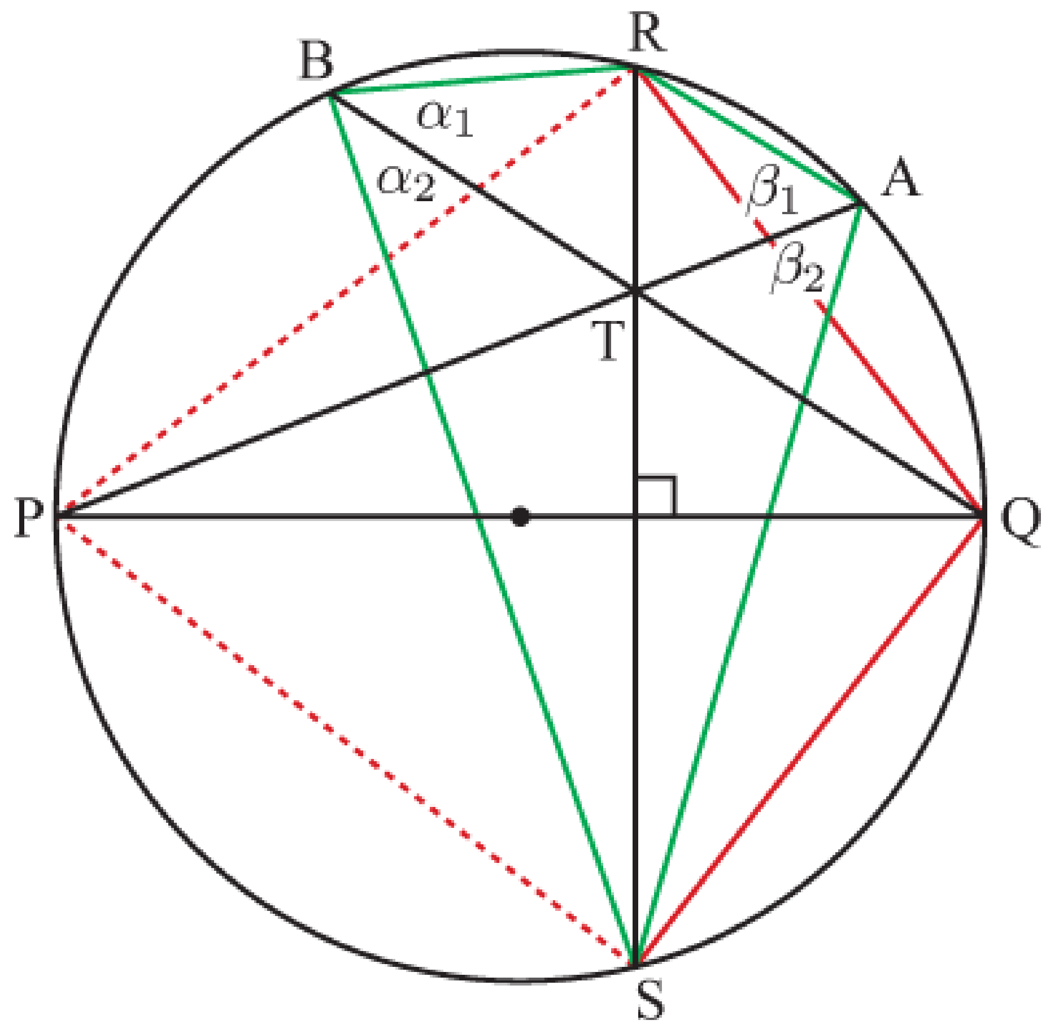
$\therefore \quad QR=\dfrac{PQ.OS}{PO}+\dfrac{PR.OS}{PO}$ {from (1) and (2)}

$\therefore \quad QR=\dfrac{(PQ+PR).OS}{PO}$

$\therefore \quad \dfrac{QR}{(PQ+PR)}=\dfrac{OS}{PO}$

$\therefore$ PO : OS = (PQ + PR) : QR

12



Chords [RQ] and [QS] are equal in length.

{RS is a chord of the circle perpendicular to the diameter.}

$\therefore$ they subtend the equal angles at the circle at B.

$\therefore \quad \alpha_1=\alpha_2$

Likewise, as chords [PR] and [PS] are equal in length, they subtend equal angles at A.

$\therefore \quad \beta_1=\beta_2$

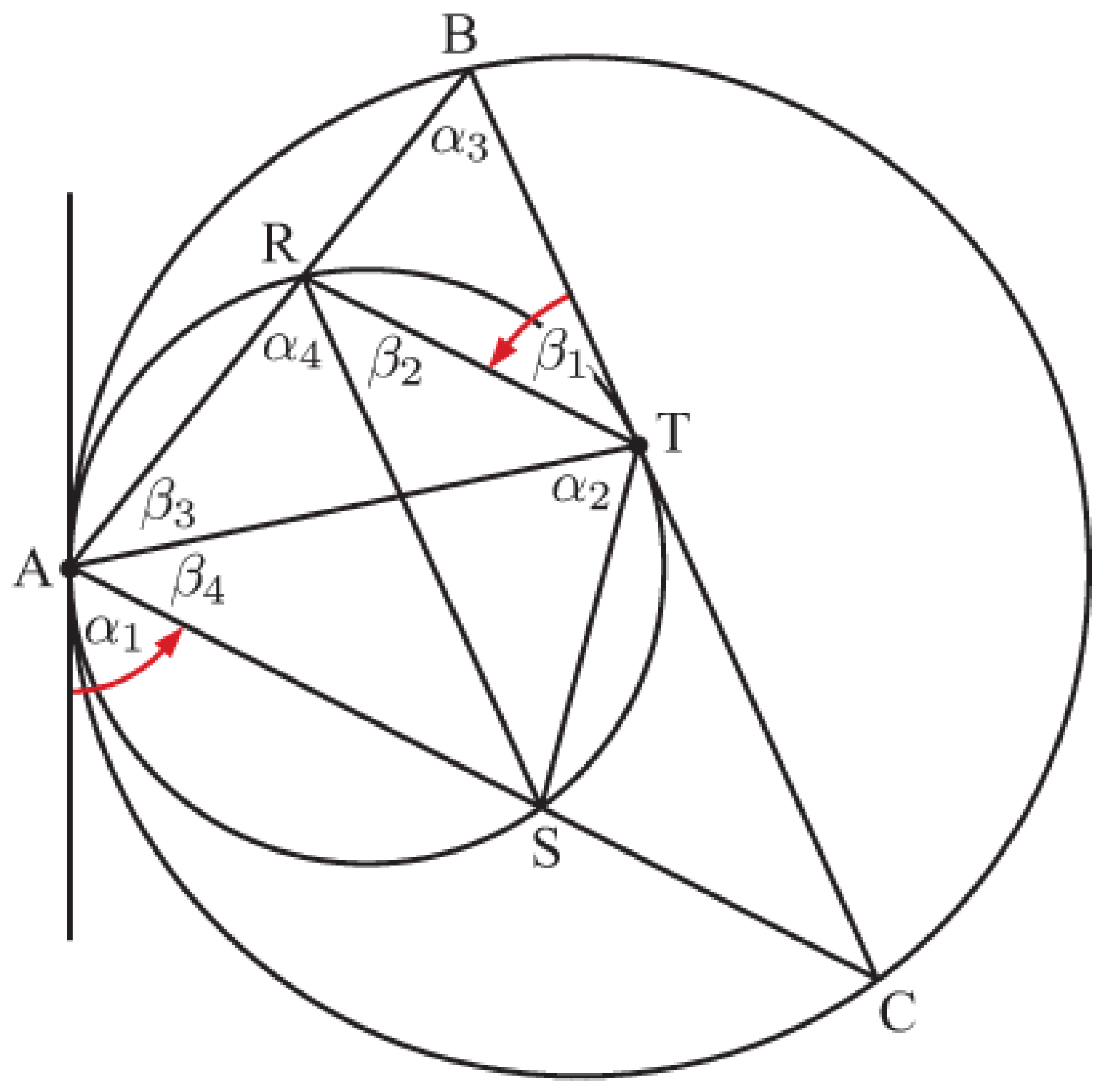
By the angle bisector theorem,

$$\frac{BR}{BS}=\frac{RT}{ST} \quad \text{and} \quad \frac{RA}{AS}=\frac{RT}{ST}$$

$\therefore \quad \dfrac{BR}{BS}=\dfrac{RA}{AS}$

$\therefore$ BR.AS = RA.BS

13



$\alpha_1=\alpha_2$ {angle between tangent and chord}

$\alpha_1=\alpha_3$ {angle between tangent and chord}

$\alpha_2=\alpha_4$ {angles subtended by the same arc}

$\therefore \quad \alpha_3=\alpha_4$

Hence, [BC] $\parallel$ [RS] {converse of equal corresponding angles}

Consequently, $\beta_1=\beta_2$ {equal alternate angles}

$\beta_1=\beta_3$ {angle between tangent and chord}

and $\beta_2=\beta_4$ {angles subtended by the same arc}

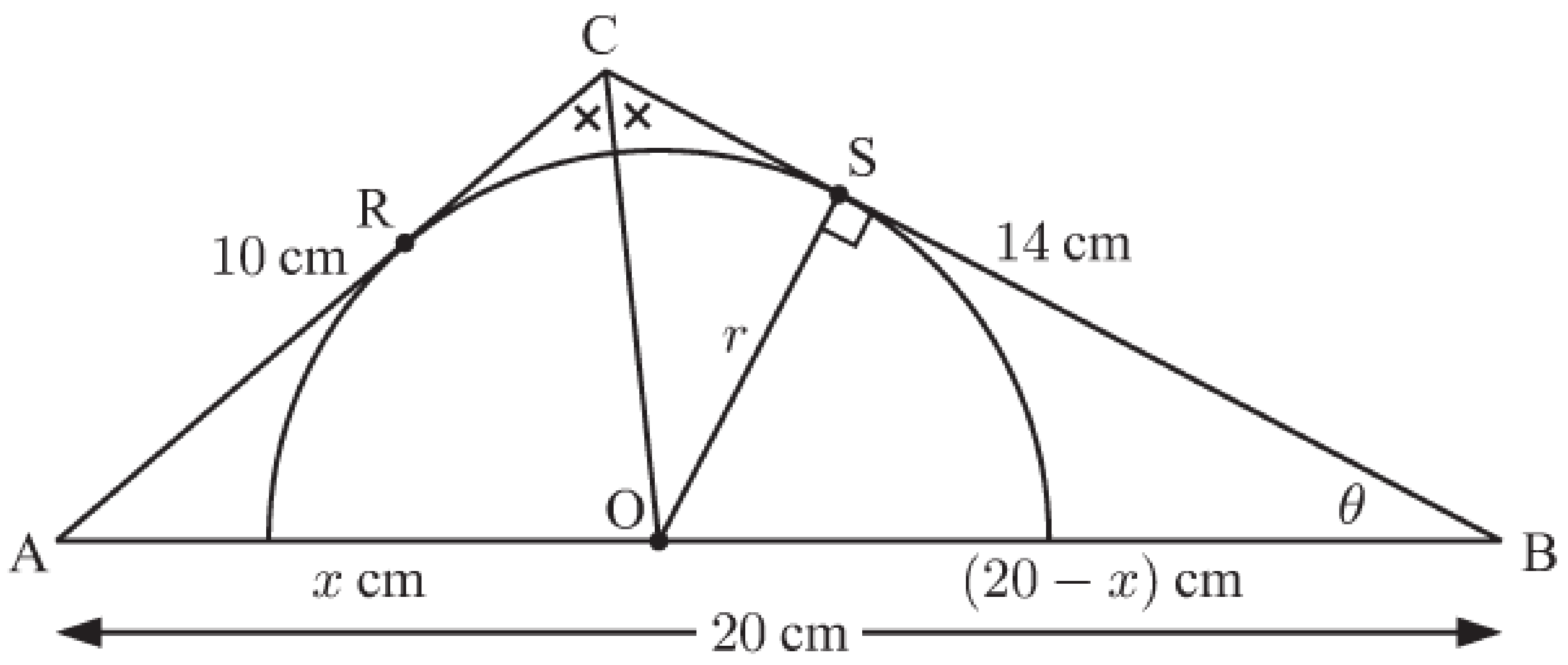
$\therefore \quad \beta_3=\beta_4$

Thus [AT] bisects $\widehat{BAC}$.

$\therefore \quad \dfrac{AB}{AC}=\dfrac{TB}{TC}$

$\therefore$ AB : AC = TB : TC

14



We join [CO] and let AO $= x$ cm.

$\therefore$ OB $= (20 - x)$ cm.

Now [CR] and [CS] are external tangents to the semi-circle.

$\therefore$ $\widehat{\text{RCO}} = \widehat{\text{SCO}}$ {tangents from an external point}

By the angle bisector theorem,

$$\frac{\text{CA}}{\text{CB}} = \frac{\text{OA}}{\text{OB}}$$

$$\therefore \frac{10}{14} = \frac{5}{7} = \frac{x}{20 - x}$$

$$\therefore 100 - 5x = 7x$$

$$\therefore 12x = 100$$

$$\therefore x = \tfrac{25}{3}$$

$$\therefore \text{OB} = \tfrac{35}{3}\text{ cm}$$

Let $\widehat{\text{ABC}} = \theta$

$$\therefore \theta = \arccos\left(\frac{20^2 + 14^2 - 10^2}{2 \times 20 \times 14}\right)$$

$$\therefore \theta = \arccos\left(\tfrac{31}{35}\right)$$

Now [OS] $\perp$ [BC] {radius-tangent theorem}

$\therefore$ in $\triangle$BOS, $\sin\theta = \dfrac{r}{\text{OB}}$

$$\therefore r = \tfrac{35}{3}\sin\left(\arccos\left(\tfrac{31}{35}\right)\right)$$

$$\therefore r \approx 5.42$$

So, the radius is 5.42 cm.

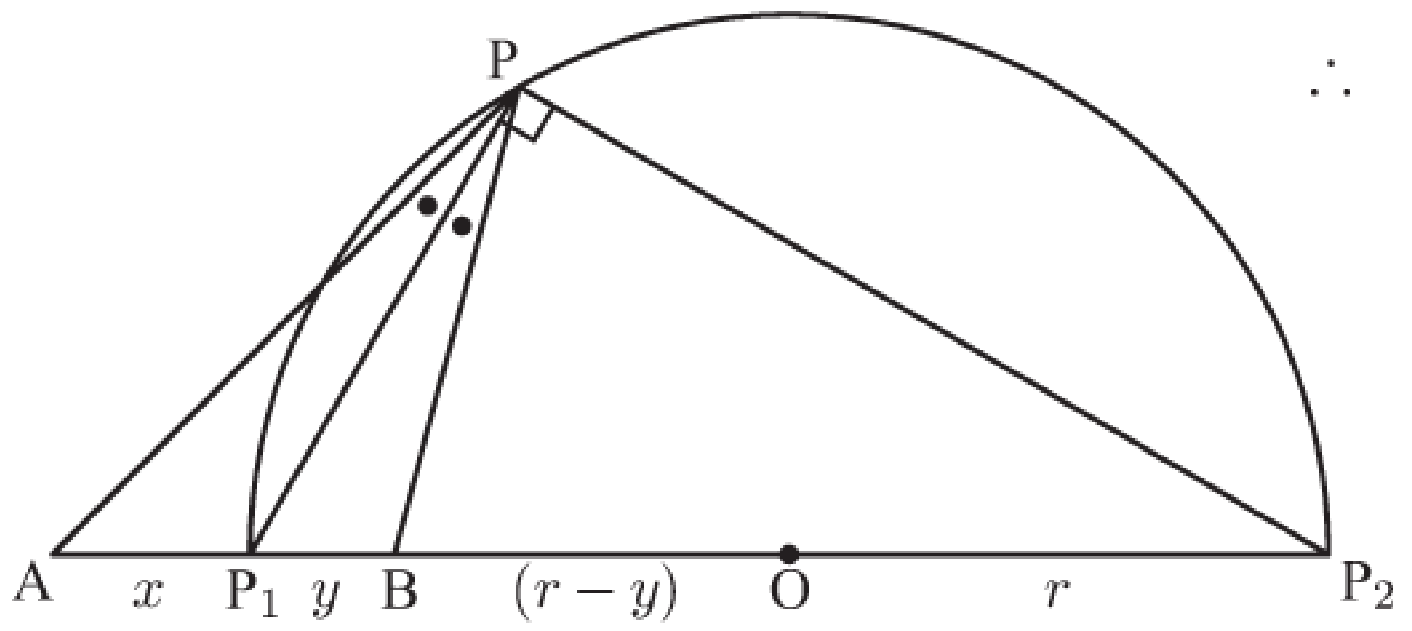
EXERCISE 21

1 **Proof of Corollary:**

Let $\text{AP}_1 = x$

$\text{P}_1\text{B} = y$

$\therefore$ BO $= r - y$



Now $\dfrac{\text{PA}}{\text{PB}} = \dfrac{\text{P}_1\text{A}}{\text{P}_1\text{B}} = \dfrac{\text{P}_2\text{A}}{\text{P}_2\text{B}}$

$$\therefore \frac{x}{y} = \frac{2r + x}{2r - y}$$

$$\therefore x(2r - y) = y(2r + x)$$

$$\therefore 2xr - xy = 2ry + xy$$

$$\therefore 2xr - 2ry = 2xy$$

$$\therefore xr - ry = xy$$

$$\therefore r(x - y) = xy$$

$$\therefore r = \frac{xy}{x - y}$$

Thus
$$\begin{aligned}\text{OA.OB} &= (r + x)(r - y)\\ &= r^2 + xr - yr - xy\\ &= r^2 + r(x - y) - xy\\ &= r^2 + \left(\frac{xy}{x - y}\right)(x - y) - xy\\ &= r^2 + xy - xy \qquad \{x \neq y\}\\ &= r^2\end{aligned}$$

2 OB $= 3$ cm and $\text{AP}_1 = 4$ cm

Now OA $= r + \text{AP}_1 = (r + 4)$ cm

But $r^2 =$ OA.OB {corollary of Apollonius' circle theorem}

$\therefore$ $r^2 = 3(r + 4)$

$\therefore$ $r^2 - 3r - 12 = 0$

$$\therefore r = \frac{3 \pm \sqrt{9 - 4(1)(-12)}}{2}$$

$$\therefore r = \frac{3 \pm \sqrt{57}}{2}$$

$$\therefore r = \frac{3 + \sqrt{57}}{2} \qquad \{\text{as } r > 0\}$$

$$\therefore r \approx 5.27\text{ cm}$$

3 **a** P(x, y), A$(-2, 0)$, and B$(4, 0)$

$\therefore$ PA $= \sqrt{(x + 2)^2 + y^2}$ and PB $= \sqrt{(x - 4)^2 + y^2}$

i
$$\frac{\text{PA}}{\text{PB}} = 1$$

$$\therefore \text{PA} = \text{PB}$$

$$\therefore \text{PA}^2 = \text{PB}^2$$

$$\therefore (x + 2)^2 + y^2 = (x - 4)^2 + y^2$$

$$\therefore \cancel{x^2} + 4x + 4 + \cancel{y^2} = \cancel{x^2} - 8x + 16 + \cancel{y^2}$$

$$\therefore 12x = 12$$

$$\therefore x = 1$$

Thus P lies on the vertical line, $x = 1$.

ii
$$\frac{\text{PA}}{\text{PB}} = \tfrac{1}{2}$$

$$\therefore \text{PB}^2 = 4[\text{PA}^2]$$

$$\therefore (x - 4)^2 + y^2 = 4[(x + 2)^2 + y^2]$$

$$\therefore x^2 - 8x + \cancel{16} + y^2 = 4x^2 + 16x + \cancel{16} + 4y^2$$

$$\therefore 3x^2 + 3y^2 + 24x = 0$$

$$\therefore x^2 + y^2 + 8x = 0$$

iii
$$\frac{\text{PA}}{\text{PB}} = 2$$

$$\therefore \text{PA}^2 = 4[\text{PB}^2]$$

$$\therefore (x + 2)^2 + y^2 = 4[(x - 4)^2 + y^2]$$

$$\therefore x^2 + 4x + 4 + y^2 = 4[x^2 - 8x + 16 + y^2]$$

$$\therefore x^2 + 4x + 4 + y^2 = 4x^2 - 32x + 64 + 4y^2$$

$$\therefore 3x^2 + 3y^2 - 36x + 60 = 0$$

$$\therefore x^2 + y^2 - 12x + 20 = 0$$

iv
$$\frac{\text{PA}}{\text{PB}} = 3$$

$$\therefore \text{PA}^2 = 9[\text{PB}^2]$$

$$\therefore (x + 2)^2 + y^2 = 9[(x - 4)^2 + y^2]$$

$$\therefore x^2 + 4x + 4 + y^2 = 9[x^2 - 8x + 16 + y^2]$$

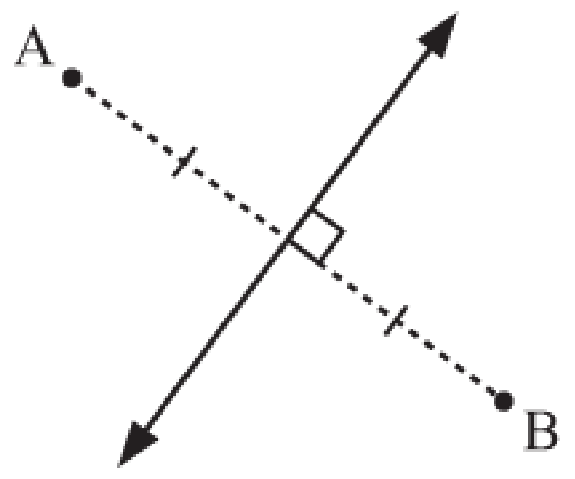
$$\therefore x^2 + 4x + 4 + y^2 = 9x^2 - 72x + 144 + 9y^2$$

$$\therefore 8x^2 + 8y^2 - 76x + 140 = 0$$

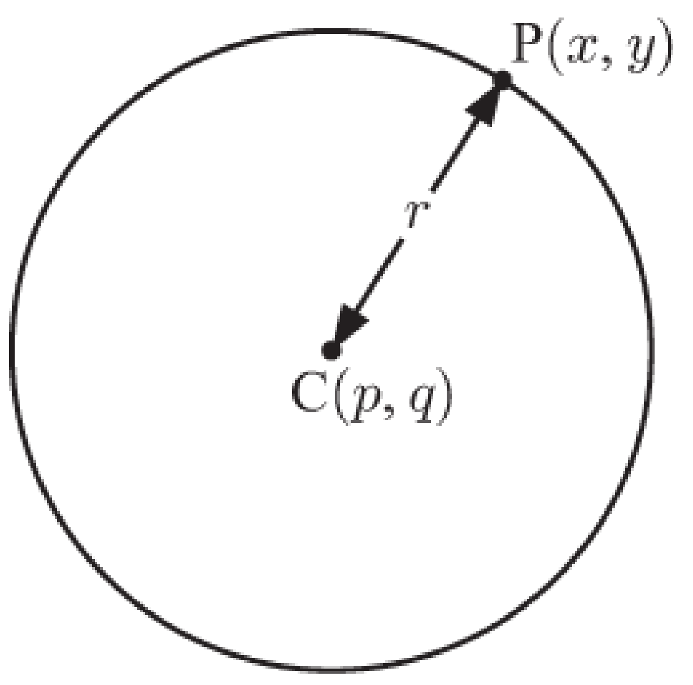
$$\therefore x^2 + y^2 - \tfrac{19}{2}x + \tfrac{35}{2} = 0$$

b When $k = 1$, $\mathrm{AP} = \mathrm{PB}$

$\therefore$ P lies on the perpendicular bisector of [AB], and this is a straight line, not a circle.



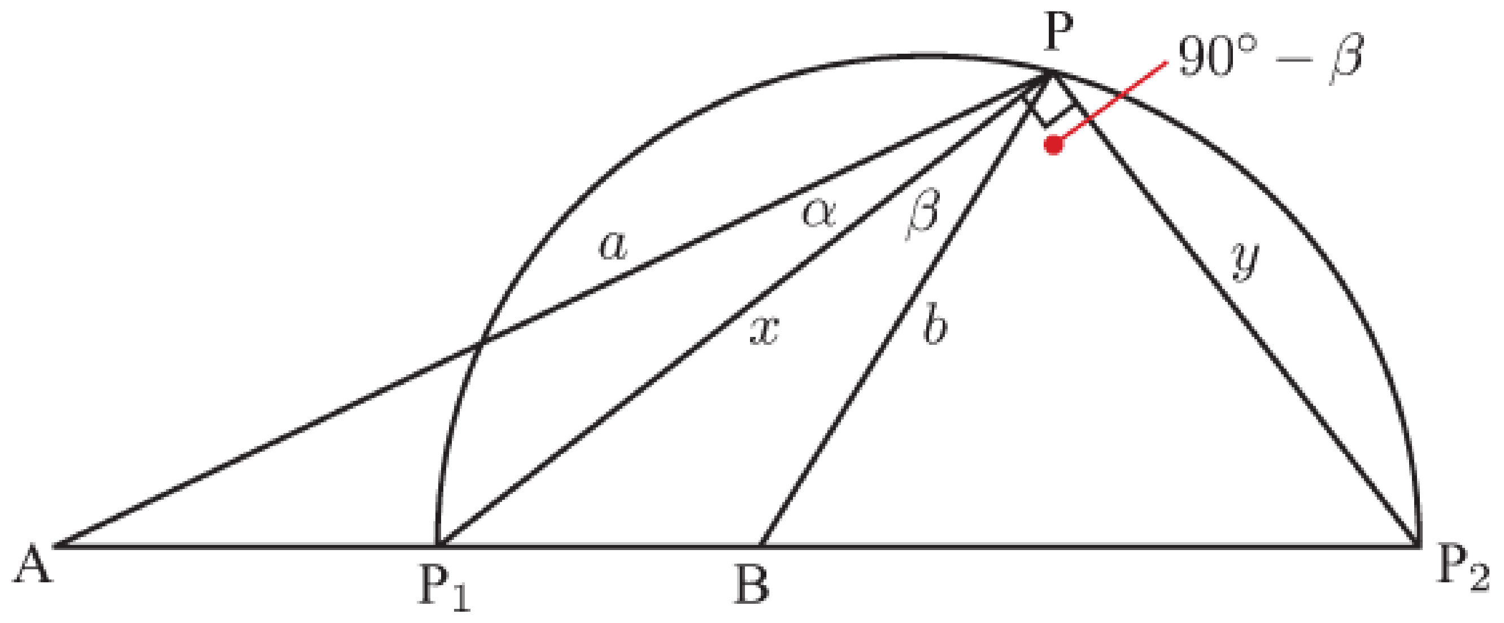
c If a circle has fixed centre $\mathrm{C}(p, q)$ and fixed radius r and $\mathrm{P}(x, y)$ moves on the circle, then $\mathrm{CP} = r$.



$$\therefore\ \mathrm{CP}^2 = r^2$$
$$\therefore\ (x-p)^2 + (y-q)^2 = r^2$$
$$\therefore\ x^2 + y^2 - 2px - 2qy + p^2 + q^2 - r^2 = 0$$

which is of the form $x^2 + y^2 + dx + ey + f = 0$ with $d = -2p$, $e = -2q$, and $f = p^2 + q^2 - r^2$.

4 a



Let $\mathrm{AP} = a$, $\mathrm{BP} = b$, $\mathrm{PP_1} = x$, $\mathrm{PP_2} = y$, $\mathrm{A\widehat{P}P_1} = \alpha$, $\mathrm{B\widehat{P}P_1} = \beta$.

As $\mathrm{P_1\widehat{P}P_2} = 90°$, $\mathrm{B\widehat{P}P_2} = 90° - \beta$.

$$\text{Now}\quad \frac{\text{area of } \triangle\mathrm{AP_1P}}{\text{area of } \triangle\mathrm{BP_1P}} = \frac{\mathrm{AP_1}}{\mathrm{BP_1}}$$

$$\text{and}\quad \frac{\text{area of } \triangle\mathrm{AP_2P}}{\text{area of } \triangle\mathrm{BP_2P}} = \frac{\mathrm{AP_2}}{\mathrm{BP_2}}$$

{area comparison theorem, altitudes are equal}

$$\therefore\ \frac{\text{area of } \triangle\mathrm{AP_1P}}{\text{area of } \triangle\mathrm{BP_1P}} = \frac{\text{area of } \triangle\mathrm{AP_2P}}{\text{area of } \triangle\mathrm{BP_2P}} \quad \left\{\text{as } \frac{\mathrm{AP_1}}{\mathrm{BP_1}} = \frac{\mathrm{AP_2}}{\mathrm{BP_2}}\right\}$$

$$\therefore\ \frac{\frac{1}{2}ax\sin\alpha}{\frac{1}{2}bx\sin\beta} = \frac{\frac{1}{2}ay\sin(90° + \alpha)}{\frac{1}{2}by\sin(90° - \beta)}$$

$$\therefore\ \frac{\sin\alpha}{\sin\beta} = \frac{\cos\alpha}{\cos\beta}$$

$$\therefore\ \frac{\sin\alpha}{\cos\alpha} = \frac{\sin\beta}{\cos\beta}$$

$$\therefore\ \tan\alpha = \tan\beta$$

$$\therefore\ \alpha = \beta$$

b As $\alpha = \beta$, the angle bisector theorem applies.

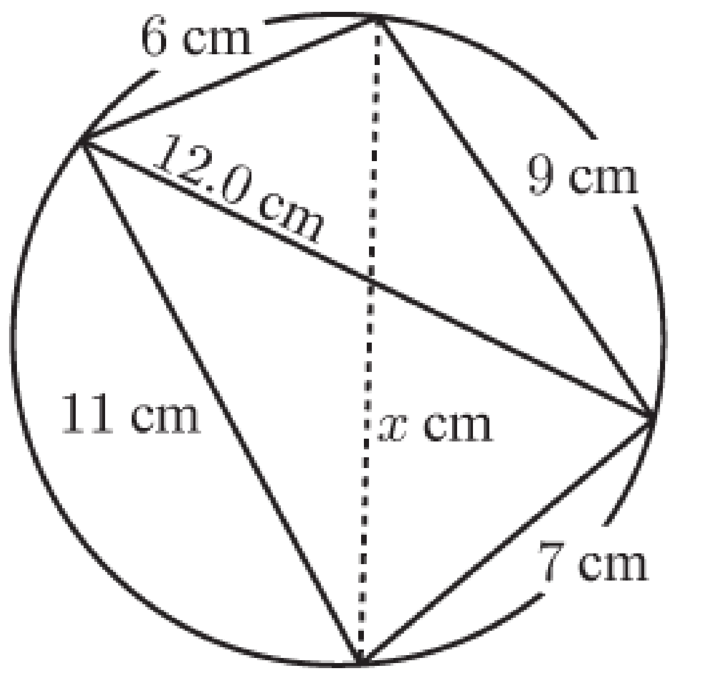
$$\therefore\ \frac{\mathrm{AP}}{\mathrm{BP}} = \frac{\mathrm{AP_1}}{\mathrm{BP_1}} \quad \left(\text{or } = \frac{\mathrm{AP_2}}{\mathrm{BP_2}}\right)$$

$$\therefore\ \frac{\mathrm{AP}}{\mathrm{BP}} = k, \text{ a positive constant}$$

{as A, B, and $\mathrm{P_1}$ are fixed}

EXERCISE 2J

1



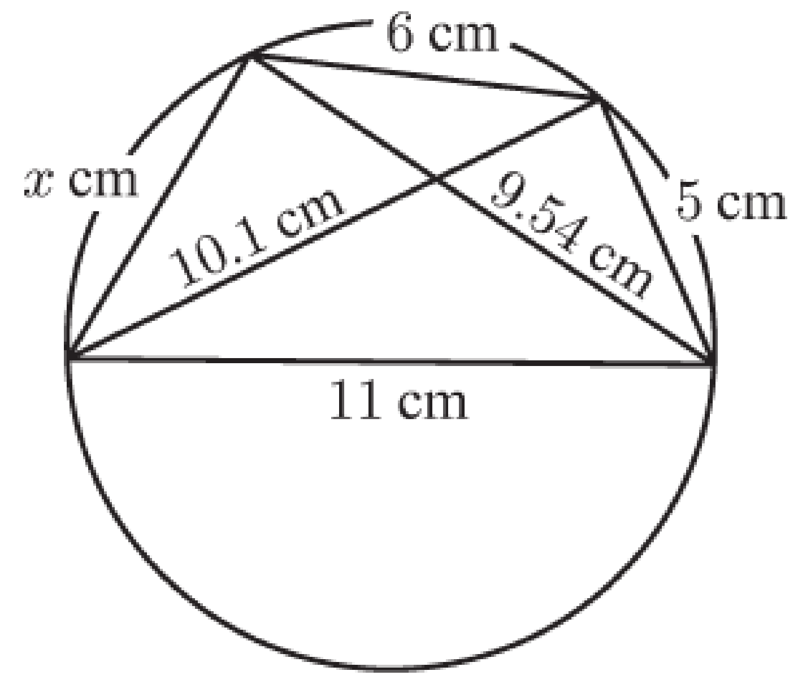
Let the other diagonal be x cm.

By Ptolemy's theorem,

$$12.0x \approx 6 \times 7 + 11 \times 9$$
$$\therefore\ 12.0x \approx 141$$
$$\therefore\ x \approx 11.8$$

$\therefore$ the other diagonal is approximately 11.8 cm long.

2



Let the 4th side be x cm.

By Ptolemy's theorem,

$$5x + 6 \times 11 \approx 10.1 \times 9.54$$
$$\therefore\ x \approx \frac{10.1 \times 9.54 - 6 \times 11}{5}$$
$$\therefore\ x \approx 6.07$$

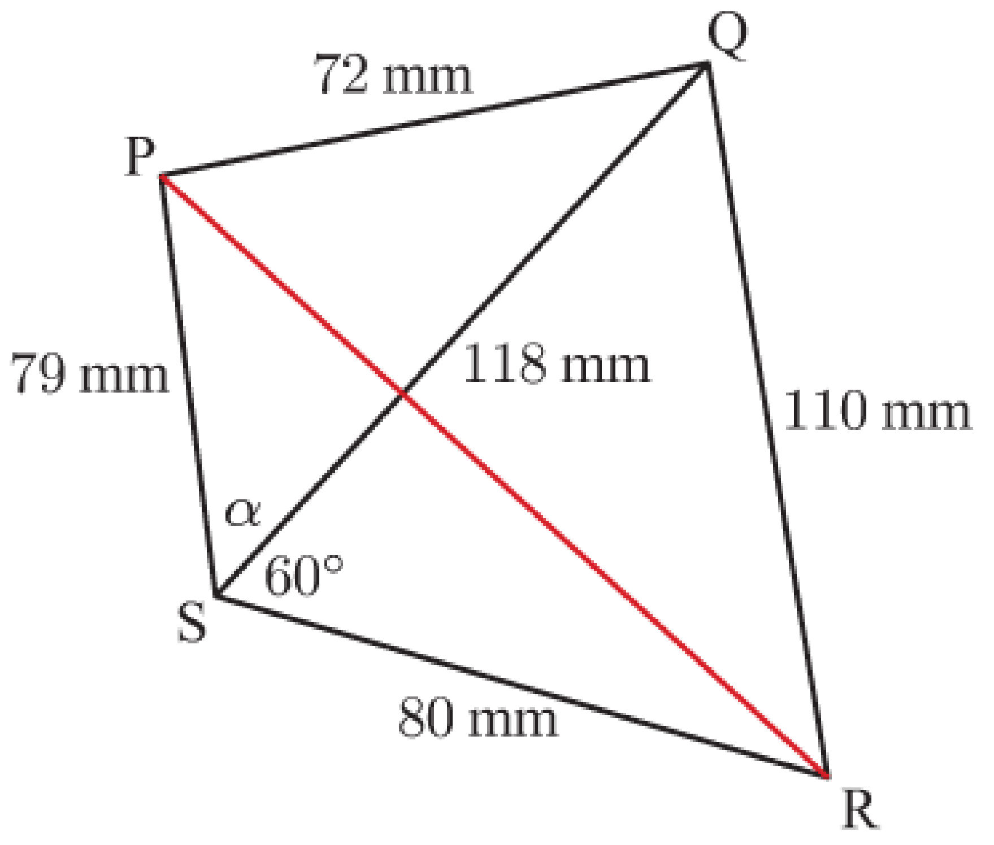
$\therefore$ the 4th side is approximately 6.07 cm long.

3 a

PQ.RS + SP.QR	and	PR.QS
$= 7 \times 13 + 9 \times 11$		$= 14 \times 12$
$= 190$		$= 168$

$\therefore$ PQRS is not a cyclic quadrilateral.

b



$$\text{In } \triangle\mathrm{PQS},\quad \cos\alpha = \frac{79^2 + 118^2 - 72^2}{2 \times 79 \times 118}$$
$$\therefore\ \alpha = \arccos\left(\tfrac{14\,981}{18\,644}\right)$$
$$\therefore\ \mathrm{P\widehat{S}R} = 60° + \arccos\left(\tfrac{14\,981}{18\,644}\right)$$

In $\triangle\mathrm{PSR}$,

$$\mathrm{PR}^2 = 79^2 + 80^2 - 2 \times 79 \times 80\cos\left(60° + \arccos\left(\tfrac{14\,981}{18\,644}\right)\right)$$

$\therefore$ $\mathrm{PR} \approx 118.65$

Now PQ.RS + SP.QR	and	SQ.PR
$= 72 \times 80 + 79 \times 110$		$\approx 118 \times 118.65$
$= 14\,450$		$\approx 14\,001$

$\therefore$ PQRS is not a cyclic quadrilateral.

4 a By Ptolemy's theorem, $xz + wy = mn$.

b $xz + wy =$ sum of areas of blue rectangles

$mn =$ area of brown rectangle

$\therefore$ using Ptolemy's theorem, blue area = brown area.

c When ABCD is a rectangle,

$x = z$, $w = y$ {opposite sides are equal}

and $m = n$ {diagonals are equal}

$\therefore$ Ptolemy's theorem becomes $x^2 + y^2 = m^2$ which is Pythagoras' theorem.

5 **a** By the Cosine rule,

$$\begin{cases} x^2 = a^2 + d^2 - 2ad\cos\theta \quad \text{and} \\ x^2 = c^2 + b^2 - 2cb\cos(180^\circ - \theta) \end{cases}$$

But $\cos(180^\circ - \theta) = -\cos\theta$

$$\therefore \quad \frac{x^2 - a^2 - d^2}{-\cancel{2}ad} = \frac{x^2 - c^2 - b^2}{\cancel{2}bc}$$

$$\therefore \quad (x^2 - a^2 - d^2)bc = -ad(x^2 - c^2 - b^2)$$

$$\therefore \quad x^2bc - a^2bc - bcd^2 = -adx^2 + ac^2d + ab^2d$$

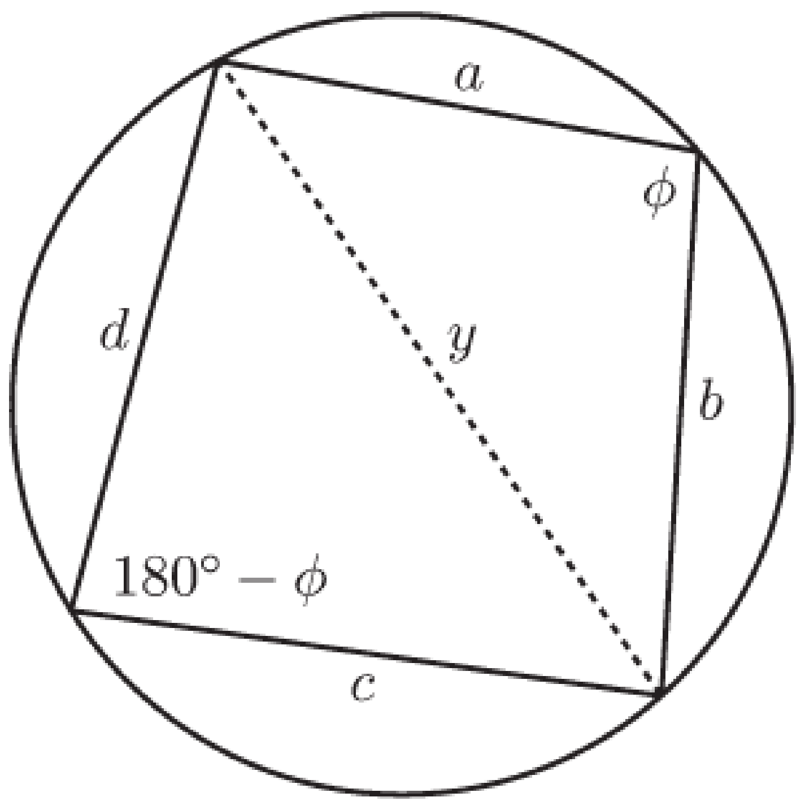
$$\therefore \quad x^2(bc + ad) = ac^2d + ab^2d + a^2bc + bcd^2$$

$$\therefore \quad x^2(bc + ad) = ac(cd + ab) + bd(ab + cd)$$

$$\therefore \quad x^2(bc + ad) = (ab + cd)(ac + bd)$$

$$\therefore \quad x^2 = \frac{(ac + bd)(ab + cd)}{(bc + ad)}$$

b



By similar reasoning,

$$y^2 = \frac{(bd + ca)(bc + da)}{cd + ba}$$

{replacing d by a, a by b, c by d, and b by c}

$$\therefore \quad y^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}$$

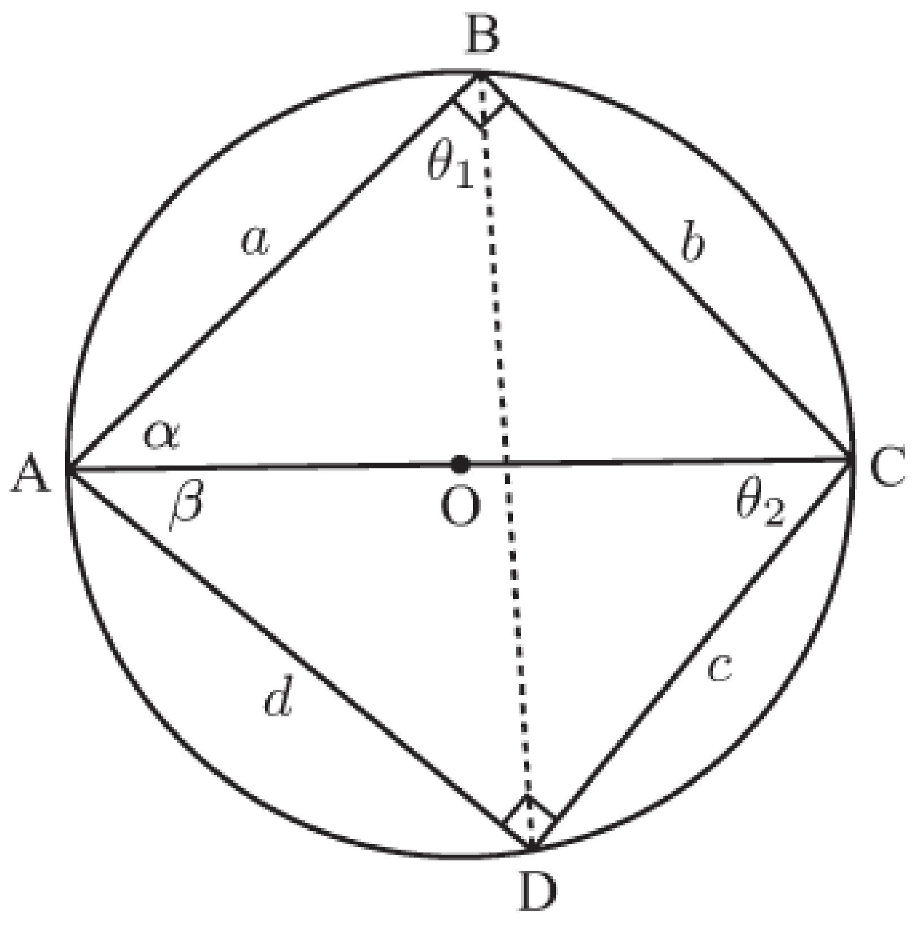
c

$$x^2y^2 = \frac{(ac + bd)(ab + cd)}{bc + ad} \cdot \frac{(ac + bd)(ad + bc)}{ab + cd}$$

$$\therefore \quad x^2y^2 = (ac + bd)^2$$

$$\therefore \quad ac + bd = xy$$

6



$r = 1 \quad \therefore \quad \text{AC} = 2$

$\widehat{\text{ABC}} = \widehat{\text{ADC}} = 90^\circ$

{angles in a semi-circle}

$$\sin\alpha = \frac{b}{2}$$

$$\cos\alpha = \frac{a}{2}$$

$$\sin\beta = \frac{c}{2}$$

$$\cos\beta = \frac{d}{2}$$

$$\therefore \quad ac + bd = 4\cos\alpha\sin\beta + 4\sin\alpha\cos\beta \quad \text{.... (1)}$$

$\theta_1 = \theta_2$ {angles subtended by the same arc}

In △ACD, $\sin\theta = \dfrac{d}{2}$

In △ABD, using the Sine rule,

$$\frac{\text{BD}}{\sin(\alpha + \beta)} = \frac{d}{\sin\theta}$$

$$\therefore \quad \text{BD} = \frac{d\sin(\alpha + \beta)}{\frac{d}{2}}$$

$$\therefore \quad \text{BD} = 2\sin(\alpha + \beta)$$

$$\text{AC.BD} = 2\text{BD}$$

$$\therefore \quad \text{AC.BD} = 4\sin(\alpha + \beta) \quad \text{.... (2)}$$

$$\text{AC.BD} = ac + bd \quad \{\text{Ptolemy's theorem}\}$$

$$\therefore \quad 4\sin(\alpha + \beta) = 4\sin\alpha\cos\beta + 4\cos\alpha\sin\beta$$

{using (1) and (2)}

$$\therefore \quad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

EXERCISE 2K.1

1 AT : TB = 3 : 7 and BS : SC = 5 : 3

By Ceva's theorem, $\dfrac{\text{AT}}{\text{TB}} \cdot \dfrac{\text{BS}}{\text{SC}} \cdot \dfrac{\text{CR}}{\text{RA}} = 1$

$$\therefore \quad \tfrac{3}{7} \times \tfrac{5}{3} \times \frac{\text{CR}}{\text{RA}} = 1$$

$$\therefore \quad \frac{\text{CR}}{\text{RA}} = \tfrac{7}{5}$$

∴ R divides [AC] in the ratio 5 : 7.

2 **a** $\text{BD} = \frac{1}{2}\text{BC} \quad \therefore \quad \text{BD} : \text{DC} = 1 : 1$

$\text{CE} = \frac{2}{3}\text{CA} \quad \therefore \quad \text{CE} : \text{EA} = 2 : 1$

By Ceva's theorem, $\dfrac{\text{AF}}{\text{FB}} \cdot \dfrac{\text{BD}}{\text{DC}} \cdot \dfrac{\text{CE}}{\text{EA}} = 1$

$$\therefore \quad \frac{\text{AF}}{\text{FB}} \times \tfrac{1}{1} \times \tfrac{2}{1} = 1$$

$$\therefore \quad \frac{\text{AF}}{\text{FB}} = \tfrac{1}{2}$$

$$\therefore \quad \text{AF} : \text{FB} = 1 : 2$$

b We have that AE : EC = 1 : 2.

We use the theorem that if two triangles have the same altitude, then the ratio of their areas is the same as the ratio of their bases.

$$\frac{\text{area of } \triangle\text{ABE}}{\text{area of } \triangle\text{CBE}} = \tfrac{1}{2} \quad \{\text{equal altitudes}\}$$

$$\frac{\text{area of } \triangle\text{AOE}}{\text{area of } \triangle\text{COE}} = \tfrac{1}{2} \quad \{\text{equal altitudes}\}$$

$$\begin{aligned} \text{area of } \triangle\text{AOB} &= \text{area of } \triangle\text{ABE} - \text{area of } \triangle\text{AOE} \\ &= \frac{\text{area of } \triangle\text{CBE}}{2} - \frac{\text{area of } \triangle\text{COE}}{2} \\ &= \frac{\text{area of } \triangle\text{CBE} - \text{area of } \triangle\text{COE}}{2} \\ &= \frac{\text{area of } \triangle\text{BOC}}{2} \end{aligned}$$

∴ area of △AOB : area of △BOC = 1 : 2

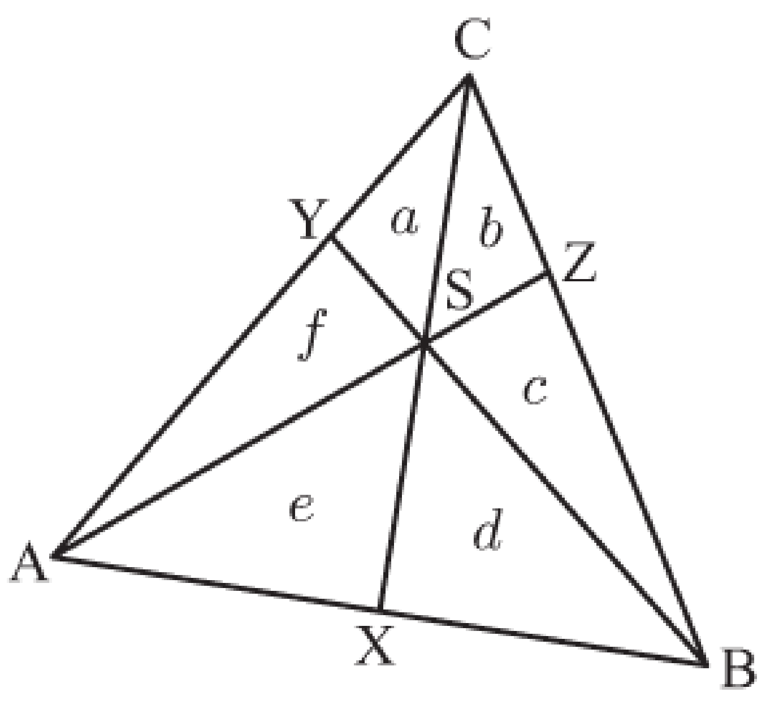
3 **a** By Ceva's theorem, $\dfrac{\text{AX}}{\text{XB}} \times \dfrac{\text{BZ}}{\text{ZC}} \times \dfrac{\text{CY}}{\text{YA}} = 1$

$$\therefore \quad \frac{\text{AX}}{\text{XB}} \times 2 \times \frac{2}{3} = 1$$

$$\therefore \quad \frac{\text{AX}}{\text{XB}} = \frac{3}{4}$$

∴ X divides [AB] in the ratio 3 : 4.

b



Now $c = 2b$

$f = \frac{3}{2}a$

$d = \frac{4}{3}e$

{areas of triangles are proportional to their bases if altitudes are equal}

Similarly, $e + d + c = 2(f + a + b)$

$\therefore\ e + \frac{4}{3}e + \cancel{2b} = 3a + 2a + \cancel{2b}$

$\therefore\ a = \frac{7}{15}e$ and so $f = \frac{7}{10}e$

Also, $4(a + f + e) = 3(b + c + d)$

$\therefore\ 4(\frac{7}{15}e + \frac{7}{10}e + e) = 3(b + 2b + \frac{4}{3}e)$

$\therefore\ \frac{26}{3}e = 4e + 9b$

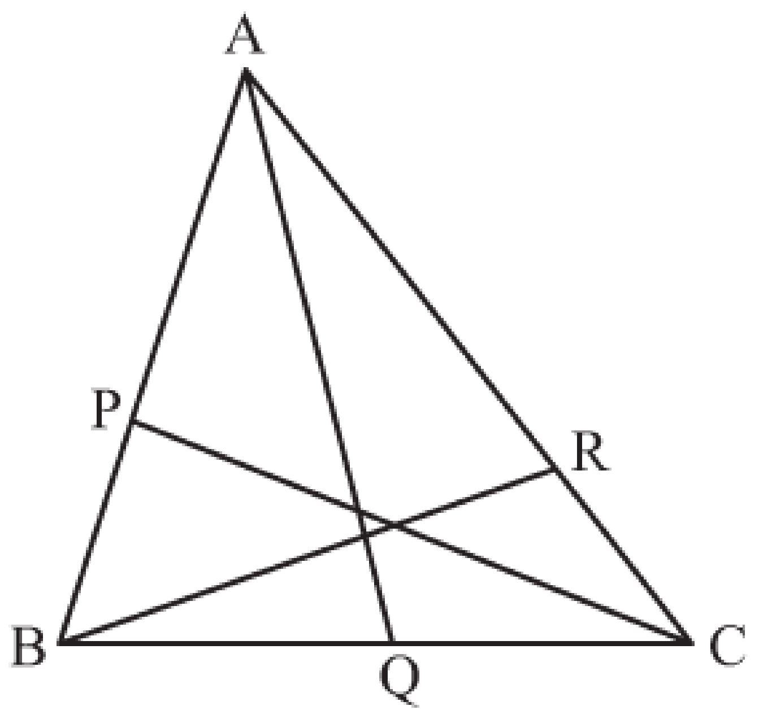
$\therefore\ b = \frac{17}{21}e$

$\therefore$ AS : SZ $= f + a : b$

$= \frac{7}{15}\cancel{e} + \frac{7}{10}\cancel{e} : \frac{17}{21}\cancel{e}$

$= 49 : 34$

4



$\text{AP} = \frac{2}{3}\text{AB}$

$\therefore$ AP : PB = 2 : 1

$\text{BQ} = \frac{3}{4}\text{BC}$

$\therefore$ BQ : QC = 3 : 1

$\text{CR} = \frac{1}{7}\text{CA}$

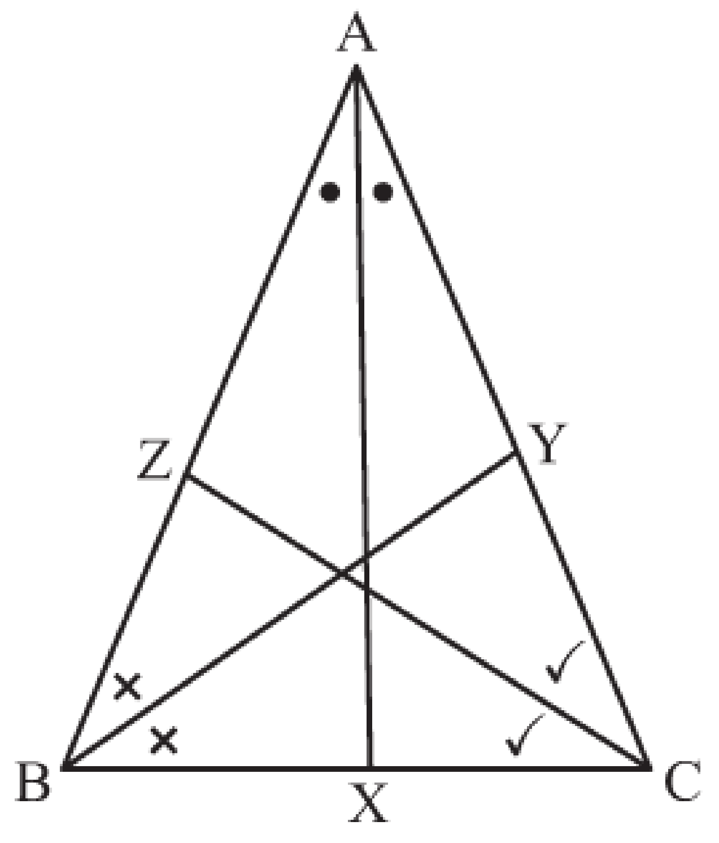
$\therefore$ CR : RA = 1 : 6

Now $\frac{\text{AP}}{\text{PB}} \cdot \frac{\text{BQ}}{\text{QC}} \cdot \frac{\text{CR}}{\text{RA}} = \frac{2}{1} \times \frac{3}{1} \times \frac{1}{6}$

$= 1$

$\therefore$ [AQ], [BR], and [CP] are concurrent.
{converse of Ceva's theorem}

5



Let the triangle have vertices A, B, and C.
The angle bisector [AX] meets [BC] at X.
[BY] meets [AC] at Y.
[CZ] meets [AB] at Z.
By the angle bisector theorem,

$\frac{\text{AB}}{\text{AC}} = \frac{\text{BX}}{\text{XC}},\ \ \frac{\text{BC}}{\text{BA}} = \frac{\text{CY}}{\text{YA}},$

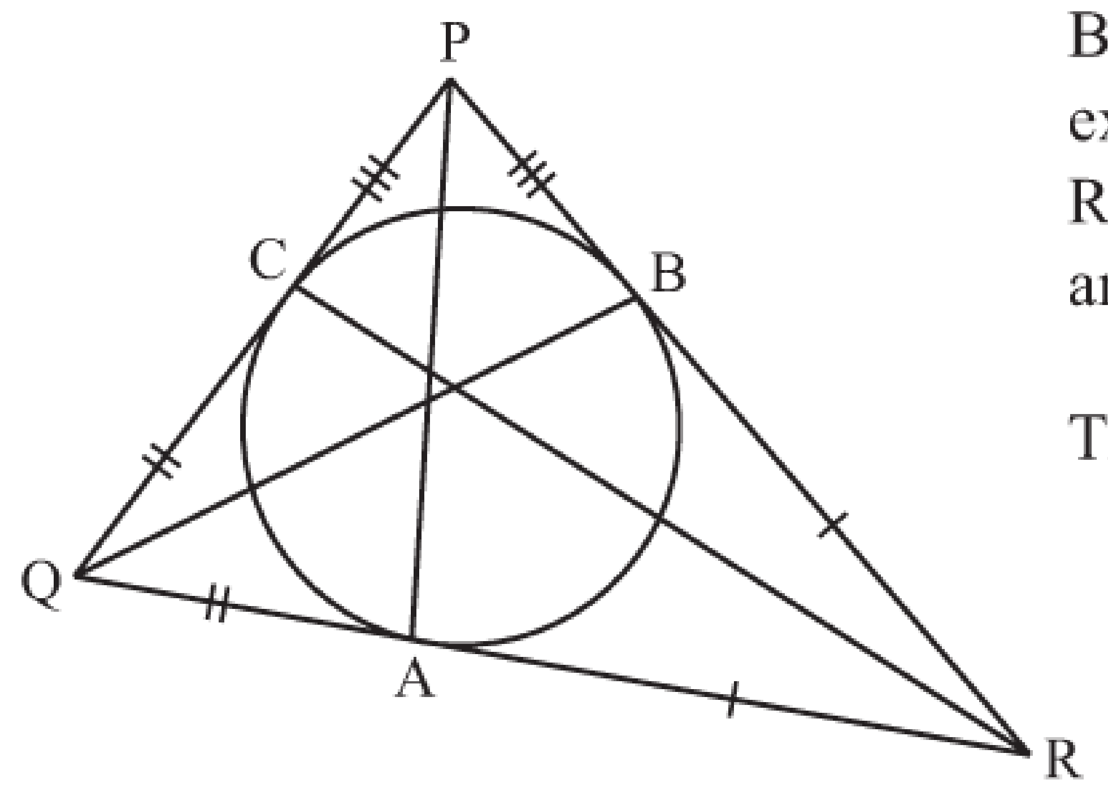
and $\frac{\text{CA}}{\text{CB}} = \frac{\text{AZ}}{\text{ZB}}.$

$\therefore\ \frac{\text{BX}}{\text{XC}} \cdot \frac{\text{CY}}{\text{YA}} \cdot \frac{\text{AZ}}{\text{ZB}} = \frac{\text{AB}}{\text{AC}} \cdot \frac{\text{BC}}{\text{BA}} \cdot \frac{\text{CA}}{\text{CB}}$

$= 1$

$\therefore$ by Ceva's theorem, [AX], [BY], and [CZ] are concurrent.

6

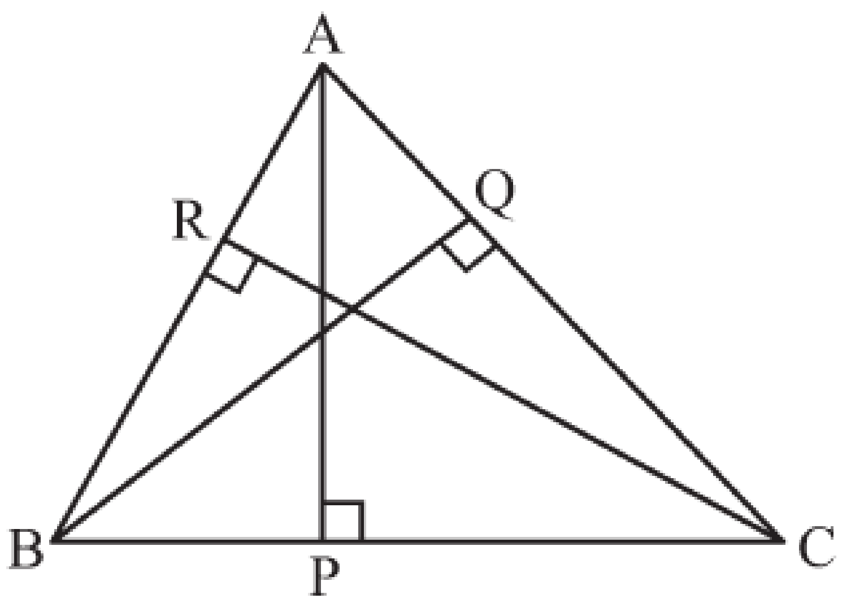


By the tangents from an external point theorem, RA = RB, QA = QC, and PB = PC.

Thus $\frac{\text{RA}}{\text{AQ}} \cdot \frac{\text{QC}}{\text{CP}} \cdot \frac{\text{PB}}{\text{BR}} = 1$

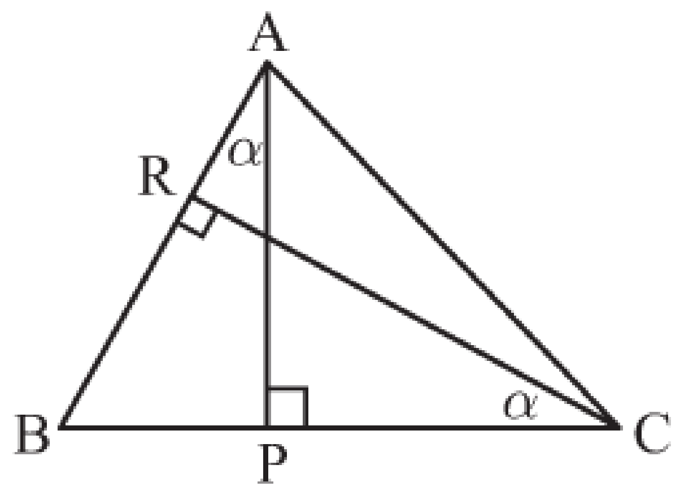
$\therefore$ [PA], [QB], and [RC] are concurrent.

7



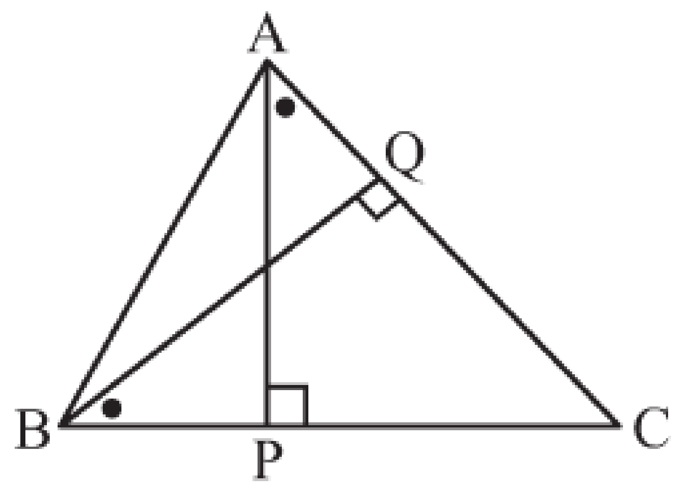
Let the triangle have vertices A, B, and C. The altitudes from A, B, and C meet [BC], [AC], and [AB] at P, Q, and R respectively.

We need to prove that $\frac{\text{AR}}{\text{RB}} \cdot \frac{\text{BP}}{\text{PC}} \cdot \frac{\text{CQ}}{\text{QA}} = 1$



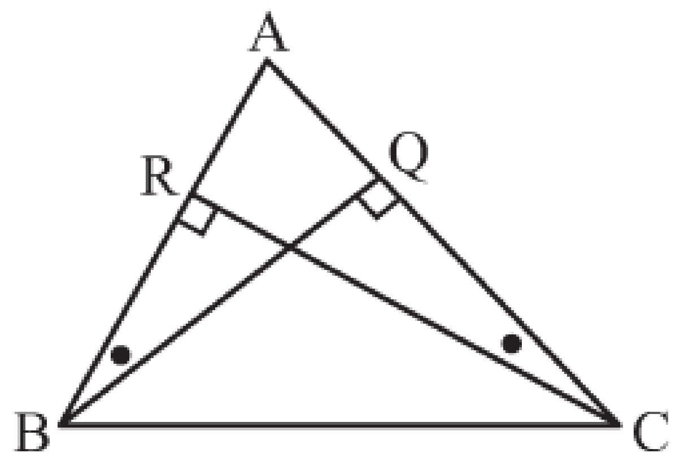
△s ABP and CBR are equiangular, and therefore similar.
{Equal angles are marked.}

$\therefore\ \frac{\text{AB}}{\text{CB}} = \frac{\text{BP}}{\text{BR}} = \frac{\text{AP}}{\text{CR}}$ .... (1)



Likewise △s BQC and APC are similar.

$\therefore\ \frac{\text{BC}}{\text{AC}} = \frac{\text{BQ}}{\text{AP}} = \frac{\text{QC}}{\text{PC}}$ .... (2)



Finally, also △s ACR and ABQ are similar.

$\therefore\ \frac{\text{AC}}{\text{AB}} = \frac{\text{CR}}{\text{BQ}} = \frac{\text{AR}}{\text{AQ}}$ .... (3)

$\therefore\ \frac{\text{AR}}{\text{RB}} \cdot \frac{\text{BP}}{\text{PC}} \cdot \frac{\text{CQ}}{\text{QA}} = \frac{\text{BP}}{\text{RB}} \cdot \frac{\text{QC}}{\text{PC}} \cdot \frac{\text{AR}}{\text{AQ}}$ {rearranging to fit (1), (2), (3)}

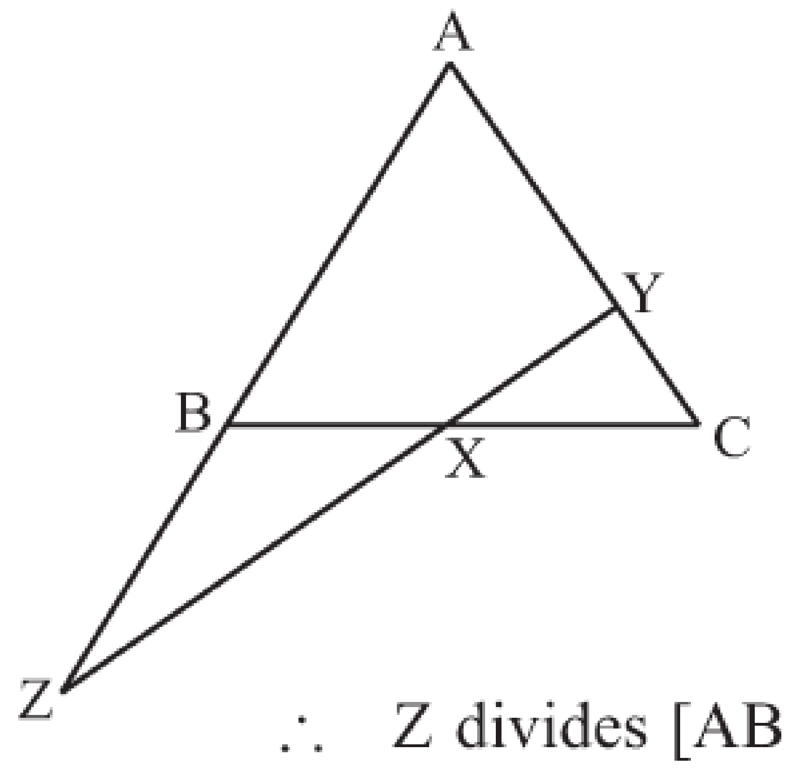
$= \frac{{}^1\cancel{\text{AB}}}{{}_1\cancel{\text{CB}}} \cdot \frac{\cancel{\text{BC}}^1}{\cancel{\text{AC}}_1} \cdot \frac{\cancel{\text{AC}}^1}{\cancel{\text{AB}}_1}$

$= 1$

So, by Ceva's theorem, [AP], [BQ], and [CR] are concurrent.

EXERCISE 2K.2

1



By Menelaus' theorem,

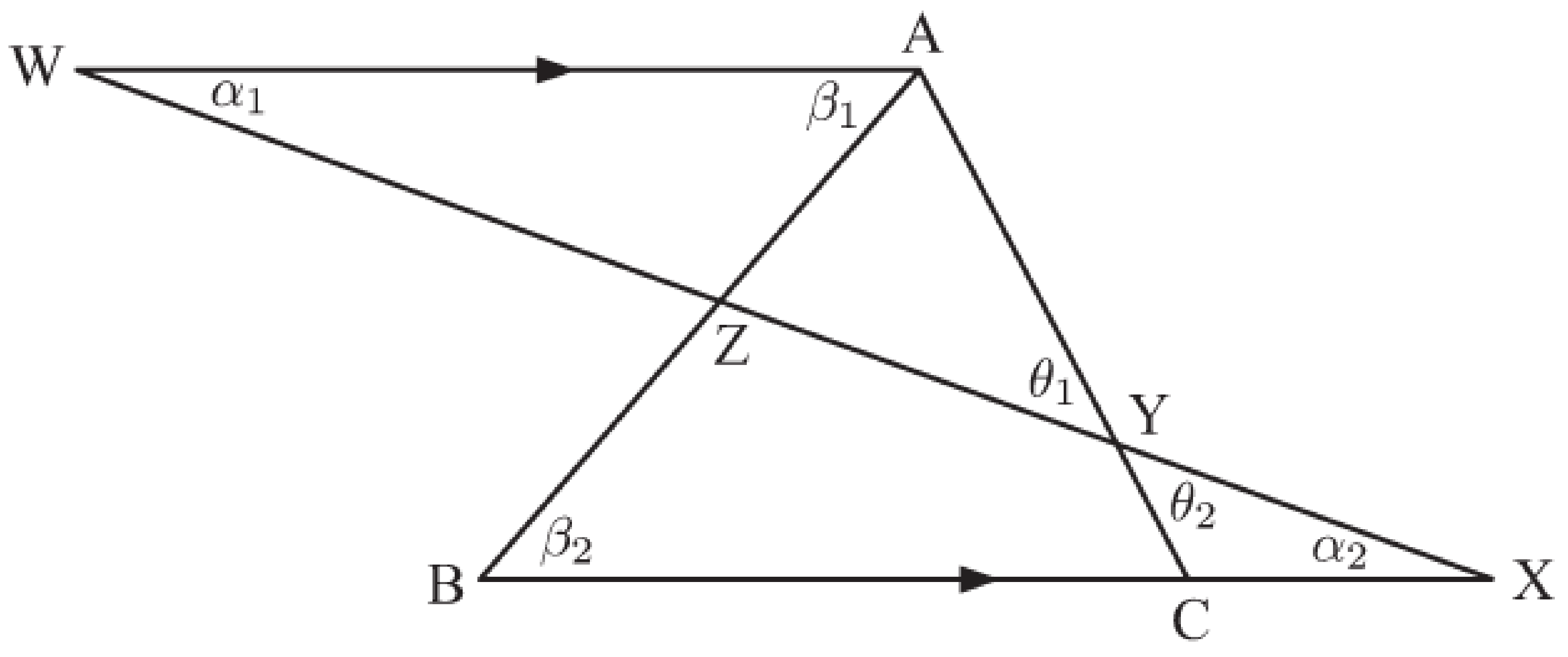
$\frac{\text{BX}}{\text{XC}} \cdot \frac{\text{CY}}{\text{YA}} \cdot \frac{\text{AZ}}{\text{ZB}} = -1$

$\therefore\ \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{\text{AZ}}{\text{ZB}} = -1$

$\therefore\ \frac{\text{AZ}}{\text{ZB}} = -\frac{10}{3}$

$\therefore$ Z divides [AB] externally in the ratio 10 : 3.

2



In △s AWZ and BXZ,

$\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {equal alternate angles}

$\therefore$ the triangles are similar

$\therefore\ \frac{\text{AZ}}{\text{BZ}} = \frac{\text{AW}}{\text{BX}}$ .... (1)

In △s AYW and CYX,

$\alpha_1 = \alpha_2$, and $\theta_1 = \theta_2$ {vertically opposite angles}

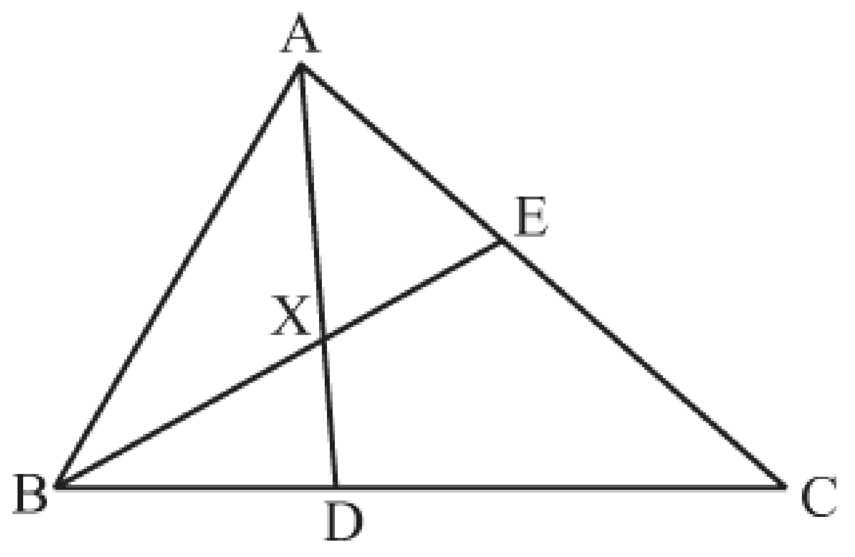
∴ the triangles are similar

$$\therefore \quad \frac{AW}{CX} = \frac{AY}{CY} \quad .... (2)$$

$$\begin{aligned} \text{Thus} \quad \frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA} &= \left(-\frac{AW}{BX}\right)\cdot\frac{BX}{XC}\cdot\left(-\frac{CX}{AW}\right) \\ &= \frac{CX}{XC} \\ &= -1 \end{aligned}$$

3 D divides [BC] in the ratio 2 : 3 $\therefore \frac{BD}{DC} = \frac{2}{3}$

E divides [CA] in the ratio 5 : 4 $\therefore \frac{CE}{EA} = \frac{5}{4}$



Let [BE] and [AD] meet at X.

Now BXE is a transversal of △ADC.

∴ by Menelaus' theorem,

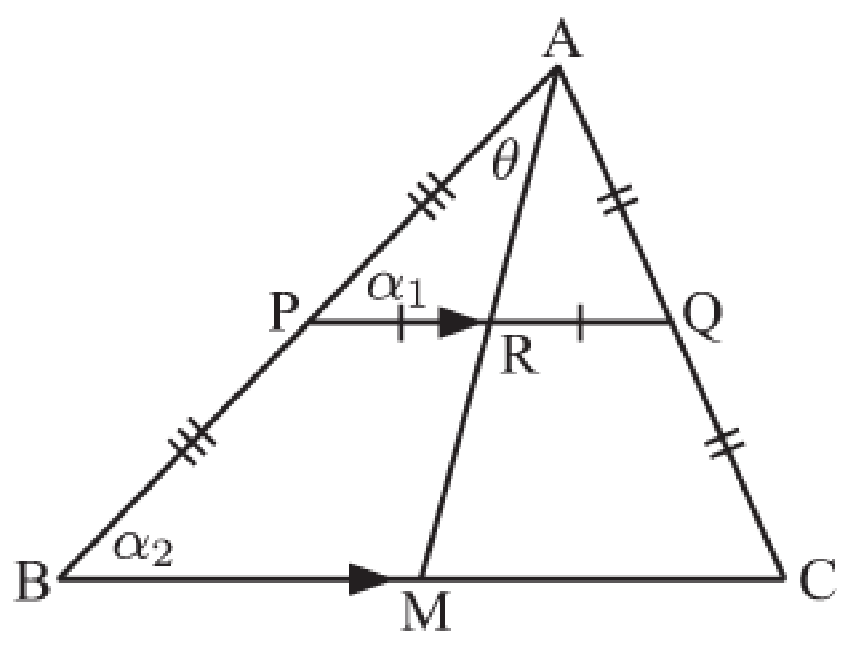
$$\frac{AX}{XD}\cdot\frac{DB}{BC}\cdot\frac{CE}{EA} = -1$$

$$\therefore \quad \frac{AX}{XD}\cdot\left(-\frac{2}{5}\right)\cdot\frac{5}{4} = -1$$

$$\therefore \quad \frac{AX}{XD} = \frac{2}{1}$$

∴ X divides [AD] in the ratio 2 : 1.

4 **a**



[PQ] ∥ [BC] {midpoint theorem}

△APR is similar to △ABM {$\alpha_1 = \alpha_2$, equal corresponding angles, and θ is common}

$$\therefore \quad \frac{PR}{BM} = \frac{AP}{AB} = \frac{1}{2}$$

∴ BM = 2(PR) .... (1)

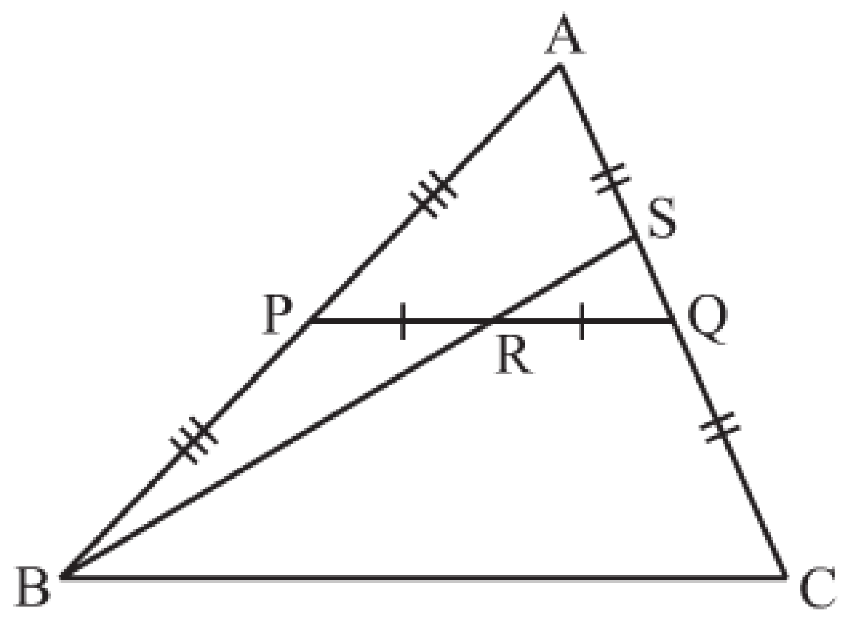
Likewise △ARQ is similar to △AMC and MC = 2(RQ) .... (2)

Using (1) and (2), as PR = RQ

∴ BM = MC

∴ M is the midpoint of [BC].

b



BRS is a transversal of △APQ. So, by Menelaus,

$$\frac{AS}{SQ}\cdot\frac{QR}{RP}\cdot\frac{PB}{BA} = -1$$

$$\therefore \quad \frac{AS}{SQ}\cdot\frac{1}{1}\cdot -\frac{1}{2} = -1$$

$$\therefore \quad \frac{AS}{SQ} = \frac{2}{1}$$

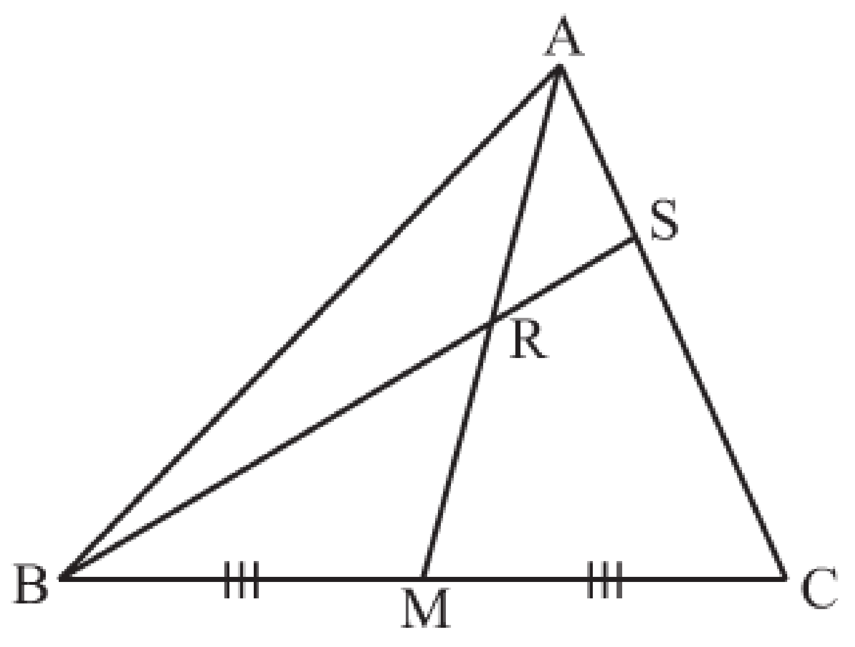
∴ AS : SQ : QC = 2 : 1 : 3

∴ AS : SC = 2 : 4

= 1 : 2

∴ S divides [AC] in the ratio 1 : 2.

c



Also ARM is a transversal of △SBC, and by Menelaus,

$$\frac{BR}{RS}\cdot\frac{SA}{AC}\cdot\frac{CM}{MB} = -1$$

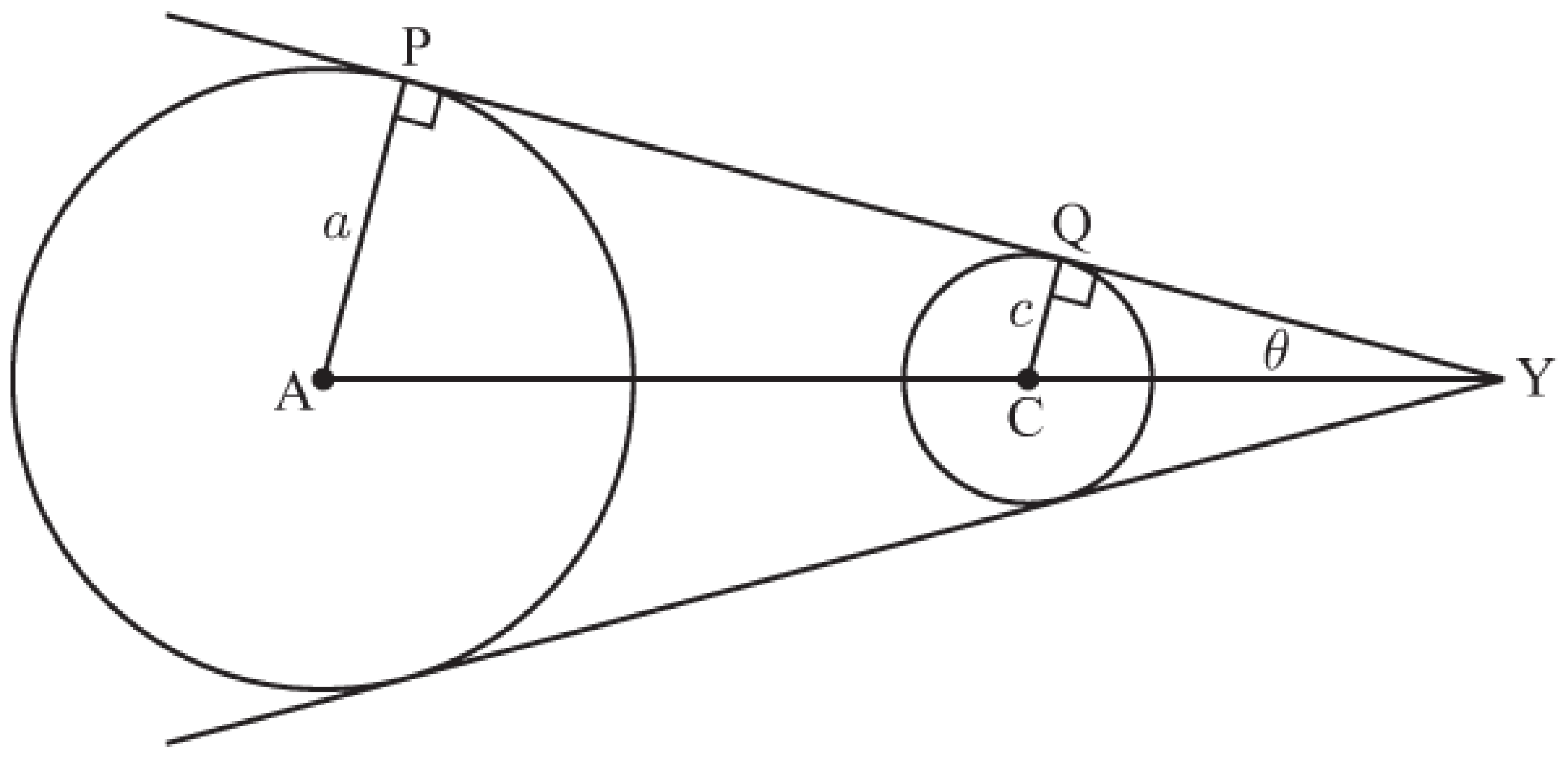
$$\therefore \quad \frac{BR}{RS}\cdot -\frac{1}{3}\cdot\frac{1}{1} = -1$$

$$\therefore \quad \frac{BR}{RS} = \frac{3}{1}$$

∴ BR : RS = 3 : 1

∴ R divides [BS] in the ratio 3 : 1.

5



△s APY and CQY are equiangular, and therefore similar.

$$\therefore \quad \frac{AY}{CY} = \frac{AP}{CQ} = \frac{a}{c}$$

Thus $\frac{AY}{CY} = \frac{a}{c}$ and likewise $\frac{BX}{CX} = \frac{b}{c}$ and $\frac{AZ}{BZ} = \frac{a}{b}$

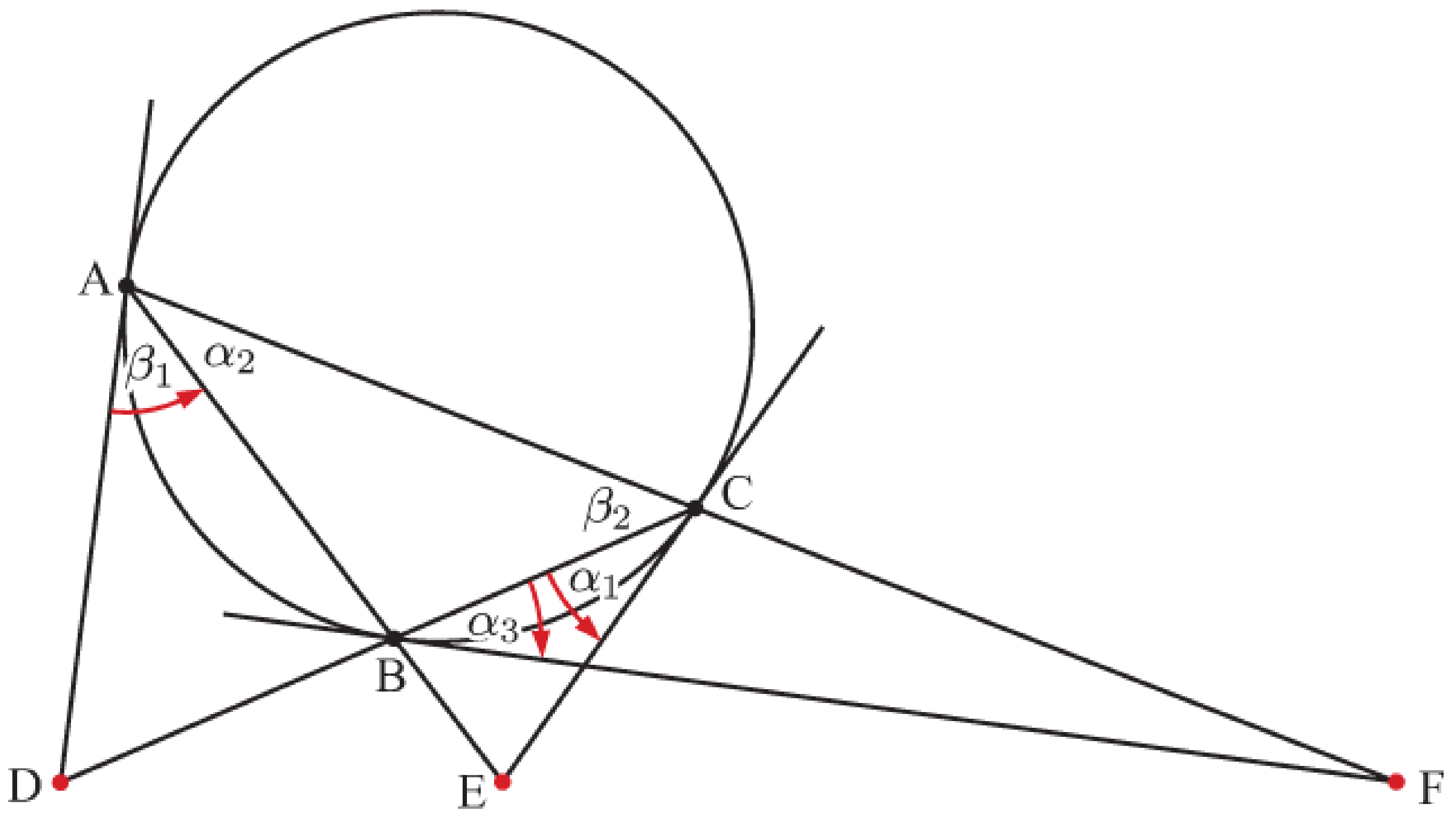
X, Y, and Z are points on the three sides produced of △ABC.

$$\frac{AY}{YC}\cdot\frac{CX}{XB}\cdot\frac{BZ}{ZA} = \left(-\frac{a}{c}\right)\cdot\left(-\frac{c}{b}\right)\cdot\left(-\frac{b}{a}\right)$$

$$\therefore \quad \frac{AY}{YC}\cdot\frac{CX}{XB}\cdot\frac{BZ}{ZA} = -1$$

∴ Y, X, and Z are collinear {converse of Menelaus' theorem}

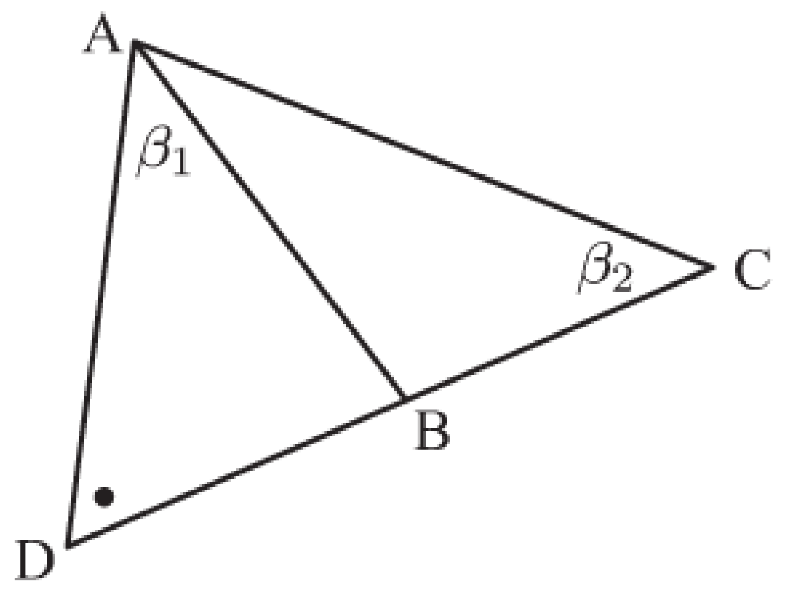
6



a By the angle between a tangent and a chord theorem,

$\alpha_1 = \alpha_2 = \alpha_3$ and $\beta_1 = \beta_2$.

Consider △s ABD, CAD:

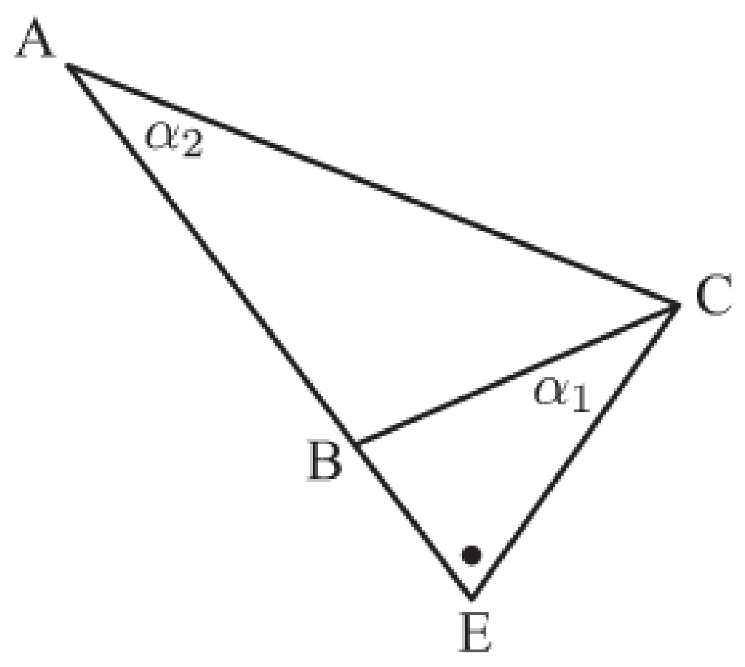


△s ABD and CAD are equiangular and therefore similar.

$$\therefore \quad \frac{\text{area of } \triangle ABD}{\text{area of } \triangle CAD} = \frac{AB^2}{CA^2} = \frac{DB}{DC} \quad .... (1)$$

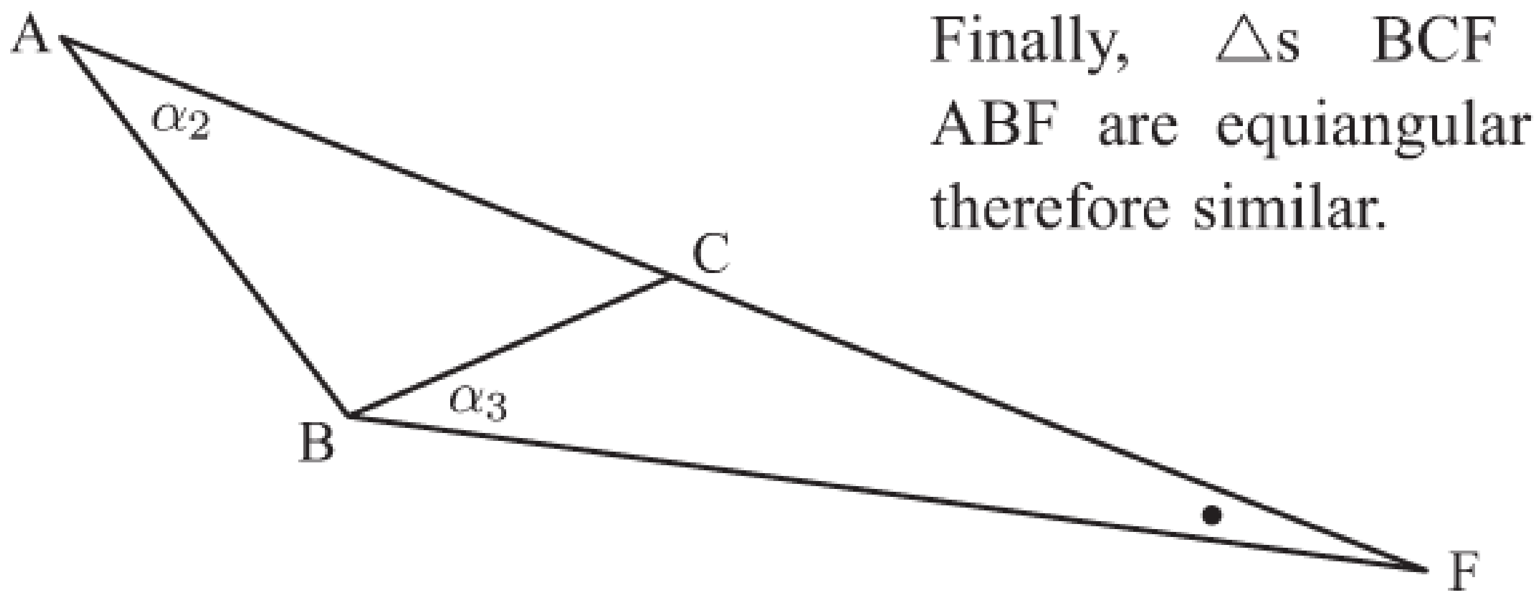
{area comparison theorem}

∴ $DB : DC = AB^2 : AC^2$

b 

Likewise, △s BCE and CAE are equiangular and therefore similar.

$$\therefore \quad \frac{\text{area of } \triangle\text{BCE}}{\text{area of } \triangle\text{CAE}} = \frac{\text{BC}^2}{\text{CA}^2} = \frac{\text{BE}}{\text{AE}} \quad .... \ (2)$$



Finally, △s BCF and ABF are equiangular and therefore similar.

$$\therefore \quad \frac{\text{area of } \triangle\text{BCF}}{\text{area of } \triangle\text{ABF}} = \frac{\text{BC}^2}{\text{AB}^2} = \frac{\text{CF}}{\text{AF}} \quad .... \ (3)$$

$$\text{Now} \quad \frac{\text{AF}}{\text{FC}} \cdot \frac{\text{CD}}{\text{DB}} \cdot \frac{\text{BE}}{\text{EA}} = \underbrace{-\frac{{}^{1}\cancel{\text{AB}^2}}{{}_{1}\cancel{\text{BC}^2}}}_{\text{from (3)}} \cdot \underbrace{-\frac{{}^{1}\cancel{\text{CA}^2}}{{}_{1}\cancel{\text{AB}^2}}}_{\text{from (1)}} \cdot \underbrace{-\frac{{}^{1}\cancel{\text{BC}^2}}{{}_{1}\cancel{\text{CA}^2}}}_{\text{from (2)}}$$

$$= -1$$

∴ by the converse of Menelaus' theorem, D, E, and F are collinear.

7 △GHI has 5 transversals and for each we can use Menelaus' theorem.

For transversal DXB, $\frac{\text{HX}}{\text{XG}} \cdot \frac{\text{GB}}{\text{BI}} \cdot \frac{\text{ID}}{\text{DH}} = -1$

For transversal AYF, $\frac{\text{HA}}{\text{AG}} \cdot \frac{\text{GF}}{\text{FI}} \cdot \frac{\text{IY}}{\text{YH}} = -1$

For transversal CZE, $\frac{\text{HE}}{\text{EG}} \cdot \frac{\text{GZ}}{\text{ZI}} \cdot \frac{\text{IC}}{\text{CH}} = -1$

For transversal ABC, $\frac{\text{HC}}{\text{CI}} \cdot \frac{\text{IB}}{\text{BG}} \cdot \frac{\text{GA}}{\text{AH}} = -1$

For transversal DEF, $\frac{\text{HD}}{\text{DI}} \cdot \frac{\text{IF}}{\text{FG}} \cdot \frac{\text{GE}}{\text{EH}} = -1$

Multiplying all of these gives

$$\frac{\text{HX}}{\text{XG}} \cdot \frac{\cancel{\text{GB}}^{1}}{{}_{1}\cancel{\text{BI}}} \cdot \frac{\cancel{\text{ID}}^{1}}{{}_{1}\cancel{\text{DH}}} \cdot \frac{\cancel{\text{HA}}^{1}}{{}_{1}\cancel{\text{AG}}} \cdot \frac{\cancel{\text{GF}}^{1}}{{}_{1}\cancel{\text{FI}}} \cdot \frac{\text{IY}}{\text{YH}} \cdot \frac{\cancel{\text{HE}}^{1}}{{}_{1}\cancel{\text{EG}}} \cdot \frac{\text{GZ}}{\text{ZI}} \cdot \frac{\cancel{\text{IC}}^{1}}{{}_{1}\cancel{\text{CH}}} \cdot \frac{\cancel{\text{HC}}^{1}}{{}_{1}\cancel{\text{CI}}} \cdot \frac{\cancel{\text{IB}}^{1}}{{}_{1}\cancel{\text{BG}}}$$
$$\cdot \frac{\cancel{\text{GA}}^{1}}{{}_{1}\cancel{\text{AH}}} \cdot \frac{\cancel{\text{HD}}^{1}}{{}_{1}\cancel{\text{DI}}} \cdot \frac{\cancel{\text{IF}}^{1}}{{}_{1}\cancel{\text{FG}}} \cdot \frac{\cancel{\text{GE}}^{1}}{{}_{1}\cancel{\text{EH}}}$$
$$= -1$$

∴ $\frac{\text{HX}}{\text{XG}} \cdot \frac{\text{GZ}}{\text{ZI}} \cdot \frac{\text{IY}}{\text{YH}} = -1$ for points X, Y, and Z on two sides of △GHI and one side produced.

∴ X, Y, and Z are collinear. {converse of Menelaus' theorem}

EXERCISE 2L

1 **a** $d = \frac{|2(3) + 5(2) + 6|}{\sqrt{4 + 25}} = \frac{22}{\sqrt{29}}$ units

b $d = \frac{|4(-1) - 3(4) - 4|}{\sqrt{16 + 9}} = \frac{|-20|}{5} = 4$ units

c $d = \frac{|3(2) - (-1) - 2|}{\sqrt{9 + 1}} = \frac{5}{\sqrt{10}}$ units

d $d = \frac{|m(-1) + (-3) - 5|}{\sqrt{m^2 + 1}} = \frac{|m + 8|}{\sqrt{m^2 + 1}}$ units

2 **a** $(1, 1)$ lies on $3x + 2y = 5$

$$\therefore \quad d = \frac{|3(1) + 2(1) + 1|}{\sqrt{9 + 4}} = \frac{6}{\sqrt{13}} \text{ units}$$

b $\left(0, \frac{-c_1}{b}\right)$ lies on $ax + by + c_1 = 0$

$$\therefore \quad d = \frac{\left|a(0) + b\left(\frac{-c_1}{b}\right) + c_2\right|}{\sqrt{a^2 + b^2}}$$

$$\therefore \quad d = \frac{|c_2 - c_1|}{\sqrt{a^2 + b^2}} \text{ units}$$

3 **a** $\frac{|3(k) - 2(-3) + 6|}{\sqrt{9 + 4}} = \sqrt{13}$

$$\therefore \quad |3k + 12| = 13$$
$$\therefore \quad 3k + 12 = \pm 13$$
$$\therefore \quad 3k = 1 \text{ or } -25$$
$$\therefore \quad k = \tfrac{1}{3} \text{ or } -\tfrac{25}{3}$$

b $\frac{|(1) + (-2) - k|}{\sqrt{1 + 1}} = \frac{|(1) - (-2) + 7|}{\sqrt{1 + 1}}$

$$\therefore \quad \frac{|-1 - k|}{\sqrt{2}} = \frac{|10|}{\sqrt{2}}$$
$$\therefore \quad |k + 1| = 10$$
$$\therefore \quad k + 1 = \pm 10$$
$$\therefore \quad k = 9 \text{ or } -11$$

4 The distance of $\text{P}(x, y)$ from $x - y - 4 = 0$ is $\frac{|x - y - 4|}{\sqrt{1 + 1}}$.

$$\therefore \quad \frac{|x - y - 4|}{\sqrt{2}} = 2\sqrt{2}$$
$$\therefore \quad |x - y - 4| = 4$$
$$\therefore \quad x - y - 4 = \pm 4$$
$$\therefore \quad x - y = 0 \quad \text{or} \quad x - y = 8$$

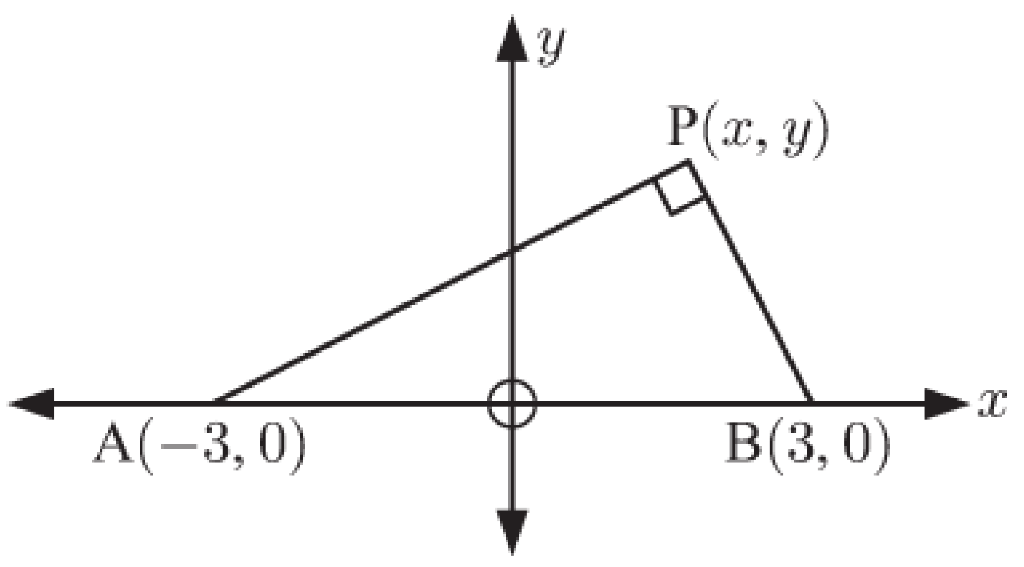
∴ $x - y = 0$ and $x - y = 8$ are the two parallel lines.

5
$$\text{PN} = \text{PS}$$
$$\therefore \quad \sqrt{(x + 1)^2 + (y - 8)^2} = \sqrt{(x - 5)^2 + (y - 4)^2}$$
$$\therefore \quad \cancel{x^2} + 2x + 1 + \cancel{y^2} - 16y + 64 = \cancel{x^2} - 10x + 25 + \cancel{y^2} - 8y + 16$$
$$\therefore \quad 12x - 8y + 24 = 0$$

∴ the locus is $3x - 2y + 6 = 0$

6 gradient of [AP]

$$= \frac{y - 0}{x - (-3)}$$
$$= \frac{y}{x + 3}$$



gradient of [BP] $= \frac{y - 0}{x - 3} = \frac{y}{x - 3}$

Thus $\left(\frac{y}{x + 3}\right)\left(\frac{y}{x - 3}\right) = -1 \qquad \{m_1 m_2 = -1\}$

$$\therefore \quad y^2 = -(x^2 - 9)$$
$$\therefore \quad y^2 = -x^2 + 9$$
$$\therefore \quad x^2 + y^2 = 9$$

∴ the locus is $x^2 + y^2 = 9$

7 **a**

$$\sqrt{(x-2)^2+(y-1)^2} = \frac{|2x-y-5|}{\sqrt{4+1}}$$

$$\therefore\ (x-2)^2+(y-1)^2 = \frac{(2x-y-5)^2}{5}$$

$$\therefore\ 5[x^2-4x+4+y^2-2y+1] = 4x^2+y^2+25 - 4xy+10y-20x$$

$$\therefore\ 5x^2 \cancel{-20x} + 5y^2 - 10y + \cancel{25} = 4x^2+y^2+\cancel{25} - 4xy+10y\cancel{-20x}$$

$$\therefore\ x^2+4xy+4y^2-20y=0$$

b

$$\frac{|3x-4y-3|}{\sqrt{9+16}} = \frac{|5x-12y-4|}{\sqrt{25+144}}$$

$$\therefore\ \frac{|3x-4y-3|}{5} = \frac{|5x-12y-4|}{13}$$

$$\therefore\ 13\,|3x-4y-3| = 5\,|5x-12y-4|$$

$$\therefore\ 13(3x-4y-3) = \pm 5(5x-12y-4)$$

$$\therefore\ 39x-52y-39 = \pm[25x-60y-20]$$

$$\therefore\ 39x-52y-39 = 25x-60y-20$$

$$\text{or}\quad 39x-52y-39 = -25x+60y+20$$

$$\therefore\ 14x+8y-19=0 \quad\text{or}\quad 64x-112y-59=0$$

which is a line pair.

8 **a** **i**

$$\text{AP} = 2\text{BP}$$

$$\therefore\ \text{AP}^2 = 4\text{BP}^2$$

$$\therefore\ (x+1)^2+y^2 = 4[(x-3)^2+y^2]$$

$$\therefore\ x^2+2x+1+y^2 = 4[x^2-6x+9+y^2]$$

$$\therefore\ 4x^2+4y^2-24x+36 = x^2+y^2+2x+1$$

$$\therefore\ 3x^2+3y^2-26x+35=0$$

ii The locus of P is a circle. {Apollonius' circle theorem}

b **i**

$$2\text{AP} = \text{BP}$$

$$\therefore\ 4\text{AP}^2 = \text{BP}^2$$

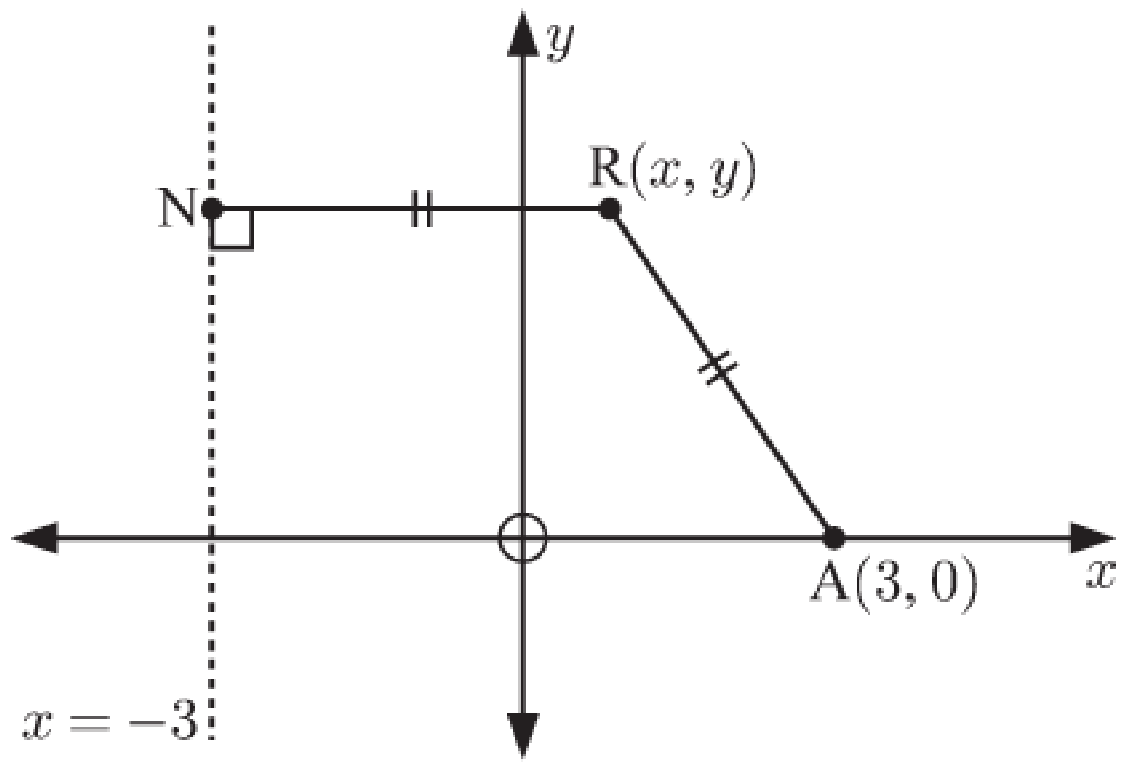
$$\therefore\ 4[x^2+2x+1+y^2] = x^2-6x+9+y^2$$

$$\therefore\ 4x^2+4y^2+8x+4-x^2-y^2+6x-9=0$$

$$\therefore\ 3x^2+3y^2+14x-5=0$$

ii Once again, by Apollonius' circle theorem, the locus of P is a circle.

9 **a**



Let N be a point on the line $x=-3$ such that

$$\text{NR} = \text{AR}$$

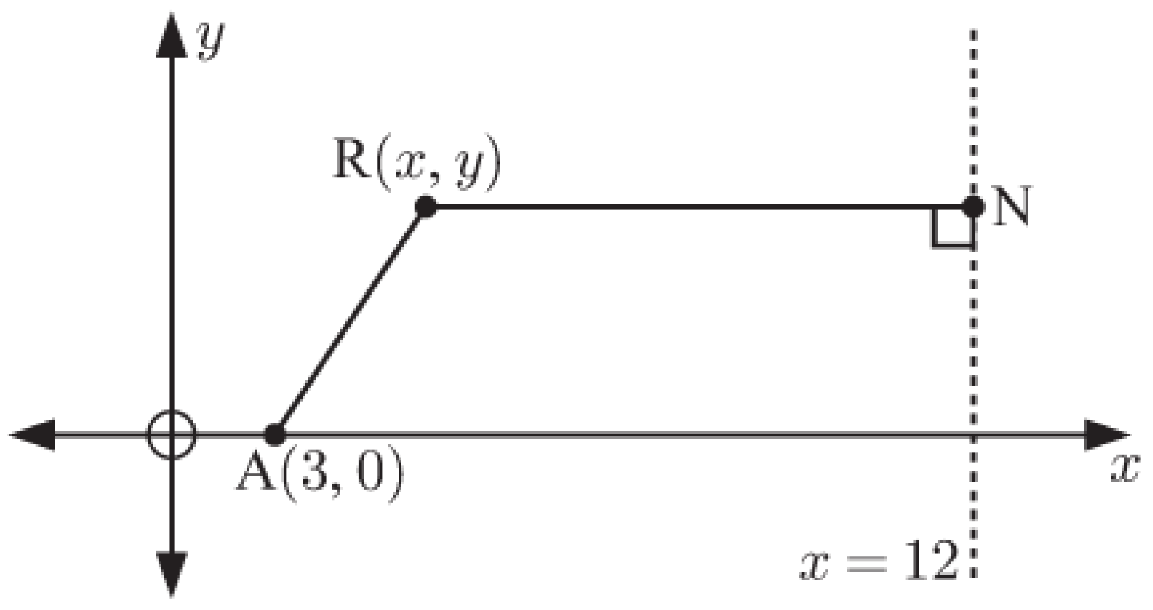
$$\therefore\ x-(-3) = \sqrt{(x-3)^2+y^2}$$

$$\therefore\ (x-3)^2+y^2 = (x+3)^2$$

$$\therefore\ \cancel{x^2}-6x+\cancel{9}+y^2 = \cancel{x^2}+6x+\cancel{9}$$

$$\therefore\ y^2 = 12x$$

b



Let N be a point on the line $x=12$ such that

$$\text{AR} = \tfrac{1}{2}\,\text{RN}$$

$$\therefore\ 4\,\text{AR}^2 = \text{RN}^2$$

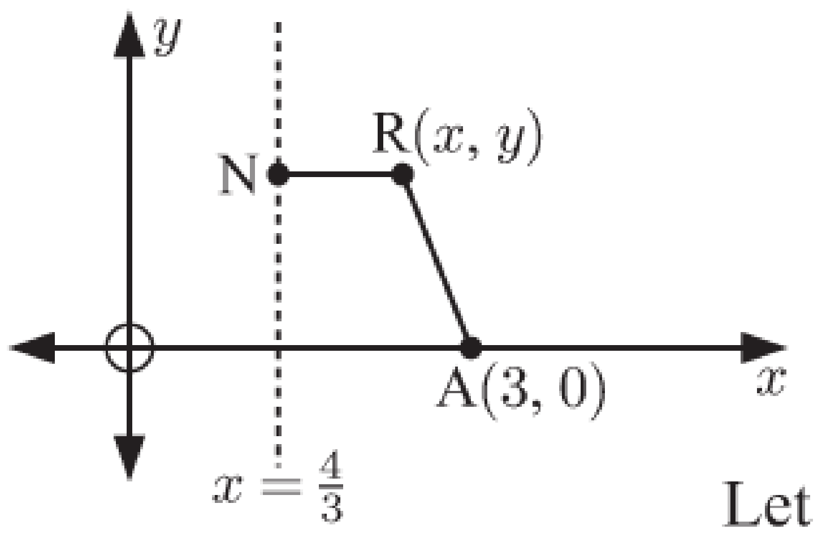
$$\therefore\ 4[(x-3)^2+y^2] = (12-x)^2$$

$$\therefore\ 4[x^2-6x+9+y^2] = 144-24x+x^2$$

$$\therefore\ 4x^2\cancel{-24x}+36+4y^2-x^2\cancel{+24x}-144=0$$

$$\therefore\ 3x^2+4y^2=108$$

c



Let N be a point on the line $x=\frac{4}{3}$ such that

$$\text{AR} = \tfrac{3}{2}\,\text{NR}$$

$$\therefore\ 4\,\text{AR}^2 = 9\,\text{NR}^2$$

$$\therefore\ 4[(x-3)^2+y^2] = 9(x-\tfrac{4}{3})^2$$

$$\therefore\ 4[x^2-6x+9+y^2] = 9[x^2-\tfrac{8}{3}x+\tfrac{16}{9}]$$

$$\therefore\ 4x^2+4y^2\cancel{-24x}+36 = 9x^2\cancel{-24x}+16$$

$$\therefore\ 5x^2-4y^2=20$$

10 **a**

$$\text{AQ}+\text{BQ}=6$$

$$\therefore\ \sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2}=6$$

$$\therefore\ \sqrt{x^2-4x+4+y^2} = 6-\sqrt{x^2+4x+4+y^2}$$

$$\therefore\ \cancel{x^2}-4x+\cancel{4}+\cancel{y^2} = 36-12\sqrt{x^2+4x+4+y^2}+\cancel{x^2}+4x+\cancel{4}+\cancel{y^2}$$

{squaring both sides}

$$\therefore\ 12\sqrt{x^2+4x+4+y^2} = 8x+36$$

$$\therefore\ 3\sqrt{x^2+4x+4+y^2} = 2x+9$$

$$\therefore\ 9(x^2+4x+4+y^2) = 4x^2+36x+81$$

{squaring both sides again}

$$\therefore\ 9x^2+\cancel{36x}+36+9y^2 = 4x^2+\cancel{36x}+81$$

$$\therefore\ 5x^2+9y^2=45$$

b

$$\text{AQ}-\text{BQ}=2$$

$$\therefore\ \text{AQ}=\text{BQ}+2$$

$$\therefore\ \sqrt{x^2-4x+4+y^2} = \sqrt{x^2+4x+4+y^2}+2$$

$$\therefore\ \cancel{x^2}-4x+\cancel{4}+\cancel{y^2} = \cancel{x^2}+4x+\cancel{4}+\cancel{y^2}+4\sqrt{x^2+4x+4+y^2}+4$$

{squaring both sides}

$$\therefore\ 4\sqrt{x^2+4x+4+y^2} = -8x-4$$

$$\therefore\ \sqrt{x^2+4x+4+y^2} = -2x-1$$

$$\therefore\ x^2+\cancel{4x}+4+y^2 = 4x^2+\cancel{4x}+1$$

{squaring both sides again}

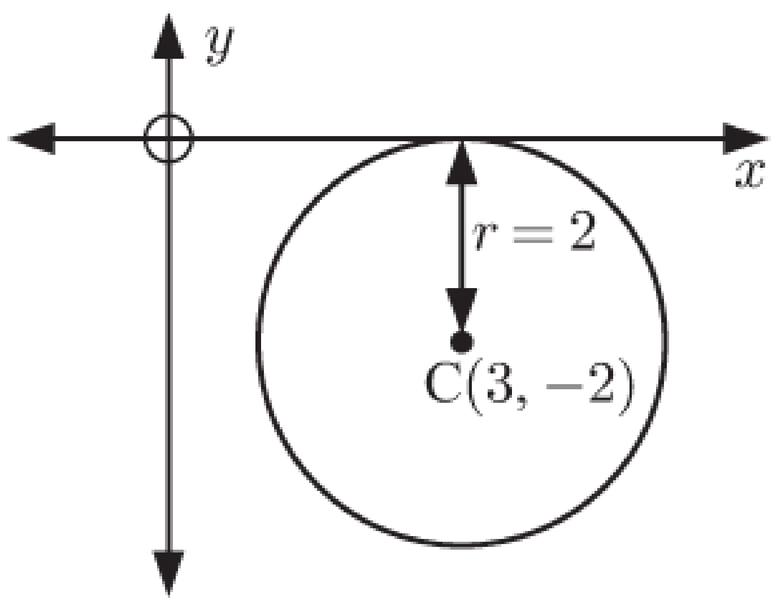
$$\therefore\ 3x^2-y^2=3$$

EXERCISE 2M.1

1 **a** Centre $(2,\ 3)$, $r = 2$ units
b Centre $(0,\ -3)$, $r = 3$ units
c Centre $(2,\ 0)$, $r = \sqrt{7}$ units

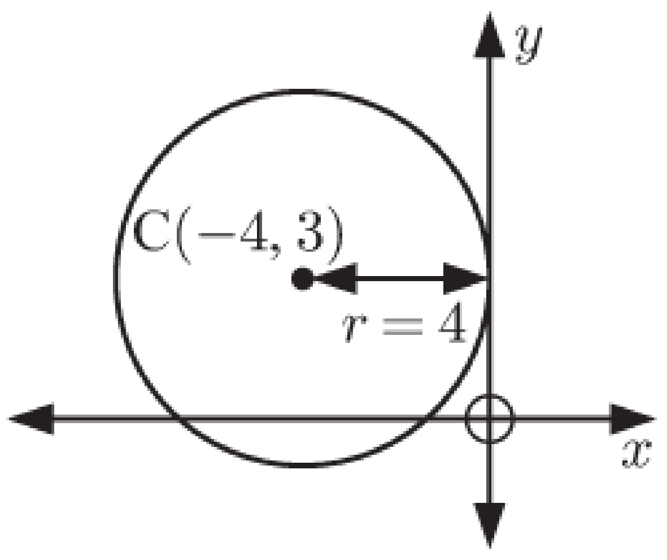
2 **a** $(x-2)^2 + (y-3)^2 = 25$ **b** $(x+2)^2 + (y-4)^2 = 1$
c $(x-4)^2 + (y+1)^2 = 3$ **d** $(x+3)^2 + (y+1)^2 = 11$

3 **a**



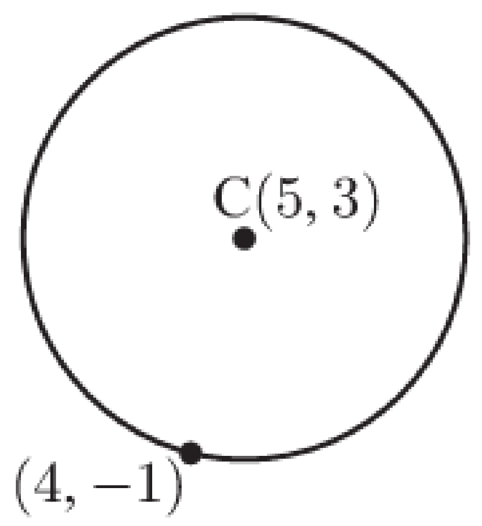
$\therefore\ (x-3)^2 + (y+2)^2 = 4$

b



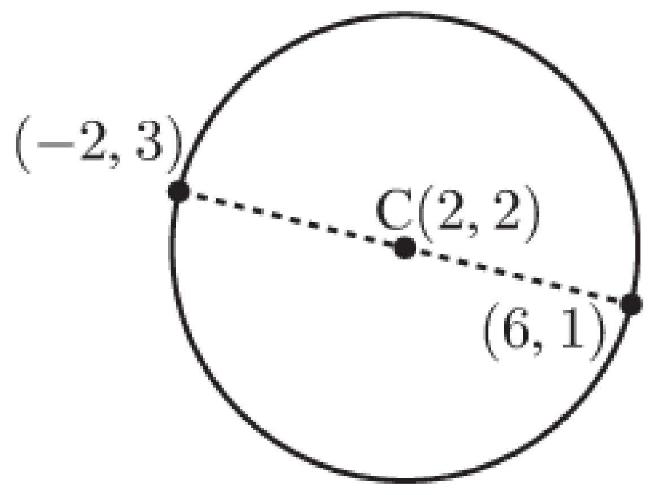
$\therefore\ (x+4)^2 + (y-3)^2 = 16$

c



$$r^2 = (5-4)^2 + (3+1)^2$$
$$\therefore\ r^2 = 1 + 16$$
$$\therefore\ r^2 = 17$$
$$\therefore\ (x-5)^2 + (y-3)^2 = 17$$

d



Centre is at $\left(\dfrac{-2+6}{2},\ \dfrac{3+1}{2}\right)$

that is, at $(2,\ 2)$ and

$$r^2 = (6-2)^2 + (1-2)^2$$
$$= 16 + 1$$
$$= 17$$
$$\therefore\ (x-2)^2 + (y-2)^2 = 17$$

e $(x+3)^2 + (y-2)^2 = 7$ {same centre and $r^2 = 7$}

4 **a** A circle, centre $(-2,\ 7)$, radius $\sqrt{5}$ units.
b The point $(-2,\ 7)$.
c Nothing, as the LHS $\geqslant 0$ for all $x,\ y \in \mathbb{R}$ and RHS < 0.

5 **a** If P is inside the circle then PC $< r$

$$\therefore\ \text{PC}^2 < r^2$$
$$\therefore\ (x-h)^2 + (y-k)^2 < r^2$$

b If $(x-h)^2 + (y-k)^2 > r^2$ then $\text{PC}^2 > r^2$

$$\therefore\ \text{PC} > r$$

$\therefore$ P lies outside the circle.

6 $(x+2)^2 + (y-3)^2 = 25$ has centre $\text{C}(-2,\ 3)$ and $r = 5$ units. Using **5** above:

a For A$(2,\ 0)$,
$$(x+2)^2 + (y-3)^2$$
$$= (2+2)^2 + (-3)^2$$
$$= 5^5$$
$$= r^2$$
$\therefore$ A lies on the circle.

b For B$(1,\ 1)$,
$$(x+2)^2 + (y-3)^2$$
$$= (1+2)^2 + (1-3)^2$$
$$= 13 \text{ which is } < 25$$
$\therefore$ B lies inside the circle.

c For D$(3,\ 0)$,
$$(x+2)^2 + (y-3)^2$$
$$= (3+2)^2 + (-3)^2$$
$$= 5^2 + 3^2 \text{ which is } > 25$$
$\therefore$ D lies outside the circle.

d For E$(4,\ 1)$,
$$(x+2)^2 + (y-3)^2$$
$$= (4+2)^2 + (1-3)^2$$
$$= 40 \text{ which is } > 25$$
$\therefore$ E lies outside the circle.

7 **a** As $(3,\ m)$ lies on $(x+1)^2 + (y-2)^2 = 25$,
$$4^2 + (m-2)^2 = 25$$
$$\therefore\ (m-2)^2 = 9$$
$$\therefore\ m - 2 = \pm 3$$
$$\therefore\ m = 5 \text{ or } -1$$

b As $(m,\ -2)$, lies on $(x+2)^2 + (y-3)^2 = 36$,
$$(m+2)^2 + 25 = 36$$
$$\therefore\ (m+2)^2 = 11$$
$$\therefore\ m + 2 = \pm\sqrt{11}$$
$$\therefore\ m = -2 \pm \sqrt{11}$$

c As $(3,\ -1)$ lies on $(x+4)^2 + (y+m)^2 = 53$,
$$7^2 + (m-1)^2 = 53$$
$$\therefore\ (m-1)^2 = 4$$
$$\therefore\ m - 1 = \pm 2$$
$$\therefore\ m = 3 \text{ or } -1$$

8 **a**
$$x^2 + y^2 + 6x - 2y - 3 = 0$$
$$\therefore\ x^2 + 6x + 9 + y^2 - 2y + 1 = 3 + 9 + 1$$
$$\therefore\ (x+3)^2 + (y-1)^2 = 13$$
which is a circle, centre $(-3,\ 1)$, $r = \sqrt{13}$ units.

b
$$x^2 + y^2 - 6x - 2 = 0$$
$$\therefore\ x^2 - 6x + 9 + y^2 = 2 + 9$$
$$\therefore\ (x-3)^2 + y^2 = 11$$
which is a circle, centre $(3,\ 0)$, $r = \sqrt{11}$ units.

c
$$x^2 + y^2 + 4y - 1 = 0$$
$$\therefore\ x^2 + y^2 + 4y + 4 = 1 + 4$$
$$\therefore\ x^2 + (y+2)^2 = 5$$
which is a circle, centre $(0,\ -2)$, $r = \sqrt{5}$ units.

d
$$x^2 + y^2 + 4x - 8y + 3 = 0$$
$$\therefore\ x^2 + 4x + 4 + y^2 - 8y + 16 = -3 + 4 + 16$$
$$\therefore\ (x+2)^2 + (y-4)^2 = 17$$
which is a circle, centre $(-2,\ 4)$, $r = \sqrt{17}$ units.

e
$$x^2 + y^2 - 4x - 6y - 3 = 0$$
$$\therefore\ x^2 - 4x + 4 + y^2 - 6y + 9 = 3 + 4 + 9$$
$$\therefore\ (x-2)^2 + (y-3)^2 = 16$$
which is a circle, centre $(2,\ 3)$, $r = 4$ units.

f
$$x^2 + y^2 - 8x = 0$$
$$\therefore\ x^2 - 8x + 16 + y^2 = 16$$
$$\therefore\ (x-4)^2 + y^2 = 16$$
which is a circle, centre $(4,\ 0)$, $r = 4$ units.

9 **a**
$$x^2 + y^2 - 12x + 8y + k = 0$$
$$\therefore\ x^2 - 12x + 36 + y^2 + 8y + 16 = -k + 36 + 16$$
$$\therefore\ (x-6)^2 + (y+4)^2 = 52 - k$$
which is a circle of radius $\sqrt{52-k}$
$$\therefore\ \sqrt{52-k} = 4$$
$$\therefore\ 52 - k = 16$$
$$\therefore\ k = 36$$

b $x^2+y^2+6x-4y=k$

$\therefore\ x^2+6x+9+y^2-4y+4=k+9+4$

$\therefore\ (x+3)^2+(y-2)^2=k+13$

which is a circle with radius $\sqrt{k+13}$

$\therefore\ \sqrt{k+13}=\sqrt{11}$

$\therefore\ k+13=11$

$\therefore\ k=-2$

c $x^2+y^2+4x-2y+k=0$

$\therefore\ x^2+4x+4+y^2-2y+1=-k+4+1$

$\therefore\ (x+2)^2+(y-1)^2=5-k$

which is a circle if $5-k>0$

$\therefore\ k<5$

10 $x^2+y^2+dx+ey+f=0$

a $x^2+dx+\left(\frac{d}{2}\right)^2+y^2+ey+\left(\frac{e}{2}\right)^2=\left(\frac{d}{2}\right)^2+\left(\frac{e}{2}\right)^2-f$

$\therefore\ \left(x+\frac{d}{2}\right)^2+\left(y+\frac{e}{2}\right)^2=\frac{d^2}{4}+\frac{e^2}{4}-f$

which is a circle centre $\left(-\frac{d}{2},\,-\frac{e}{2}\right)$

and radius $=\sqrt{\frac{d^2}{4}+\frac{e^2}{4}-f}$

provided $\frac{d^2}{4}+\frac{e^2}{4}-f>0$

$\therefore\ \frac{d^2}{4}+\frac{e^2}{4}>f$

$\therefore\ d^2+e^2>4f$

b $3x^2+3y^2+6x-9y+2=0$

$\therefore\ x^2+y^2+2x-3y+\frac{2}{3}=0$

which is a circle centre $(-1,\,\frac{3}{2})$ and

$r=\sqrt{\frac{4}{4}+\frac{9}{4}-\frac{2}{3}}$ $\{d=2,\ e=-3,\ f=\frac{2}{3}\}$

$\therefore\ r=\sqrt{\frac{31}{12}}$ units

c **i** When $d^2+e^2=4f$, $r=0$

$\therefore$ the equation represents the point $\left(-\frac{d}{2},\,-\frac{e}{2}\right)$.

ii When $d^2+e^2<4f$, $\sqrt{\frac{d^2}{4}+\frac{e^2}{4}-f}$ is a non-real complex number.

$\therefore$ the equation has no meaning.

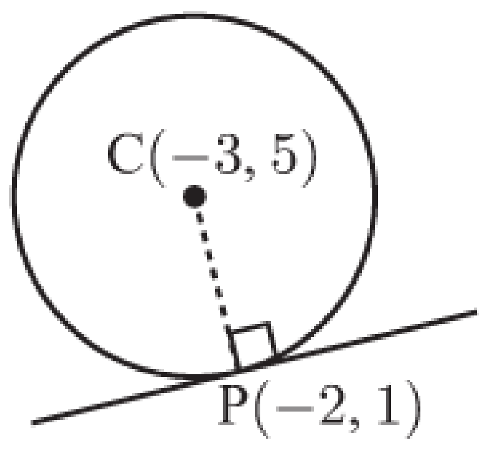
EXERCISE 2M.2

1 **a** $x^2+y^2+6x-10y+17=0$

$\therefore\ x^2+6x+9+y^2-10y+25=9+25+17$

$\therefore\ (x+3)^2+(y-5)^2=51$

has centre $C(-3,\,5)$.



Gradient of $[CP]=\frac{1-5}{-2+3}$

$=-\frac{4}{1}$

$\therefore$ gradient of tangent $=\frac{1}{4}$

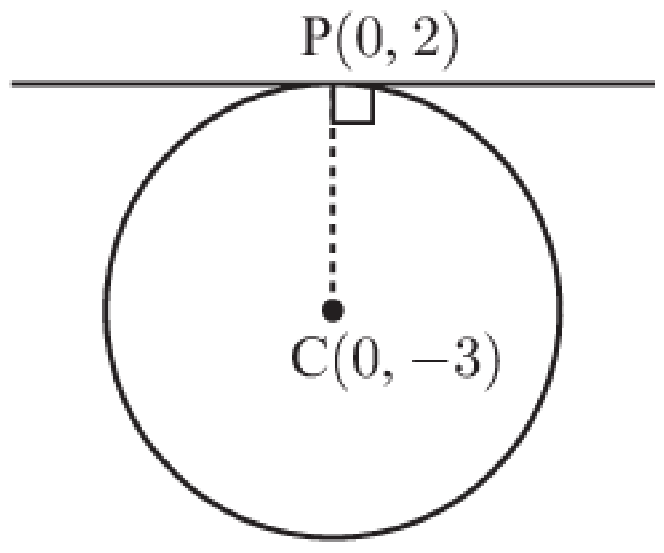
$\therefore$ tangent has equation $x-4y=(-2)-4(1)$ at $P(-2,\,1)$

$\therefore$ the tangent is $x-4y=-6$

b $x^2+y^2+6y=16$

$\therefore\ x^2+(x+3)^2=25$

which has centre $C(0,\,-3)$.



Gradient of $[CP]=\frac{2-(-3)}{0-0}$

which is undefined.

$\therefore$ gradient of tangent is 0

$\therefore$ its equation is $y=2$.

2 **a** $x^2+y^2-24x-16y+111=0$

$\therefore\ x^2-24x+144+y^2-16y+64=-111+144+64$

$\therefore\ (x-12)^2+(y-8)^2=97$

$\therefore$ the centre of the lake is at $(12,\,8)$, and the radius is $\sqrt{97}$ m.

$\therefore$ diameter $=2\sqrt{97}\approx 19.7$ m

b Gradient of $[AC]=\frac{8-4}{12-3}=\frac{4}{9}$

$\therefore$ gradient of tangent $=-\frac{9}{4}$

$\therefore$ equation is $9x+4y=9(3)+4(4)$

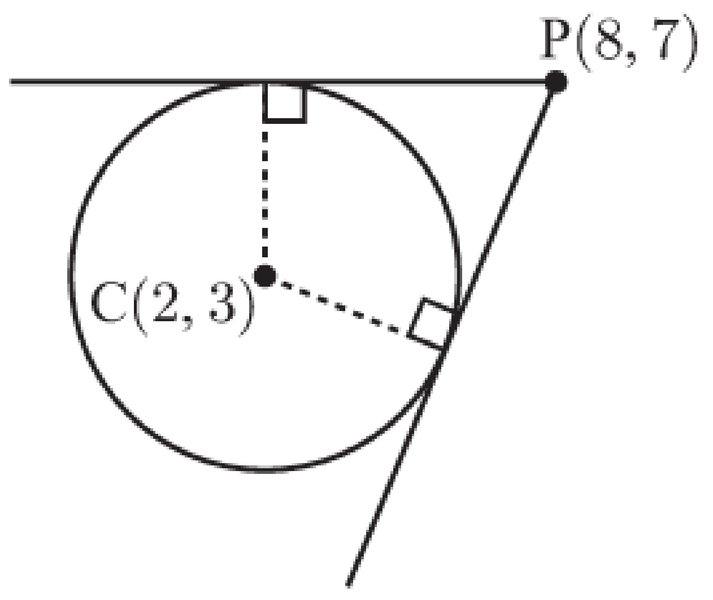
that is, $9x+4y=43$

3 Let a tangent have equation $y=mx+c$.

As $(8,\,7)$ lies on it

$7=8m+c$

$\therefore\ c=7-8m$



$\therefore$ the tangent is $y=mx+7-8m$

$\therefore\ \frac{|m(2)-(3)+7-8m|}{\sqrt{m^2+1}}=4$

{distance of $(2,\,3)$ to the line is 4 units}

$\therefore\ |4-6m|=4\sqrt{m^2+1}$

$\therefore\ (4-6m)^2=16(m^2+1)$

$\therefore\ \cancel{16}-48m+36m^2-16m^2-\cancel{16}=0$

$\therefore\ 20m^2-48m=0$

$\therefore\ m(20m-48)=0$

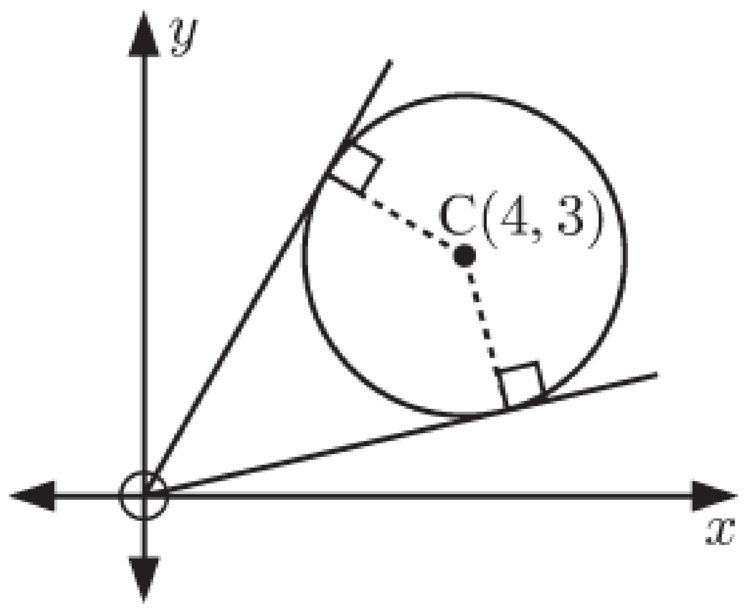
$\therefore\ m=0$ or $\frac{12}{5}$

$\therefore$ tangents have equations $y=7$ and $y=\frac{12}{5}x+7-\frac{96}{5}$

$\therefore$ the equations are $y=7$ and $12x-5y=61$.

4 Let a tangent have equation $y=mx$.

Now the distance of $(4,\,3)$ to a tangent is 2 units.



$\therefore\ \frac{|m(4)-(3)|}{\sqrt{m^2+1}}=2$

$\therefore\ |4m-3|=2\sqrt{m^2+1}$

$\therefore\ (4m-3)^2=4(m^2+1)$

$\therefore\ 16m^2-24m+9=4m^2+4$

$\therefore\ 12m^2-24m+5=0$

$\therefore\ m\approx 0.236$ or 1.76

$\therefore$ tangents have equations $y\approx 0.236x$ and $y\approx 1.76x$.

5 Let the other tangent have equation $y = mx$.
Now the distance of $(3,\ 4)$ to $mx - y = 0$ is equal to the distance of $(3,\ 4)$ to $x - 2y = 0$.

$$\therefore\ \frac{|m(3)-(4)|}{\sqrt{m^2+1}} = \frac{|(3)-2(4)|}{\sqrt{1+4}}$$

$$\therefore\ \sqrt{5}\,|3m-4| = \sqrt{m^2+1}\times 5$$

$$\therefore\ 5(3m-4)^2 = 25(m^2+1)$$

$$\therefore\ 5(9m^2-24m+16) - 25m^2 - 25 = 0$$

$$\therefore\ 45m^2 - 120m + 80 - 25m^2 - 25 = 0$$

$$\therefore\ 20m^2 - 120m + 55 = 0$$

$$\therefore\ 4m^2 - 24m + 11 = 0$$

$$\therefore\ (2m-1)(2m-11) = 0$$

$$\therefore\ m = \tfrac{1}{2} \text{ or } \tfrac{11}{2}$$

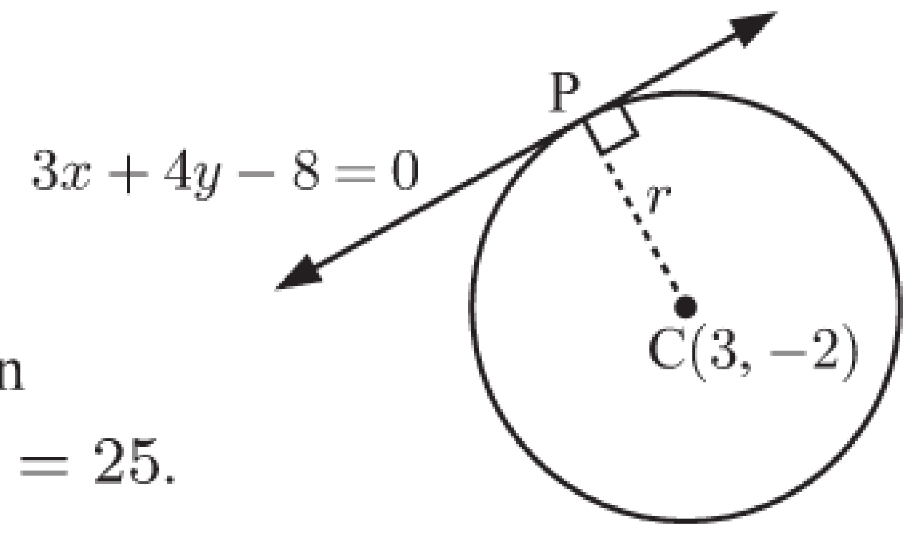
$\therefore$ the other tangent is $y = \frac{11}{2}x$.

6 **a** $r = \dfrac{|3(3)-4(-2)+8|}{\sqrt{9+16}}$

$= \frac{25}{5}$

$= 5$ units

$\therefore$ the circle has equation
$(x-3)^2 + (y+2)^2 = 25$.



b Now the gradient of the tangent is $\frac{3}{4}$.
Let P be the point of contact of the tangent and the circle.
$\therefore$ [CP] has gradient $-\frac{4}{3}$
$\therefore$ [CP] has equation $4x + 3y = 4(3) + 3(-2)$
that is, $4x + 3y = 6$
P is at the intersection of $4x+3y=6$ and $3x-4y=-8$.
Solving simultaneously gives $x=0,\ y=2$.
$\therefore$ point of contact is at $(0,\ 2)$.

7

$$x^2 + y^2 - 4x + 2y = 0$$

$$\therefore\ x^2 - 4x + 4 + y^2 + 2y + 1 = 4 + 1$$

$$\therefore\ (x-2)^2 + (y+1)^2 = 5$$

$\therefore$ the centre is $(2,\ -1)$ and $r = \sqrt{5}$ units.

a Tangent case is when

$$d = \frac{|3(2)+4(-1)-k|}{\sqrt{9+16}} = \sqrt{5}$$

$$\therefore\ |2-k| = 5\sqrt{5}$$

$$\therefore\ 2-k = \pm 5\sqrt{5}$$

$$\therefore\ k = 2 \pm 5\sqrt{5}$$

b A secant occurs when $d < \sqrt{5}$

$$\therefore\ \frac{|2-k|}{5} < \sqrt{5}$$

$$\therefore\ |2-k| < 5\sqrt{5}$$

$$\therefore\ -5\sqrt{5} < 2-k < 5\sqrt{5}$$

$$\therefore\ -2-5\sqrt{5} < -k < -2+5\sqrt{5}$$

$$\therefore\ 2-5\sqrt{5} < k < 2+5\sqrt{5}$$

{× (−1) and reverse inequality signs}

c An external line occurs when $d > \sqrt{5}$

$\therefore\ k > 2+5\sqrt{5}$ or $k < 2-5\sqrt{5}$.

8 **a** Let M have coordinates $(X,\ Y)$.
As M is the midpoint of [OA], A has coordinates $(2X,\ 2Y)$.
But $\text{AC} = r$

$$\therefore\ \sqrt{(2X-r)^2 + (2Y-0)^2} = r$$

$$\therefore\ (2X-r)^2 + 4Y^2 = r^2 \quad \{\text{squaring both sides}\}$$

$\therefore$ the Cartesian equation of the locus of M is
$(2x-r)^2 + 4y^2 = r^2$.

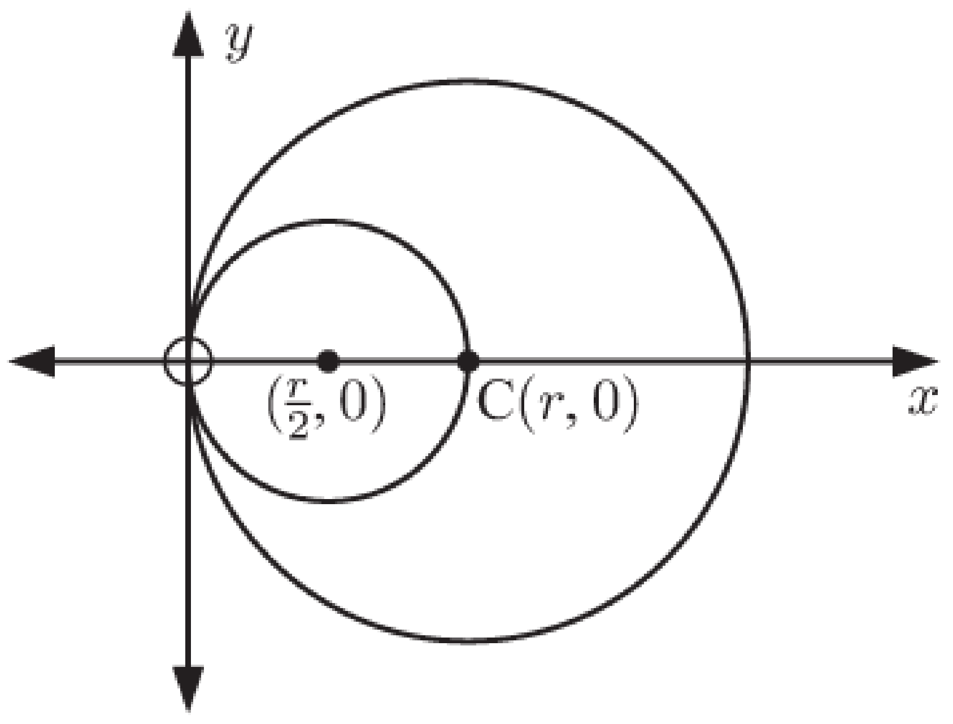
b

$$(2x-r)^2 + 4y^2 = r^2$$

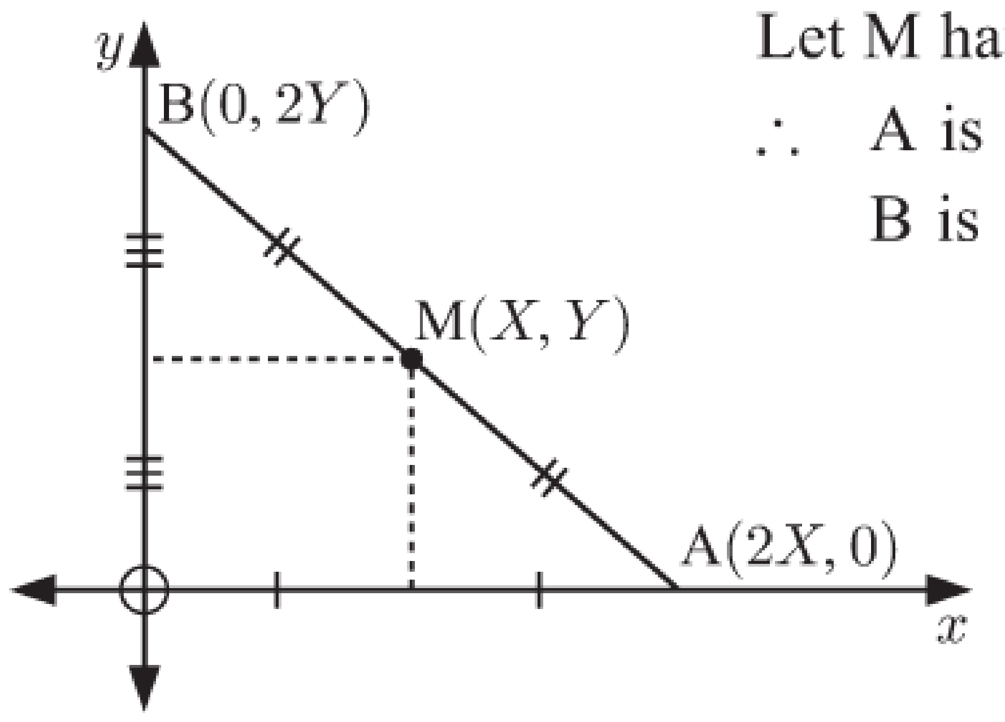
$$\therefore\ 4\left(x - \frac{r}{2}\right)^2 + 4y^2 = r^2$$

$$\therefore\ \left(x - \frac{r}{2}\right)^2 + y^2 = \left(\frac{r}{2}\right)^2$$

which is a circle centre $\left(\frac{r}{2},\ 0\right)$ and radius $\frac{r}{2}$ units.



9 **a**



Let M have coordinates $(X,\ Y)$.
$\therefore$ A is $(2X,\ 0)$ and
B is $(0,\ 2Y)$.

But $\text{AB} = p$

$$\therefore\ \sqrt{(2X-0)^2 + (0-2Y)^2} = p$$

$$\therefore\ \sqrt{4X^2 + 4Y^2} = p$$

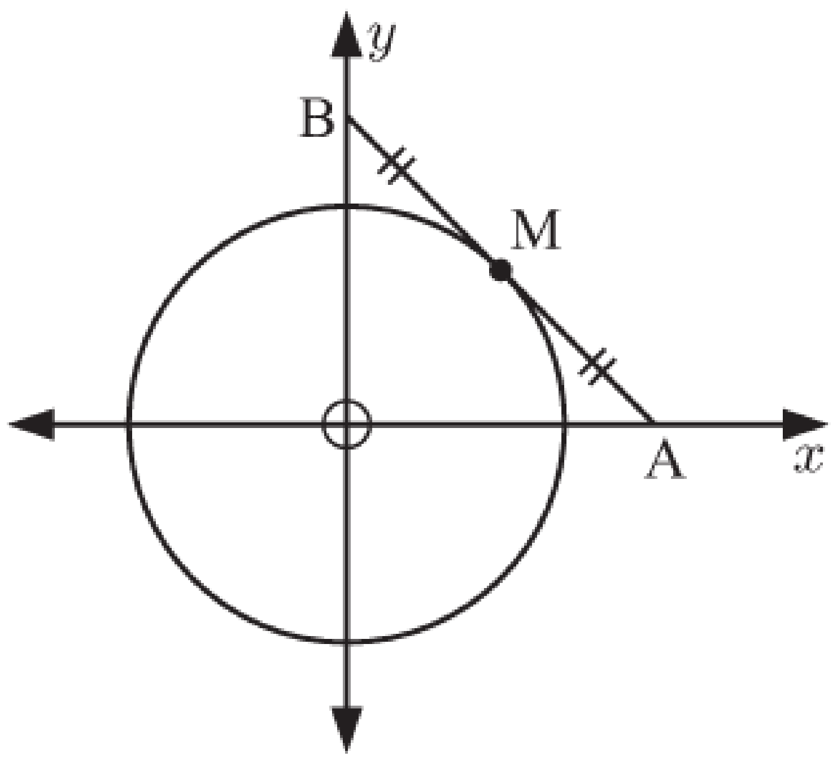
$$\therefore\ 4X^2 + 4Y^2 = p^2$$

$$\therefore\ X^2 + Y^2 = \left(\frac{p}{2}\right)^2$$

$\therefore$ the Cartesian equation of the locus of M is

$$x^2 + y^2 = \left(\frac{p}{2}\right)^2.$$

b This is the equation of a circle, centre $(0,\ 0)$, radius $\frac{p}{2}$ units.



10 **a** If $\dfrac{\text{AP}}{\text{BP}} = 3$, $\text{AP} = 3\text{BP}$

$$\therefore\ \sqrt{(x-1)^2 + y^2} = 3\sqrt{(x-5)^2 + y^2}$$

$$\therefore\ (x-1)^2 + y^2 = 9[(x-5)^2 + y^2]$$

$$\therefore\ 9(x^2 - 10x + 25 + y^2) = x^2 - 2x + 1 + y^2$$

$$\therefore \quad 9x^2 - 90x + 225 + 9y^2 - x^2 + 2x - 1 - y^2 = 0$$
$$\therefore \quad 8x^2 + 8y^2 - 88x + 224 = 0$$
$$\therefore \quad x^2 + y^2 - 11x + 28 = 0$$

Completing the square on x:

$$x^2 - 11x + (\tfrac{11}{2})^2 + y^2 = -28 + (\tfrac{11}{2})^2$$
$$\therefore \quad (x - \tfrac{11}{2})^2 + y^2 = (\tfrac{3}{2})^2$$

So, we have a circle, centre $(\frac{11}{2}, 0)$ with $r = \frac{3}{2}$ units.

b If $\dfrac{\text{AP}}{\text{BP}} = \frac{1}{3}$, $\text{BP} = 3\text{AP}$

$$\therefore \quad \text{BP}^2 = 9\text{AP}^2$$
$$\therefore \quad x^2 - 10x + 25 + y^2 = 9[x^2 - 2x + 1 + y^2]$$
$$\therefore \quad x^2 - 10x + 25 + y^2 = 9x^2 - 18x + 9 + 9y^2$$
$$\therefore \quad 8x^2 + 8y^2 - 8x - 16 = 0$$
$$\therefore \quad x^2 + y^2 - x - 2 = 0$$

Completing the square on x:

$$x^2 - x + (\tfrac{1}{2})^2 + y^2 = 2 + (\tfrac{1}{2})^2 = \tfrac{9}{4}$$
$$\therefore \quad (x - \tfrac{1}{2})^2 + y^2 = (\tfrac{3}{2})^2$$

So, we have a circle, centre $(\frac{1}{2}, 0)$ with $r = \frac{3}{2}$ units.

c If $\dfrac{\text{AP}}{\text{BP}} = 1$, $\text{AP} = \text{BP}$

$$\therefore \quad \text{AP}^2 = \text{BP}^2$$
$$\therefore \quad \cancel{x^2} - 2x + 1 + \cancel{y^2} = \cancel{x^2} - 10x + 25 + \cancel{y^2}$$
$$\therefore \quad 8x = 24$$
$$\therefore \quad x = 3$$

which is a vertical line.

11 **a** As $\dfrac{\text{AP}}{\text{BP}} = 2$, $\text{AP}^2 = 4\text{BP}^2$

$$\therefore \quad (x-2)^2 + y^2 = 4[(x-6)^2 + y^2]$$
$$\therefore \quad x^2 - 4x + 4 + y^2 = 4[x^2 - 12x + 36 + y^2]$$
$$\therefore \quad 4x^2 - 48x + 144 + 4y^2 - x^2 + 4x - 4 - y^2 = 0$$
$$\therefore \quad 3x^2 + 3y^2 - 44x + 140 = 0$$
$$\therefore \quad x^2 + y^2 - \tfrac{44}{3}x + \tfrac{140}{3} = 0$$
$$\therefore \quad x^2 - \tfrac{44}{3}x + (\tfrac{22}{3})^2 + y^2 = -\tfrac{140}{3} + (\tfrac{22}{3})^2 = \tfrac{64}{9}$$
$$\therefore \quad (x - \tfrac{22}{3})^2 + y^2 = (\tfrac{8}{3})^2$$

$\therefore$ the locus of P is a circle, centre $(\frac{22}{3}, 0)$ and radius $\frac{8}{3}$ units.

b The circle cuts the x-axis when $y = 0$

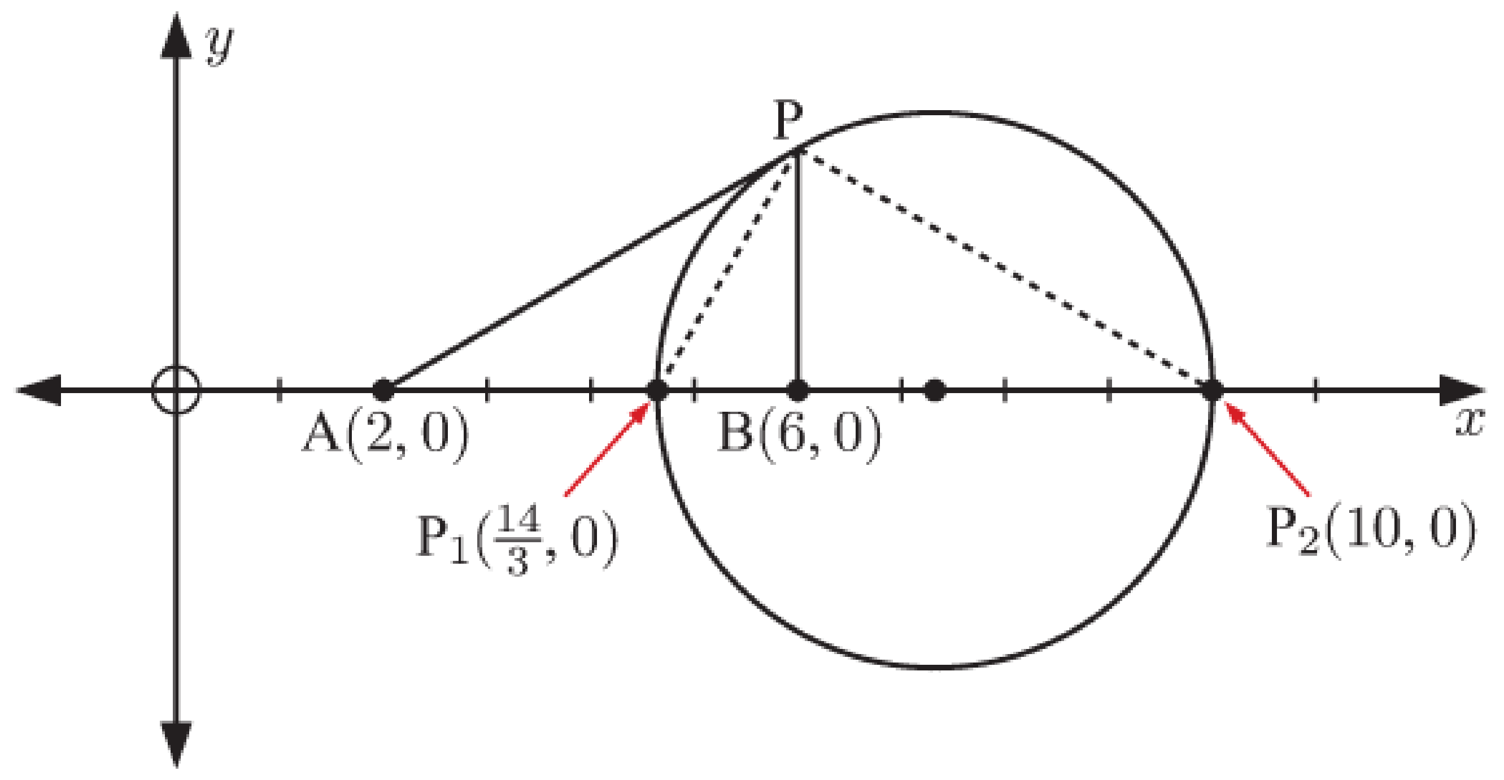
$$\therefore \quad (x - \tfrac{22}{3})^2 = (\tfrac{8}{3})^2$$
$$\therefore \quad x - \tfrac{22}{3} = \pm\tfrac{8}{3}$$
$$\therefore \quad x = 10 \text{ or } \tfrac{14}{3}$$

P_1 is $(\frac{14}{3}, 0)$ and $\text{P}_2(10, 0)$.

c
$$\frac{\text{AP}_1}{\text{BP}_1} = \frac{\frac{14}{3} - 2}{6 - \frac{14}{3}} \quad \text{and} \quad \frac{\text{AP}_2}{\text{BP}_2} = \frac{8}{4} = 2$$
$$= \frac{\frac{8}{3}}{\frac{4}{3}}$$
$$= 2$$

Thus $\dfrac{\text{AP}}{\text{BP}} = \dfrac{\text{AP}_1}{\text{BP}_1} = \dfrac{\text{AP}_2}{\text{BP}_2}$.

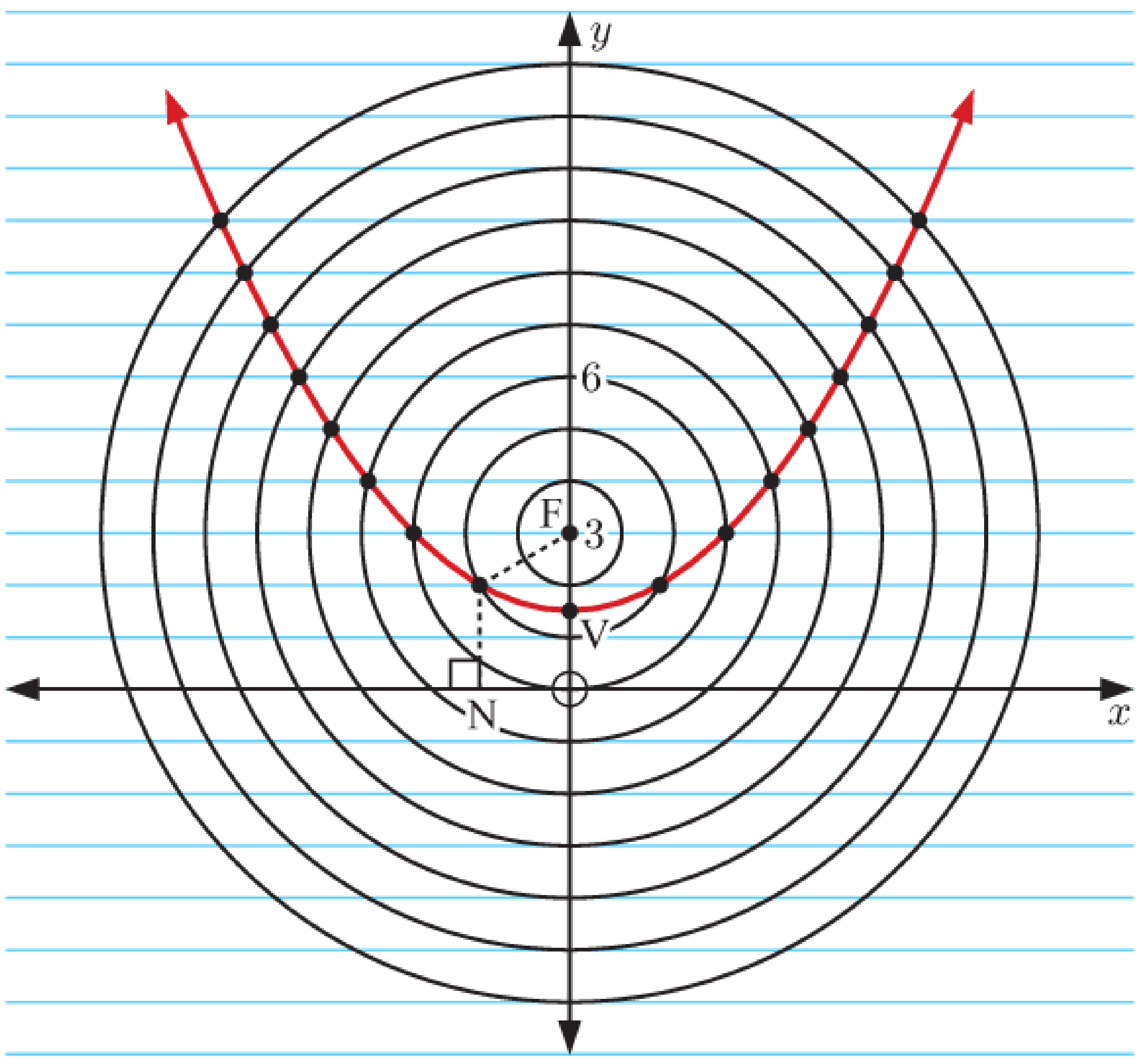
d



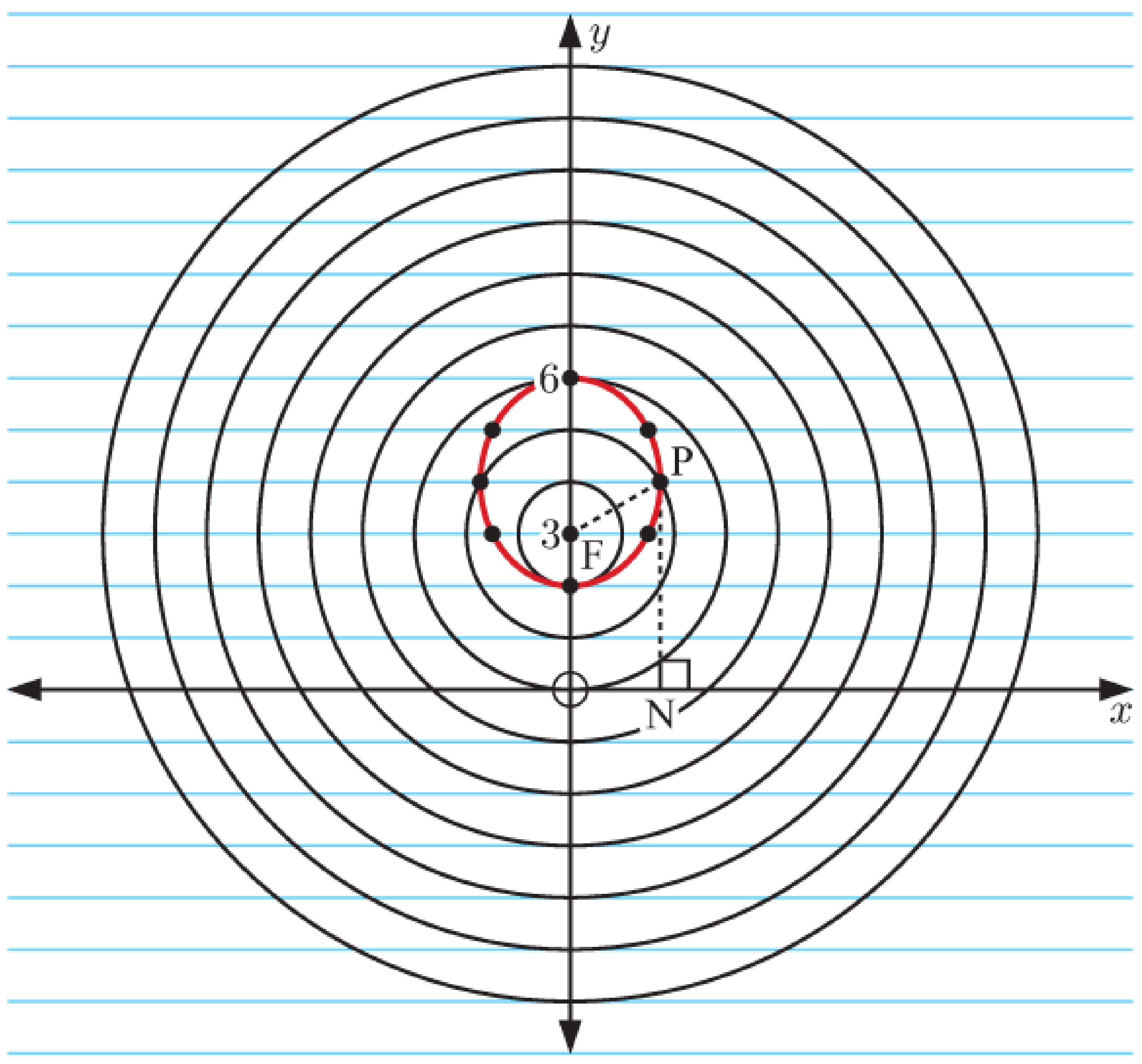
Thus, by the converse of the angle bisector theorem $[\text{PP}_1]$ bisects $\text{A}\widehat{\text{P}}\text{B}$ and $[\text{PP}_2]$ bisects exterior angle $\text{A}\widehat{\text{P}}\text{B}$.

EXERCISE 2N.1

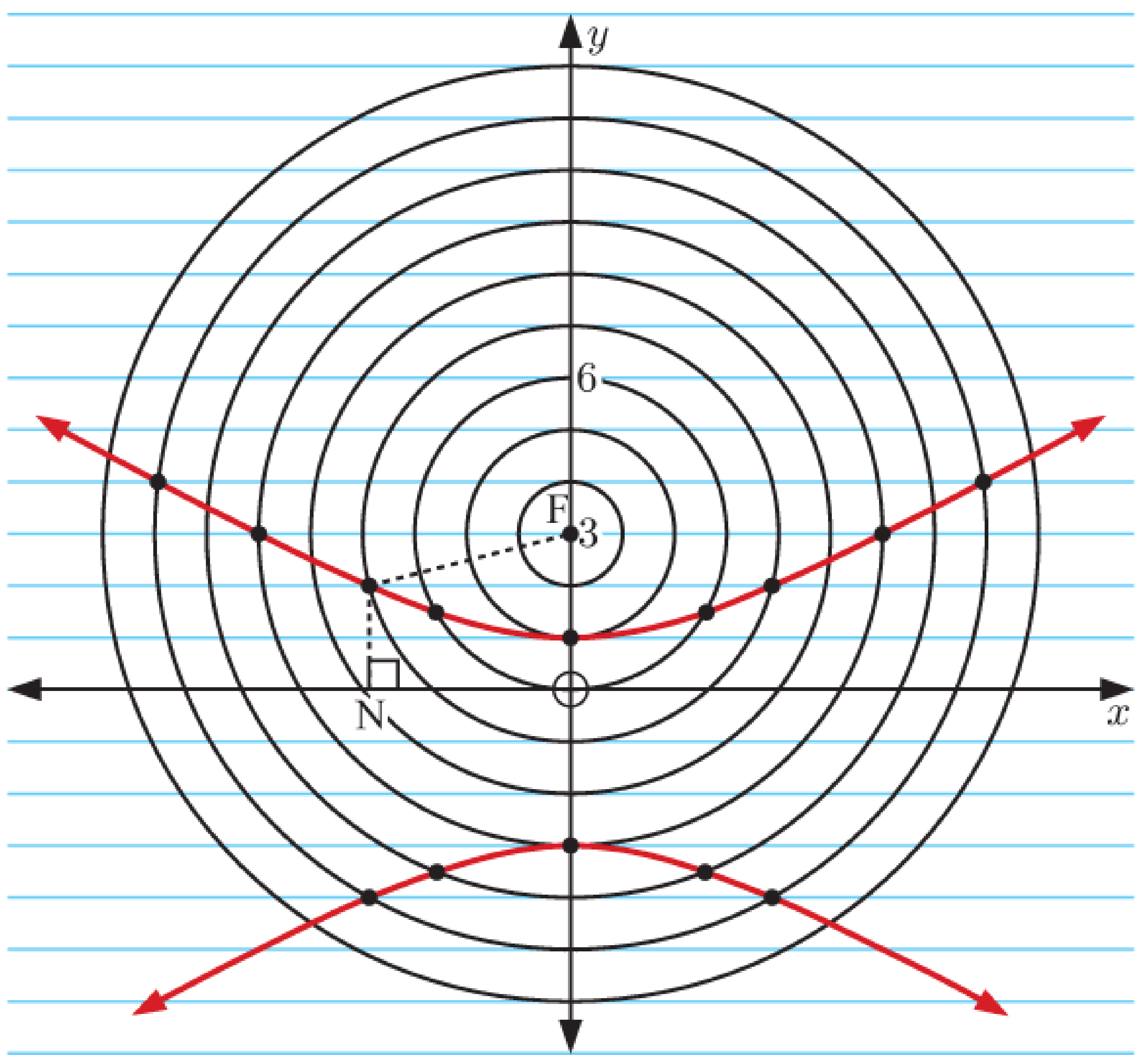
1 **a**



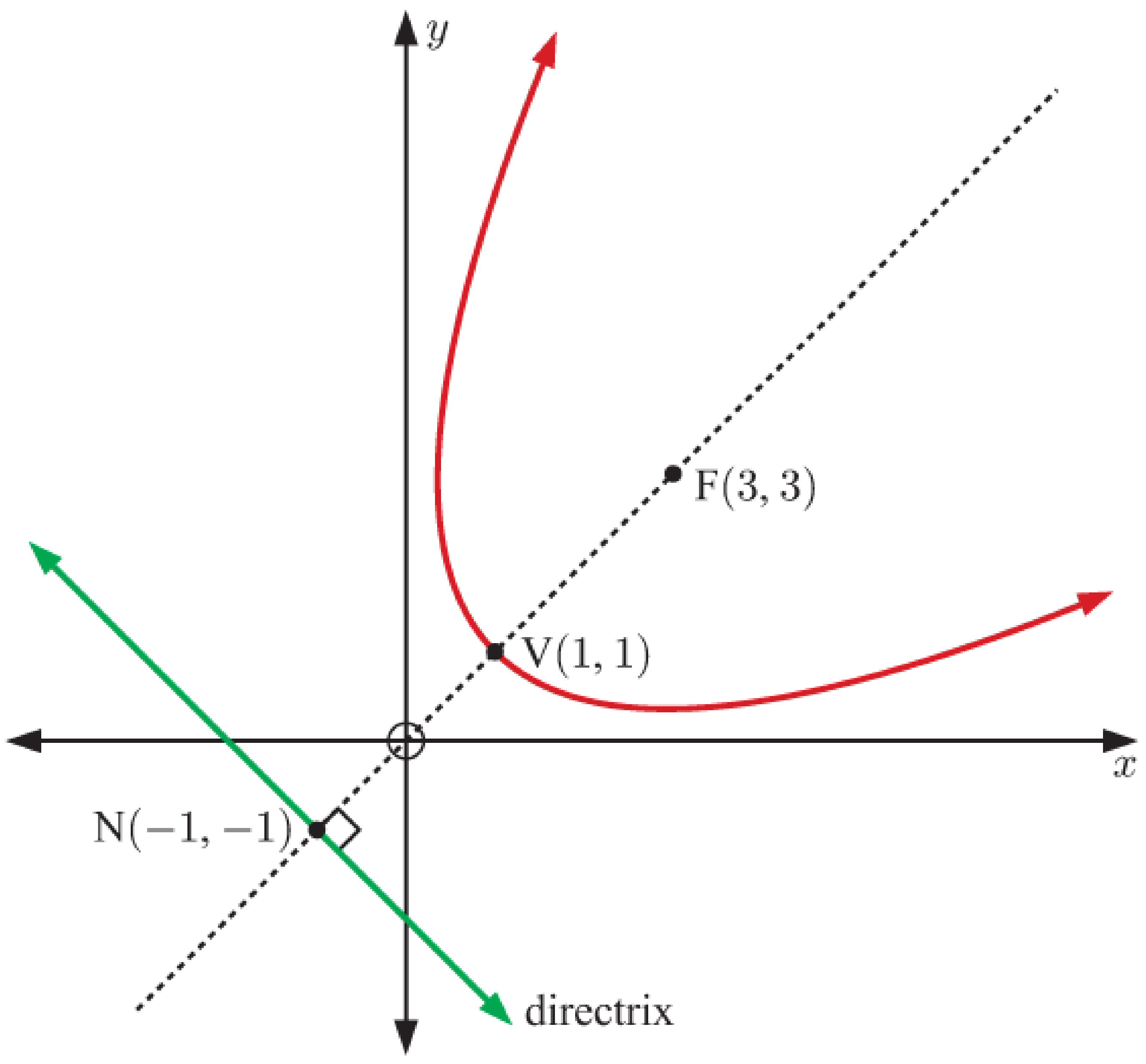
b



c



2 **a**



b The vertex is midway between the focus and the directrix.

$\therefore$ N$(-1, -1)$ lies on the directrix.

[VN] has gradient 1 and is perpendicular to the directrix.

$\therefore$ the directrix has gradient -1

and equation $x + y = (-1) + (-1)$

$\therefore$ $x + y = -2$

c P(x, y) is any point on the parabola and PF = PN.

$$\therefore \sqrt{(x-3)^2 + (y-3)^2} = \frac{|x+y+2|}{\sqrt{1+1}}$$

$$\therefore \sqrt{x^2 - 6x + 9 + y^2 - 6y + 9} = \frac{|x+y+2|}{\sqrt{2}}$$

$$\therefore 2(x^2 + y^2 - 6x - 6y + 18) = (x+y+2)^2$$

$$\therefore 2x^2 + 2y^2 - 12x - 12y + 36 = x^2 + y^2 + 4 + 2xy + 4x + 4y$$

$$\therefore x^2 + y^2 - 16x - 16y - 2xy + 32 = 0$$

3 Let P(x, y) be any point on the parabola.

The focus is F$(1, 1)$ and N is the foot of the perpendicular from P to the directrix $x + y = 4$.

PF = PN

$$\therefore \sqrt{(x-1)^2 + (y-1)^2} = \frac{|x+y-4|}{\sqrt{1+1}}$$

$$\therefore 2(x^2 - 2x + 1 + y^2 - 2y + 1) = (x+y-4)^2$$

$$\therefore 2x^2 + 2y^2 - 4x - 4y + 4 = x^2 + y^2 + 16 - 8x - 8y + 2xy$$

$$\therefore x^2 + y^2 + 4x + 4y - 2xy = 12$$

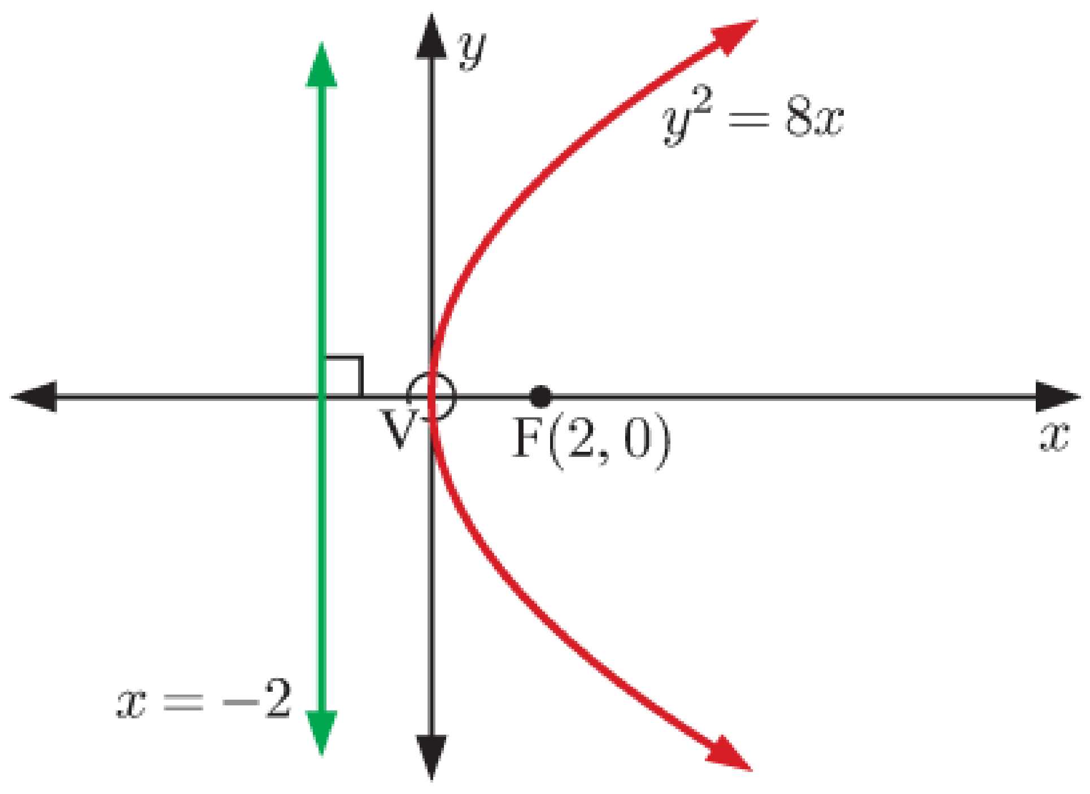
EXERCISE 2N.2

1 **a** $y^2 = 8x$

$\therefore$ $4a = 8$

$\therefore$ $a = 2$

$\therefore$ the focus is F$(2, 0)$ and the directrix is $x = -2$.



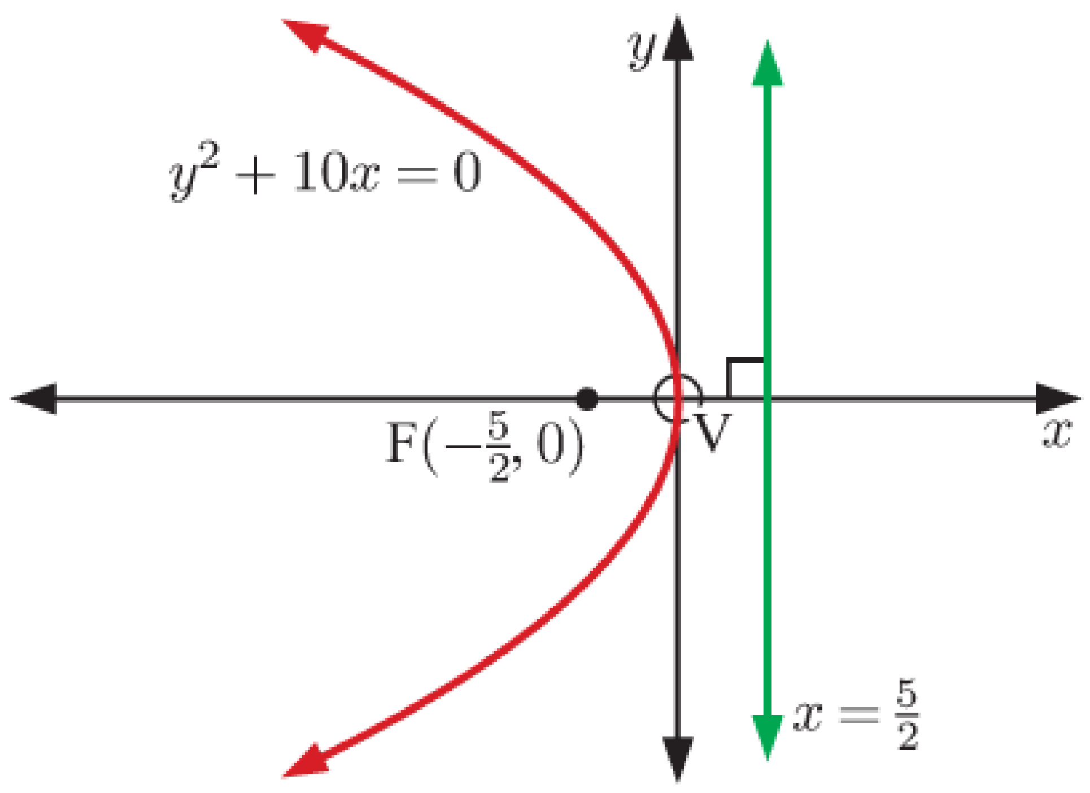
b $y^2 + 10x = 0$

$\therefore$ $y^2 = -10x$

$\therefore$ $4a = -10$

$\therefore$ $a = -\frac{5}{2}$

$\therefore$ the focus is F$(-\frac{5}{2}, 0)$ and the directrix is $x = \frac{5}{2}$.

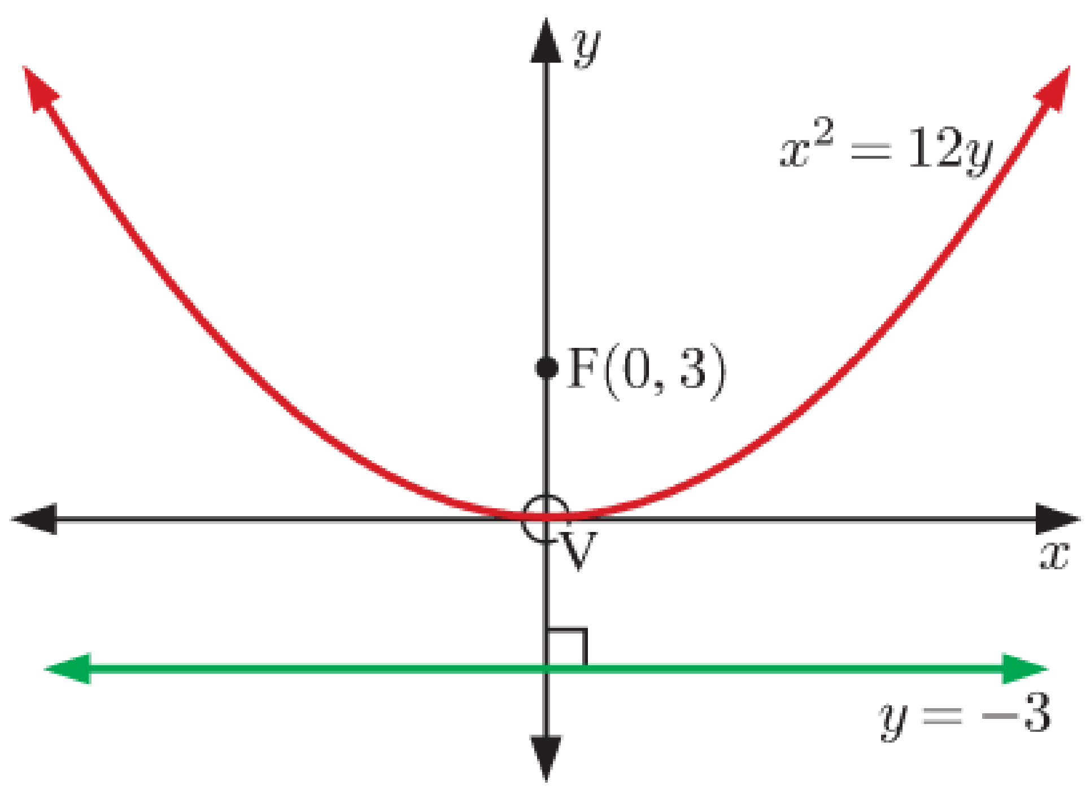


c $x^2 = 12y$

$\therefore$ $4a = 12$

$\therefore$ $a = 3$

$\therefore$ the focus is F$(0, 3)$ and the directrix is $y = -3$.



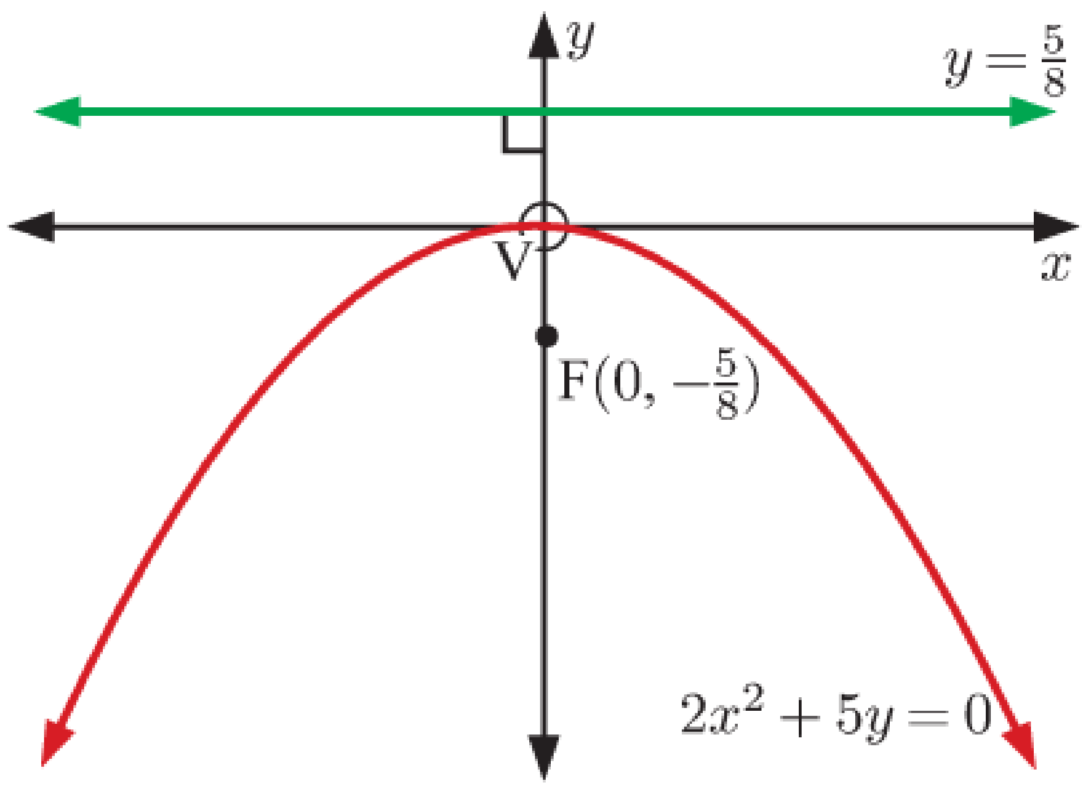
d $2x^2 + 5y = 0$

$\therefore$ $x^2 = -\frac{5}{2}y$

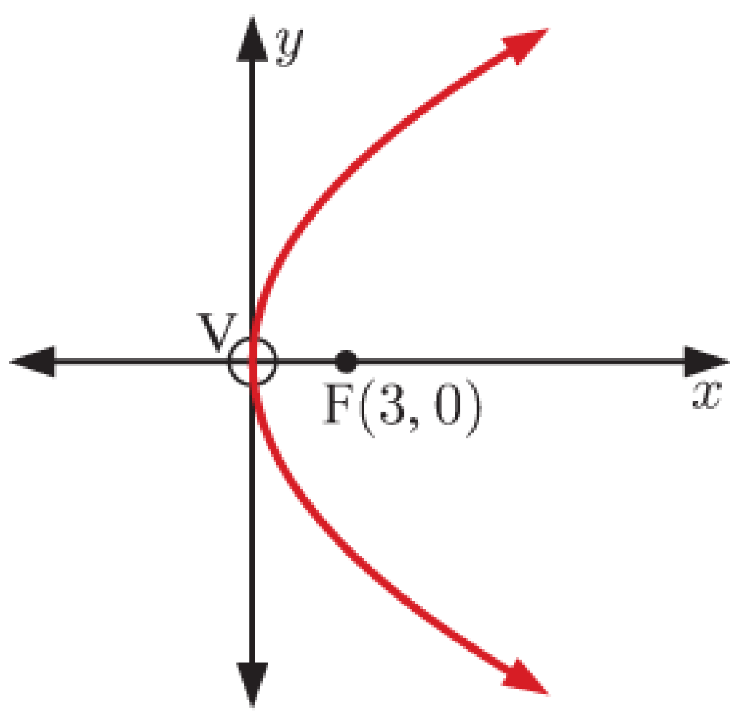
$\therefore$ $4a = -\frac{5}{2}$

$\therefore$ $a = -\frac{5}{8}$

$\therefore$ the focus is F$(0, -\frac{5}{8})$ and the directrix is $y = \frac{5}{8}$.



2 **a**

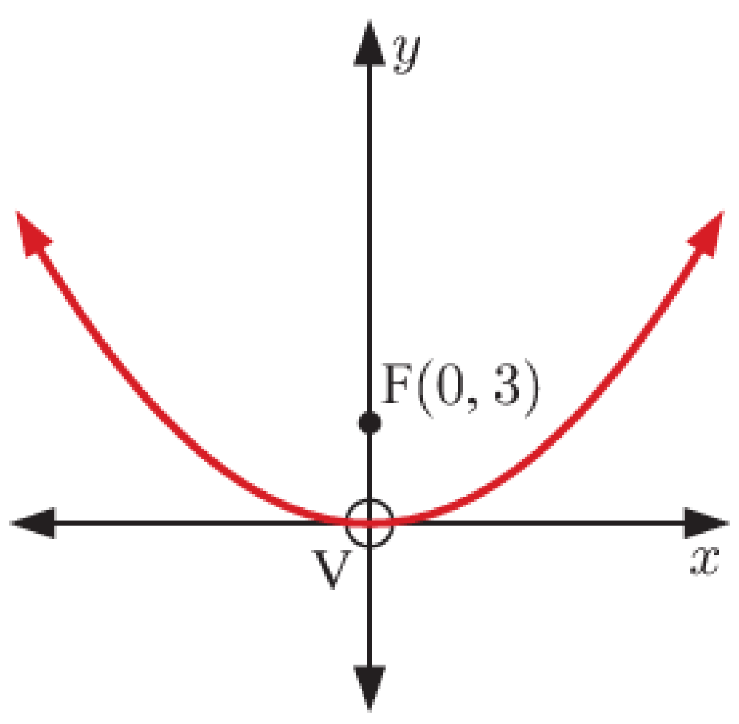


$a = 3$ and $y^2 = 4ax$

$\therefore$ $y^2 = 4(3)x$

$\therefore$ $y^2 = 12x$

b

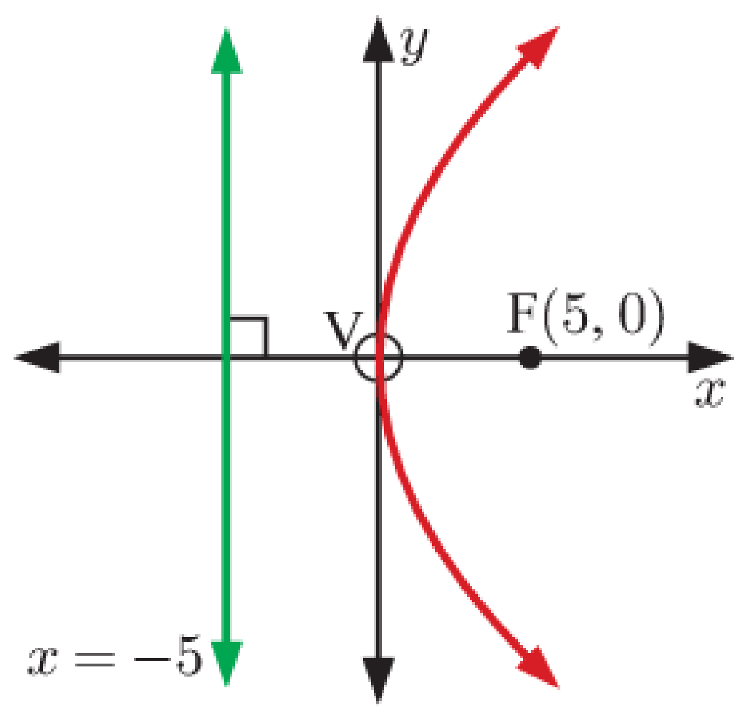


$a = 3$ and $x^2 = 4ay$

$\therefore$ $x^2 = 4(3)y$

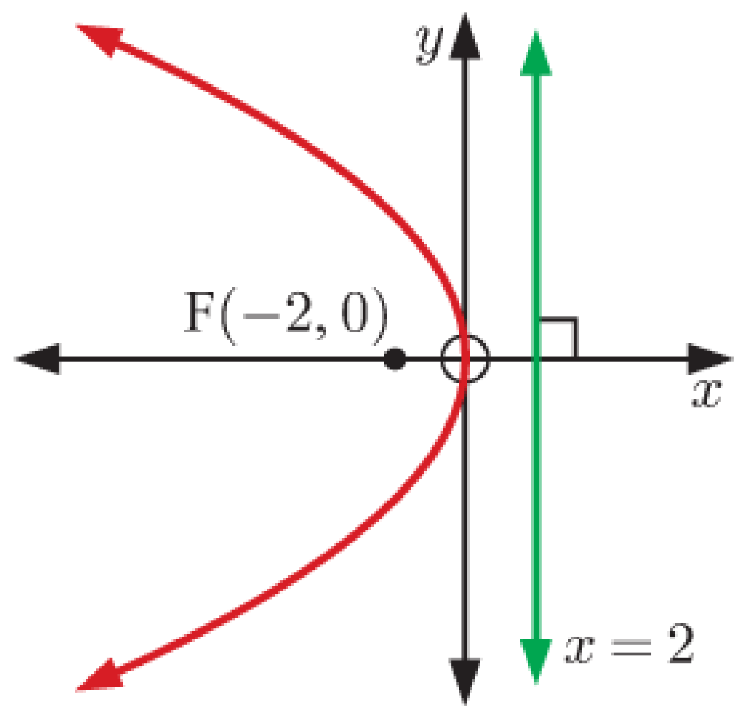
$\therefore$ $x^2 = 12y$

c



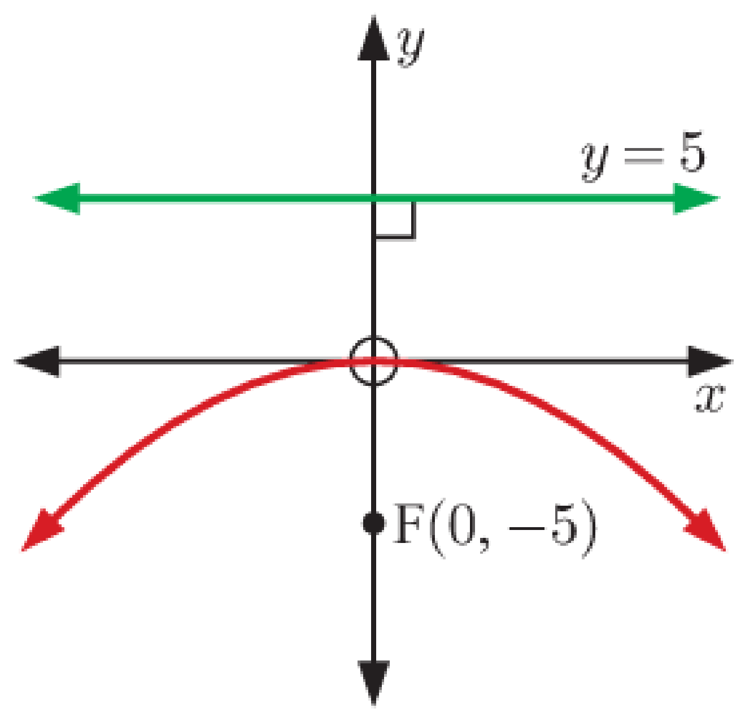
$$\begin{aligned} -a = -5 \quad &\text{and} \quad y^2 = 4ax \\ \therefore\ a = 5 \quad &\text{and} \quad y^2 = 4(5)x \\ &\therefore\ y^2 = 20x \end{aligned}$$

d



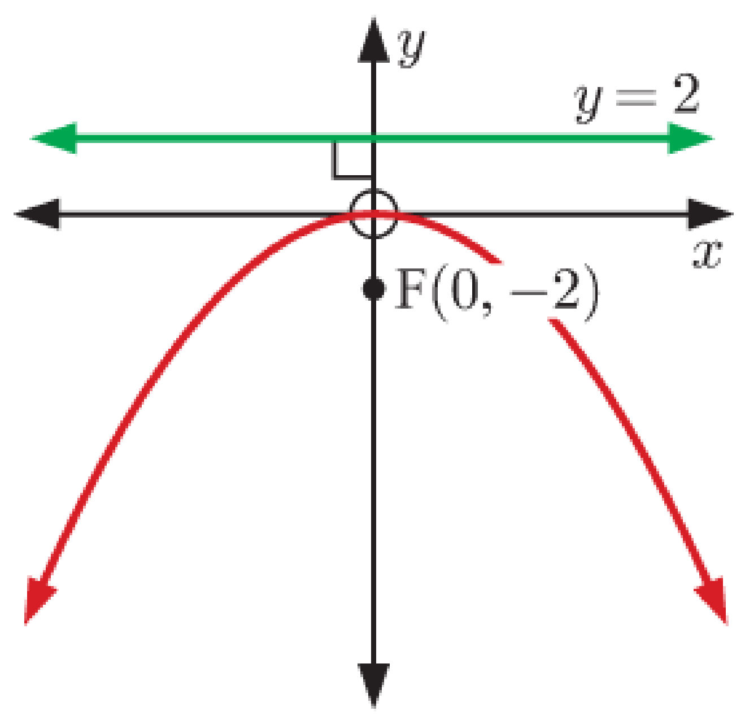
$$\begin{aligned} a = -2 \quad \text{and} \quad & y^2 = 4ax \\ \therefore\ & y^2 = 4(-2)x \\ \therefore\ & y^2 = -8x \end{aligned}$$

e



$$\begin{aligned} a = -5 \quad \text{and} \quad & x^2 = 4ay \\ \therefore\ & x^2 = 4(-5)y \\ \therefore\ & x^2 = -20y \end{aligned}$$

f



$$\begin{aligned} a = -2 \quad \text{and} \quad & x^2 = 4ay \\ \therefore\ & x^2 = 4(-2)y \\ \therefore\ & x^2 = -8y \end{aligned}$$

3 **a**

$$\begin{aligned} y^2 &= 4ax \\ \therefore\ \frac{d}{dx}(y^2) &= \frac{d}{dx}(4ax) \\ \therefore\ 2y\frac{dy}{dx} &= 4a \\ \therefore\ \frac{dy}{dx} &= \frac{2a}{y} \end{aligned}$$

$\therefore$ the gradient of the tangent at (x_1, y_1) is $\dfrac{2a}{y_1}$.

$\therefore$ the tangent at (x_1, y_1) is

$$2ax - y_1y = 2a(x_1) - y_1(y_1)$$

which is $2ax - y_1y = 2ax_1 - {y_1}^2$

which is $2ax - y_1y = -2ax_1$ $\{$as ${y_1}^2 = 4ax_1\}$

b The normal at (x_1, y_1) has gradient $-\dfrac{y_1}{2a}$.

$\therefore$ its equation is $y_1x + 2ay = y_1(x_1) + 2a(y_1)$

which is $y_1x + 2ay = x_1y_1 + 2ay_1$

c The normal cuts the x-axis when $y = 0$

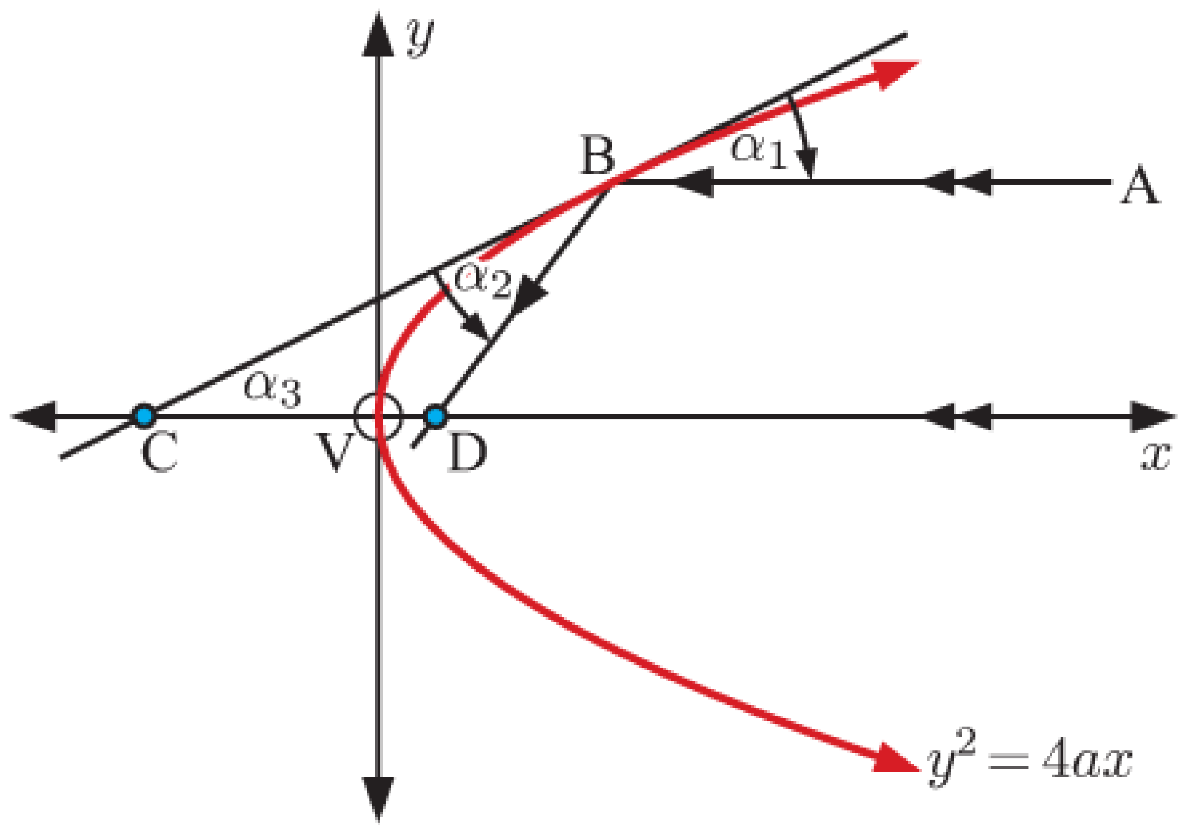
$$\begin{aligned} \therefore\ y_1x + 2a(0) &= x_1y_1 + 2ay_1 \\ \therefore\ y_1x &= y_1(x_1 + 2a) \end{aligned}$$

$\therefore$ when $(x_1, y_1) \neq (0, 0)$, $x = x_1 + 2a$

$\therefore$ $x > 2a$ $\{$as $a > 0$ $\therefore$ $x_1 > 0\}$

When $(x_1, y_1) = (0, 0)$, the tangent to the parabola is the y-axis and the normal is the x-axis. So it does not make sense to talk about the x-intercept in this case.

4 **a**



$\alpha_1 = \alpha_2$ $\{$given$\}$

$\alpha_1 = \alpha_3$ $\{$equal corresponding angles$\}$

$\therefore$ $\alpha_2 = \alpha_3$

$\therefore$ $\triangle$BCD is isosceles

$\{$converse of isosceles triangle theorem$\}$

b Consequently CD = BD $\{$from **a**$\}$

If B is (x_1, y_1), then the tangent at B has equation

$2ax - y_1y = -2ax_1$ $\{$from **3 a**$\}$.

This tangent cuts the x-axis when $y = 0$

$\therefore$ at $(-x_1, 0)$.

If we let D be $(t, 0)$, then

$$\begin{aligned} t - (-x_1) &= \sqrt{(t - x_1)^2 + {y_1}^2} \\ \therefore\ (t + x_1)^2 &= (t - x_1)^2 + {y_1}^2 \\ \therefore\ \cancel{t^2} + 2tx_1 + \cancel{{x_1}^2} &= \cancel{t^2} - 2tx_1 + \cancel{{x_1}^2} + 4ax_1 \\ &\qquad \{\text{as } y^2 = 4ax\} \\ \therefore\ 4tx_1 &= 4ax_1 \\ \therefore\ t &= a \end{aligned}$$

$\therefore$ D is $(a, 0)$, which is the focus.

5 **a** **i** When $x = 4$, $y^2 = 8(4)$

$$\begin{aligned} \therefore\ y^2 &= 32 \\ \therefore\ y &= \pm 4\sqrt{2} \end{aligned}$$

$\therefore$ P is $(4, 4\sqrt{2})$

$y^2 = 8x$

$\therefore$ $4a = 8$

$\therefore$ $a = 2$ $\therefore$ the focus is F(2, 0).

$\therefore$ [FP] has gradient $\dfrac{4\sqrt{2} - 0}{4 - 2} = \dfrac{4\sqrt{2}}{2} = \dfrac{2\sqrt{2}}{1}$

$\therefore$ the equation of the focal chord is

$$2\sqrt{2}x - y = 2\sqrt{2}(2) - (0)$$

which is $2\sqrt{2}x - y = 4\sqrt{2}$

The focal chord meets the parabola when

$$\begin{aligned} (2\sqrt{2}x - 4\sqrt{2})^2 &= 8x \\ \therefore\ 8x^2 - 32x + 32 - 8x &= 0 \\ \therefore\ 8x^2 - 40x + 32 &= 0 \\ \therefore\ x^2 - 5x + 4 &= 0 \\ \therefore\ (x - 1)(x - 4) &= 0 \\ \therefore\ x &= 1 \text{ or } 4 \end{aligned}$$

$\therefore$ Q has x-coordinate 1

$$\begin{aligned} \therefore\ y^2 &= 8(1) \\ \therefore\ y &= \pm 2\sqrt{2} \end{aligned}$$

$\therefore$ Q is $(1, -2\sqrt{2})$.

ii Using **3 a**, the tangent at $P(4,\ 4\sqrt{2})$ is

$$2(2)x - (4\sqrt{2})y = -2(2)(4)$$

which is $4x - 4\sqrt{2}y = -16$

which is $x - \sqrt{2}y = -4$ .... (1)

and the tangent at $Q(1,\ -2\sqrt{2})$ is

$$2(2)x - (-2\sqrt{2})y = -2(2)(1)$$

which is $4x + 2\sqrt{2}y = -4$

which is $2x + \sqrt{2}y = -2$ .... (2)

iii Solving (1) and (2) simultaneously,

$$3x = -6$$

$$\therefore\ x = -2$$

$\therefore$ the tangents at P and Q meet where $x = -2$ and $a = 2$

$\therefore$ the directrix is $x = -2$

$\therefore$ the tangents meet on the directrix.

The gradient of the tangent at P is $\frac{1}{\sqrt{2}}$

and the gradient of the tangent at Q is $-\frac{2}{\sqrt{2}}$

where $(\frac{1}{\sqrt{2}}) \times (-\frac{2}{\sqrt{2}}) = -1$.

$\therefore$ the tangents are at right angles to each other.

b Let P be $(x_1,\ y_1)$.

Using **3 a**, the tangent at P is $4x - y_1 y = -4x_1$.

F is $(2,\ 0)$ $\therefore$ [FP] has gradient $\dfrac{y_1}{x_1 - 2}$.

$\therefore$ the focal chord through P has equation

$$y_1 x - (x_1 - 2)y = 2y_1$$

$$\therefore\ (x_1 - 2)y = y_1(x - 2)$$

$$\therefore\ y = \frac{y_1(x-2)}{(x_1-2)}$$

This chord meets the parabola when

$$\left(\frac{y_1(x-2)}{x_1-2}\right)^2 = 8x$$

$$\therefore\ \frac{{y_1}^2(x-2)^2}{(x_1-2)^2} = 8x$$

$$\therefore\ \frac{8x_1(x-2)^2}{(x_1-2)^2} = 8x \quad \{\text{as } y^2 = 8x\}$$

$$\therefore\ x_1(x-2)^2 - x(x_1-2)^2 = 0$$

$$\therefore\ x_1(x^2 - 4x + 4) - x({x_1}^2 - 4x_1 + 4) = 0$$

$$\therefore\ x_1 x^2 + (-\cancel{4x_1} - {x_1}^2 + \cancel{4x_1} - 4)x + 4x_1 = 0$$

$$\therefore\ x_1 x^2 - ({x_1}^2 + 4)x + 4x_1 = 0$$

$$\therefore\ (x_1 x - 4)(x - x_1) = 0$$

$$\therefore\ x = \frac{4}{x_1} \text{ or } x_1$$

$\therefore$ Q has x-coordinate $\dfrac{4}{x_1}$

and $y^2 = 8x$

$$\therefore\ y^2 = \frac{32}{x_1}$$

$$\therefore\ y = \pm\frac{\sqrt{32}}{\sqrt{x_1}}$$

$\therefore$ Q is $\left(\dfrac{4}{x_1},\ -\dfrac{\sqrt{32}}{\sqrt{x_1}}\right)$.

Using **3 a**, the tangent at Q is

$$4x + \frac{\sqrt{32}}{\sqrt{x_1}}y = -\frac{16}{x_1}$$

$$\therefore\ \frac{\sqrt{32}}{\sqrt{x_1}}y = -4\left(x + \frac{4}{x_1}\right)$$

$$\therefore\ y = -4\left(x + \frac{4}{x_1}\right)\left(\frac{\sqrt{x_1}}{\sqrt{32}}\right) \quad \text{.... (1)}$$

The tangent at P is

$$4x - y_1 y = -4x_1, \quad \text{where } y_1 = \sqrt{8}\sqrt{x_1}$$

$$\therefore\ 4x - \sqrt{8}\sqrt{x_1}y = -4x_1$$

$$\therefore\ \sqrt{8}\sqrt{x_1}y = 4(x + x_1)$$

$$\therefore\ y = \frac{4(x + x_1)}{\sqrt{8}\sqrt{x_1}} \quad \text{.... (2)}$$

Equating y in (1) and (2):

$$-\cancel{4}\left(x + \frac{4}{x_1}\right)\left(\frac{\sqrt{x_1}}{\sqrt{32}}\right) = \frac{\cancel{4}(x + x_1)}{\sqrt{8}\sqrt{x_1}}$$

$$\therefore\ -\left(x + \frac{4}{x_1}\right)x_1 = \frac{\sqrt{32}}{\sqrt{8}}(x + x_1)$$

$$\therefore\ -x_1 x - 4 = 2x + 2x_1$$

$$\therefore\ (2 + x_1)x = -4 - 2x_1$$

$$\therefore\ x = \frac{-2(2 + x_1)}{2 + x_1}$$

$$\therefore\ x = -2$$

$\therefore$ the tangents at P and Q meet where $x = -2$

$\therefore$ the tangents meet on the directrix.

The gradient of the tangent at P is $\dfrac{4}{\sqrt{8}\sqrt{x_1}} = \dfrac{\sqrt{2}}{\sqrt{x_1}}$,

the gradient of the tangent at Q is $-\dfrac{4\sqrt{x_1}}{\sqrt{32}} = -\dfrac{\sqrt{x_1}}{\sqrt{2}}$,

and $\left(\dfrac{\sqrt{2}}{\sqrt{x_1}}\right) \times \left(-\dfrac{\sqrt{x_1}}{\sqrt{2}}\right) = -1$

$\therefore$ the tangents are at right angles to each other.

EXERCISE 2N.3

1 a i

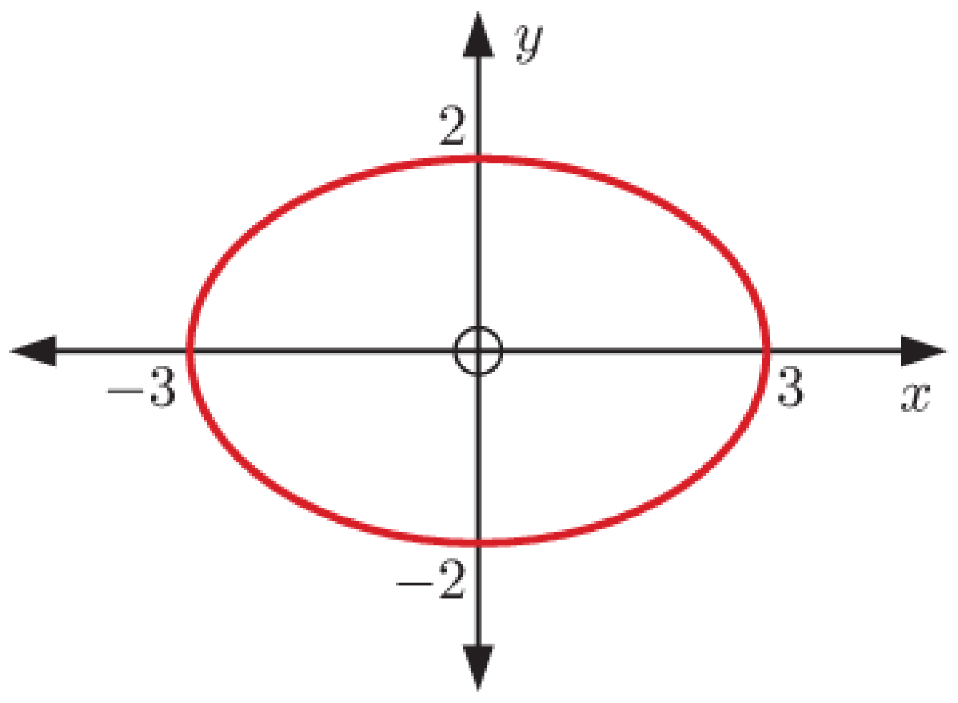
$$4x^2 + 9y^2 = 36$$

$$\therefore\ \frac{x^2}{9} + \frac{y^2}{4} = 1$$

$\therefore$ $a^2 = 9$ and $b^2 = 4$

$\therefore$ the x-intercepts are ± 3 and the y-intercepts are ± 2.

ii



iii

$$b^2 = a^2(1 - e^2)$$

$$\therefore\ 4 = 9(1 - e^2)$$

$$\therefore\ 1 - e^2 = \tfrac{4}{9}$$

$$\therefore\ e^2 = \tfrac{5}{9}$$

$$\therefore\ e = \tfrac{\sqrt{5}}{3} \quad \{\text{as } e > 0\}$$

iv $ae = 3 \times \frac{\sqrt{5}}{3} = \sqrt{5}$ and $\frac{a}{e} = \frac{3}{\frac{\sqrt{5}}{3}} = \frac{9}{\sqrt{5}}$

The focus $(\sqrt{5}, 0)$ has corresponding directrix $x = \frac{9}{\sqrt{5}}$.

The focus $(-\sqrt{5}, 0)$ has corresponding directrix $x = -\frac{9}{\sqrt{5}}$.

b **i**

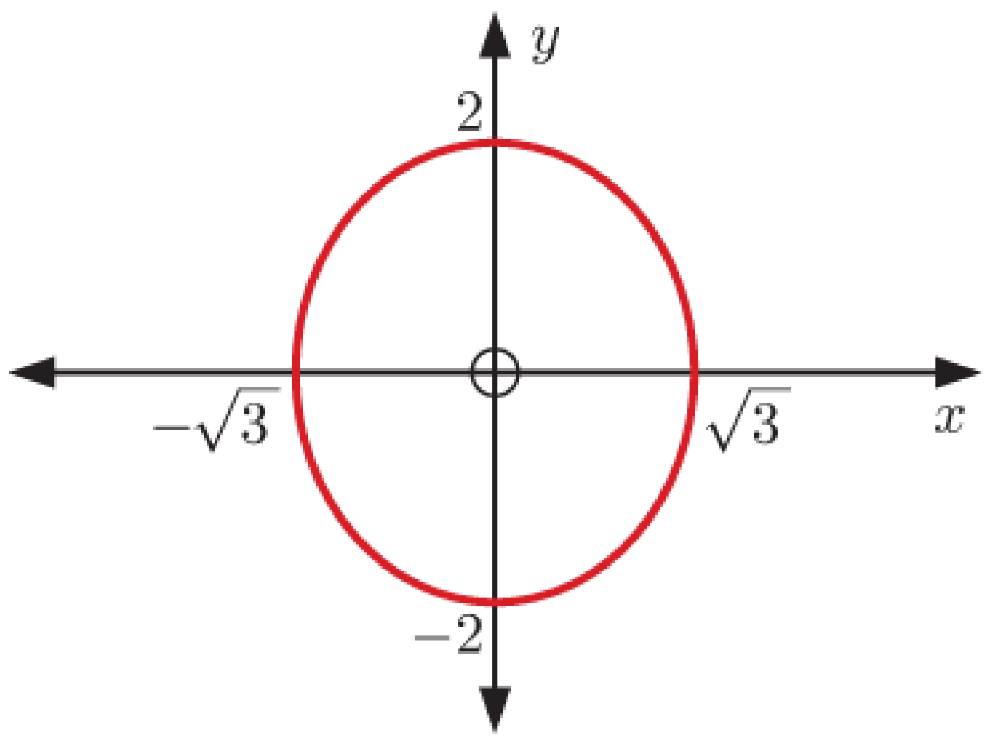
$$4x^2 + 3y^2 = 12$$

$$\therefore \quad \frac{x^2}{3} + \frac{y^2}{4} = 1$$

$\therefore \quad a^2 = 4$ and $b^2 = 3$

$\therefore$ the x-intercepts are $\pm\sqrt{3}$ and the y-intercepts are ± 2.

ii



iii

$$b^2 = a^2(1 - e^2)$$

$$\therefore \quad 3 = 4(1 - e^2)$$

$$\therefore \quad 1 - e^2 = \tfrac{3}{4}$$

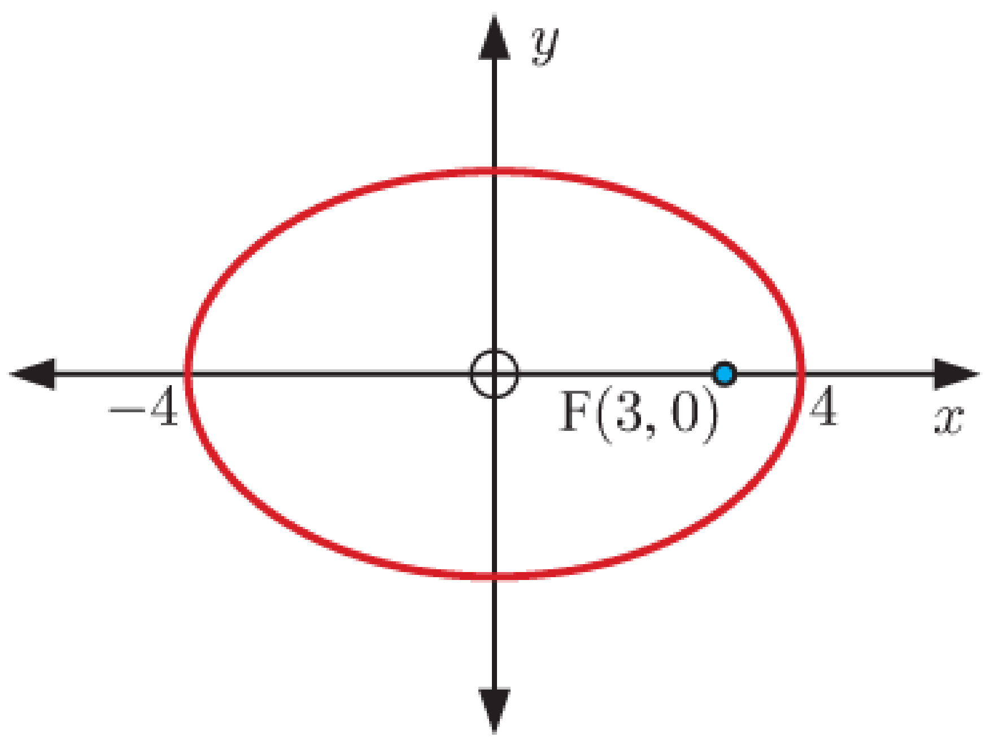
$$\therefore \quad e^2 = \tfrac{1}{4}$$

$$\therefore \quad e = \tfrac{1}{2} \quad \{\text{as } e > 0\}$$

iv $ae = 2 \times \frac{1}{2} = 1$ and $\frac{a}{e} = \frac{2}{\frac{1}{2}} = 4$

The focus $(0, 1)$ has corresponding directrix $y = 4$.
The focus $(0, -1)$ has corresponding directrix $y = -4$.

2 **a**



$a = 4$ and $ae = 3$

$$\therefore \quad e = \tfrac{3}{4}$$

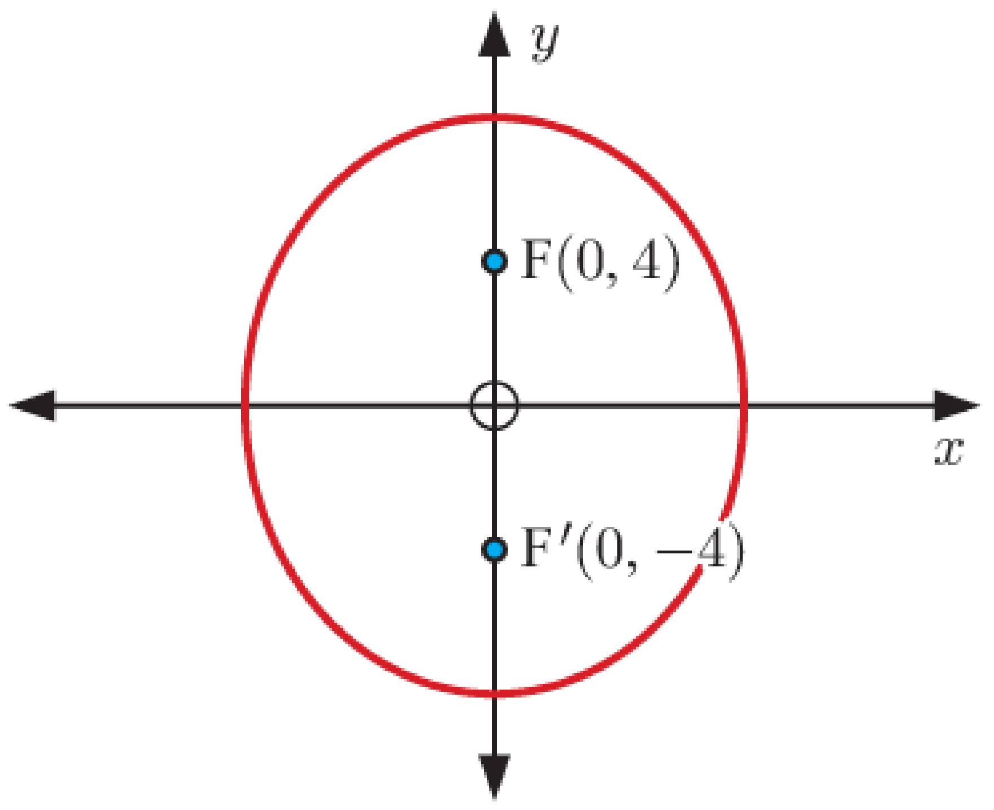
$$b^2 = a^2(1 - e^2)$$

$$\therefore \quad b^2 = 16(1 - \tfrac{9}{16})$$

$$\therefore \quad b^2 = 7$$

$\therefore$ the ellipse has equation $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$.

b



$ae = 4$ and $e = \frac{1}{2}$

$$\therefore \quad a(\tfrac{1}{2}) = 4$$

$$\therefore \quad a = 8$$

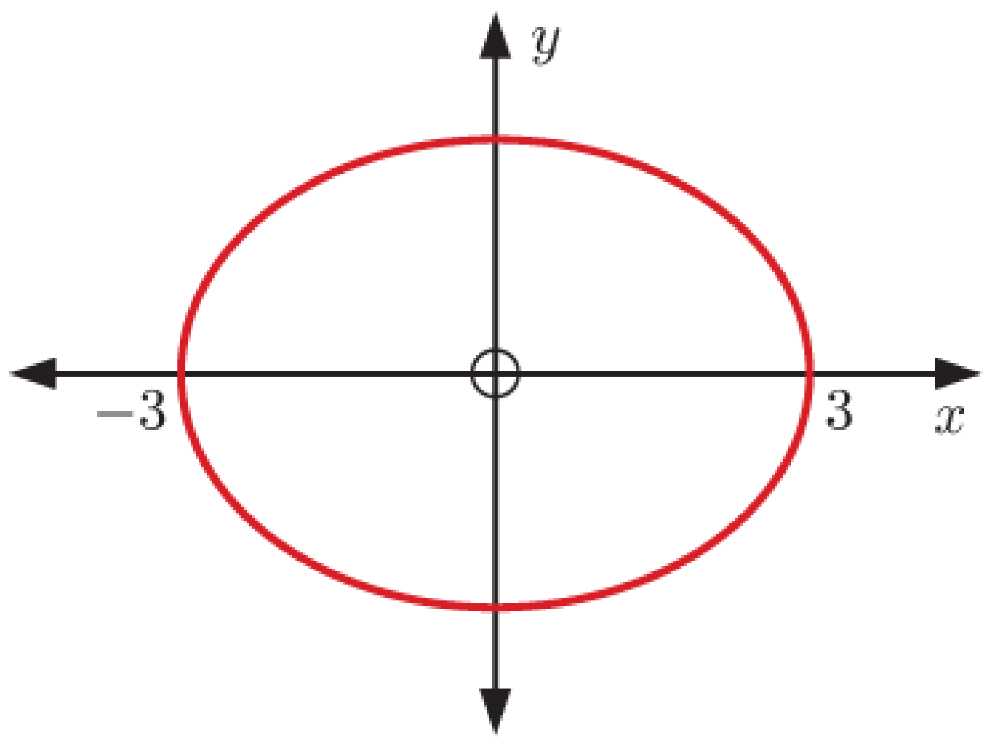
$$b^2 = a^2(1 - e^2)$$

$$\therefore \quad b^2 = 64(1 - \tfrac{1}{4})$$

$$\therefore \quad b^2 = 48$$

$\therefore$ the ellipse has equation $\dfrac{x^2}{48} + \dfrac{y^2}{64} = 1$.

c



$a = 3$ and $e = \frac{2}{3}$

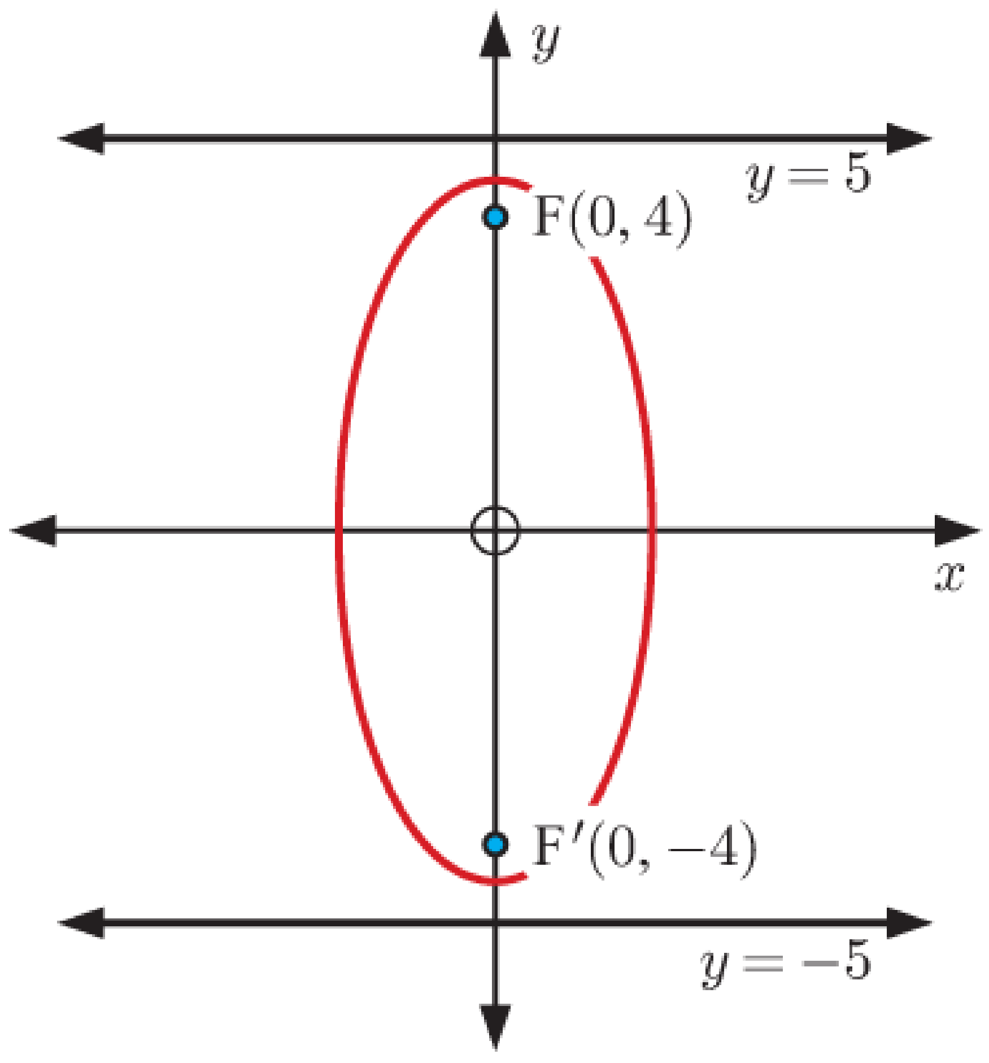
$$b^2 = a^2(1 - e^2)$$

$$\therefore \quad b^2 = 9(1 - \tfrac{4}{9})$$

$$\therefore \quad b^2 = 5$$

$\therefore$ the ellipse has equation $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$.

d



$ae = 4$ and $\frac{a}{e} = 5$

$$ae\left(\frac{a}{e}\right) = 20$$

$$\therefore \quad a^2 = 20$$

$$ae = 4$$

$$\therefore \quad e = \tfrac{4}{\sqrt{20}}$$

$$\therefore \quad e = \tfrac{2}{\sqrt{5}}$$

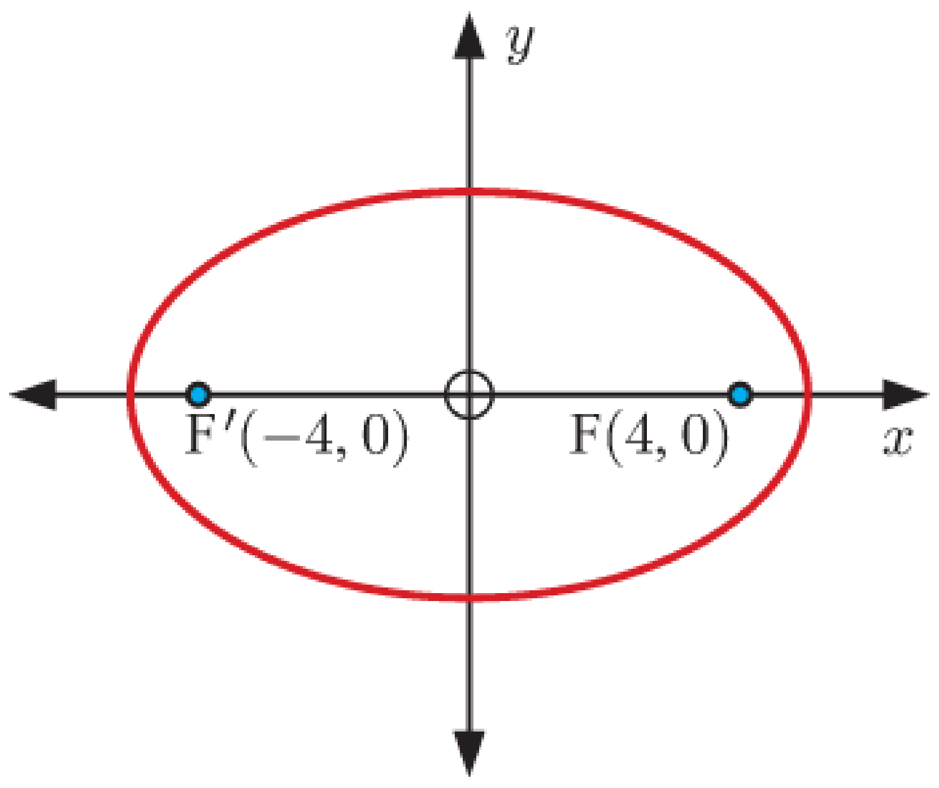
$$b^2 = a^2(1 - e^2)$$

$$\therefore \quad b^2 = 20(1 - \tfrac{4}{5})$$

$$\therefore \quad b^2 = 4$$

$\therefore$ the ellipse has equation $\dfrac{x^2}{4} + \dfrac{y^2}{20} = 1$.

e



$$\mathrm{PF} + \mathrm{PF}' = 10$$

$$\therefore \quad 2a = 10$$

{focal-distance property}

$$\therefore \quad a = 5$$

and $ae = 4$

$$\therefore \quad e = \tfrac{4}{5}$$

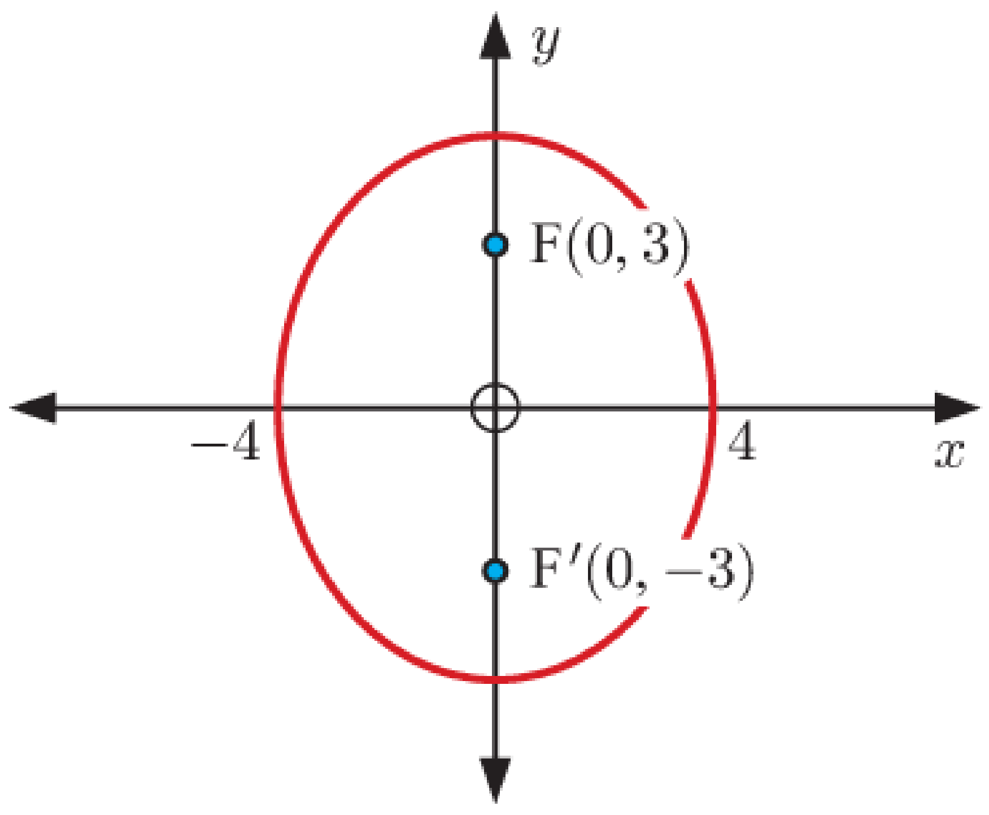
$$b^2 = a^2(1 - e^2)$$

$$\therefore \quad b^2 = 25(1 - \tfrac{16}{25})$$

$$\therefore \quad b^2 = 9$$

$\therefore$ the ellipse has equation $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$.

f



$ae = 3$ and $b = 4$

$$b^2 = a^2(1 - e^2)$$

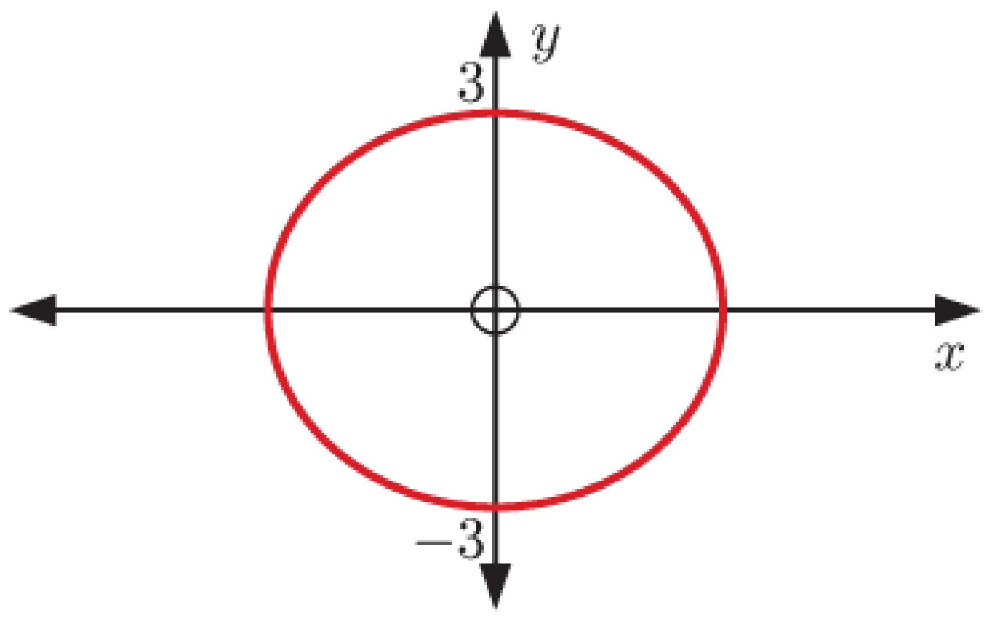
$$\therefore \quad b^2 = a^2 - a^2e^2$$

$$\therefore \quad 16 = a^2 - 9$$

$$\therefore \quad a^2 = 25$$

$\therefore$ the ellipse has equation $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$.

g



$b = 3$ and $e = \frac{1}{2}$

$$b^2 = a^2(1 - e^2)$$

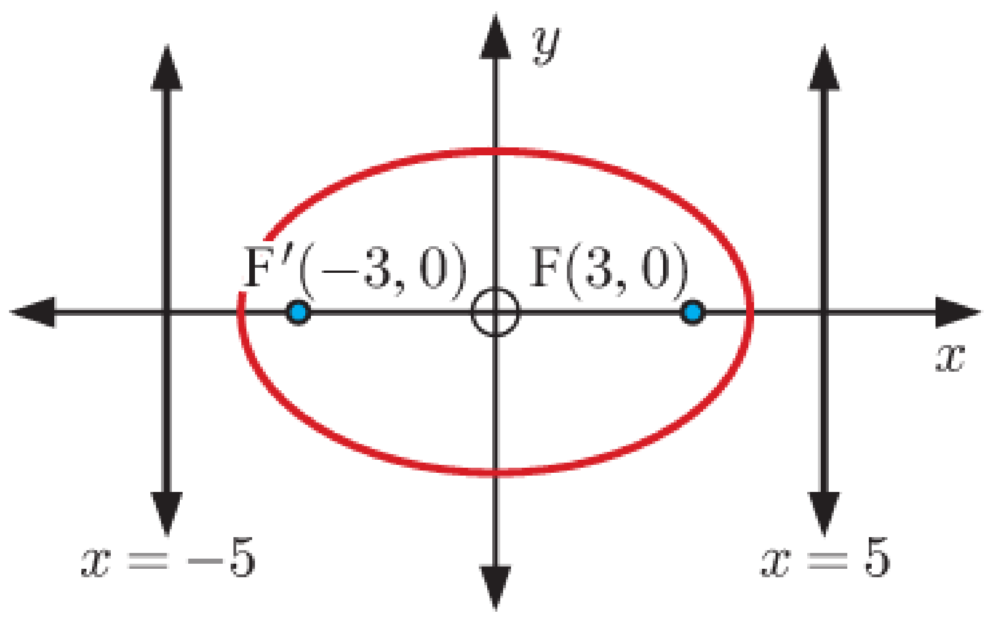
$$\therefore \quad 9 = a^2(1 - \tfrac{1}{4})$$

$$\therefore \quad a^2 = 9 \times \tfrac{4}{3}$$

$$\therefore \quad a^2 = 12$$

$\therefore$ the ellipse has equation $\dfrac{x^2}{12} + \dfrac{y^2}{9} = 1$.

h



$$ae = 3 \quad \text{and} \quad \frac{a}{e} = 5$$

$$\therefore\ ae\left(\frac{a}{e}\right) = 15$$

$$\therefore\ a^2 = 15$$

$$b^2 = a^2(1 - e^2)$$

$$\therefore\ b^2 = a^2 - a^2e^2$$

$$\therefore\ b^2 = 15 - 9$$

$$\therefore\ b^2 = 6$$

∴ the ellipse has equation $\dfrac{x^2}{15} + \dfrac{y^2}{6} = 1$.

3 The latus rectum meets the ellipse when $x = ae$.

$$\therefore\ \frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore\ \frac{y^2}{b^2} = 1 - e^2$$

$$\therefore\ y^2 = b^2(1 - e^2)$$

$$\therefore\ y^2 = b^2 \times \frac{b^2}{a^2} \quad \{\text{as } b^2 = a^2(1 - e^2)\}$$

$$\therefore\ y = \pm\frac{b^2}{a}$$

∴ the length of the latus rectum is $\dfrac{2b^2}{a}$.

4 Let N be (X, Y).

∴ A is $(3X, 0)$, B is $(0, \dfrac{3Y}{2})$

$$\text{AB} = k$$

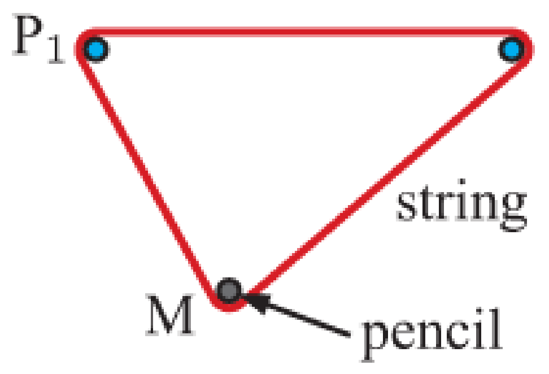
$$\therefore\ \sqrt{(3X - 0)^2 + (0 - \frac{3Y}{2})^2} = k$$

$$\therefore\ 9X^2 + \frac{9Y^2}{4} = k^2$$

$$\therefore\ \frac{9X^2}{k^2} + \frac{9Y^2}{4k^2} = 1$$

So the locus of N is an ellipse with equation $\dfrac{9x^2}{k^2} + \dfrac{9y^2}{4k^2} = 1$.

5



Make a loop with the string and place it over the pegs P_1 and P_2. Place a pencil on the inner side of the string and pull the string taut as the pencil moves. The pencil draws an ellipse.

This works because of the focal-distance property.

P_1P_2 is fixed, and $MP_1 + MP_2$ is a constant as the loop's length is fixed.

6 **a**

$$\frac{x^2}{3} + \frac{y^2}{12} = 1$$

$$\therefore\ \frac{2x}{3} + \frac{2y}{12}\frac{dy}{dx} = 0 \quad \{\text{implicit differentiation}\}$$

$$\therefore\ \frac{y}{12}\frac{dy}{dx} = -\frac{x}{3}$$

$$\therefore\ \frac{dy}{dx} = -\frac{4x}{y}$$

When $x = \sqrt{2}$, $\dfrac{(\sqrt{2})^2}{3} + \dfrac{y^2}{12} = 1$

$$\therefore\ \frac{y^2}{12} = 1 - \tfrac{2}{3}$$

$$\therefore\ y^2 = 12 \times \tfrac{1}{3}$$

$$\therefore\ y^2 = 4$$

$$\therefore\ y = \pm 2$$

At $(\sqrt{2}, 2)$, $\dfrac{dy}{dx} = -\dfrac{4(\sqrt{2})}{(2)} = -2\sqrt{2}$.

∴ the equation of the tangent is

$$2\sqrt{2}x + y = 2\sqrt{2}(\sqrt{2}) + 2$$

which is $2\sqrt{2}x + y = 6$

At $(\sqrt{2}, -2)$, $\dfrac{dy}{dx} = -\dfrac{4(\sqrt{2})}{(-2)} = 2\sqrt{2}$.

∴ the equation of the tangent is

$$2\sqrt{2}x - y = 2\sqrt{2}(\sqrt{2}) - (-2)$$

which is $2\sqrt{2}x - y = 6$

b The gradient of the normal is $\dfrac{y}{4x}$.

When $y = 2$, $\dfrac{x^2}{3} + \dfrac{2^2}{12} = 1$

$$\therefore\ \frac{x^2}{3} = 1 - \tfrac{1}{3}$$

$$\therefore\ \frac{x^2}{3} = \tfrac{2}{3}$$

$$\therefore\ x^2 = 2$$

$$\therefore\ x = \pm\sqrt{2}$$

At $(\sqrt{2}, 2)$, the gradient of the normal is $\dfrac{(2)}{4(\sqrt{2})} = \dfrac{1}{2\sqrt{2}}$.

∴ the equation of the normal is

$$x - 2\sqrt{2}y = (\sqrt{2}) - 2\sqrt{2}(2)$$

which is $x - 2\sqrt{2}y = -3\sqrt{2}$

At $(-\sqrt{2}, 2)$, the gradient of the normal is

$$\frac{(2)}{4(-\sqrt{2})} = -\frac{1}{2\sqrt{2}}.$$

∴ the equation of the normal is

$$x + 2\sqrt{2}y = (-\sqrt{2}) + 2\sqrt{2}(2)$$

which is $x + 2\sqrt{2}y = 3\sqrt{2}$

7 **a**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{.... (1)}$$

$$\therefore\ \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \{\text{implicit differentiation}\}$$

$$\therefore\ \frac{y}{b^2}\frac{dy}{dx} = -\frac{x}{a^2}$$

$$\therefore\ \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

At (x_1, y_1), $\dfrac{dy}{dx} = -\dfrac{b^2x_1}{a^2y_1}$

∴ the equation of the tangent is

$$b^2x_1x + a^2y_1y = b^2x_1(x_1) + a^2y_1(y_1)$$

$$\therefore\ \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} \quad \{\text{dividing by } a^2b^2\}$$

$$\therefore\ \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1 \quad \{\text{using (1)}\}$$

b The end of the latus rectum in the first quadrant is

$\left(ae, \dfrac{b^2}{a}\right)$. {using **3**}

The equation of the tangent is $\dfrac{(ae)x}{a^2} + \dfrac{\left(\frac{b^2}{a}\right)y}{b^2} = 1$

which is $\dfrac{ex}{a} + \dfrac{y}{a} = 1$

which is $ex + y = a$.

c The gradient of the normal at (x_1, y_1) is $\dfrac{a^2y_1}{b^2x_1}$.

$\therefore$ the normal has equation

$$a^2y_1x - b^2x_1y = a^2y_1(x_1) - b^2x_1(y_1)$$

which is $a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1$

EXERCISE 2N.4

1 a i

$$25x^2 - 16y^2 = 400$$

$$\therefore \frac{x^2}{16} - \frac{y^2}{25} = 1$$

cuts the x-axis when $y = 0$

$$\therefore x^2 = 16$$

$$\therefore x = \pm 4$$

$\therefore$ the hyperbola cuts the x-axis at $(4, 0)$ and $(-4, 0)$ but does not cut the y-axis.

ii $a^2 = 16$ and $b^2 = 25$

$$b^2 = a^2(e^2 - 1)$$

$$\therefore 25 = 16(e^2 - 1)$$

$$\therefore e^2 - 1 = \tfrac{25}{16}$$

$$\therefore e^2 = \tfrac{41}{16}$$

$$\therefore e = \tfrac{\sqrt{41}}{4} \quad \{\text{as } e > 0\}$$

$$\therefore ae = 4 \times \tfrac{\sqrt{41}}{4} = \sqrt{41}$$

$$\text{and} \quad \frac{a}{e} = \frac{4}{\frac{\sqrt{41}}{4}} = \tfrac{16}{\sqrt{41}}$$

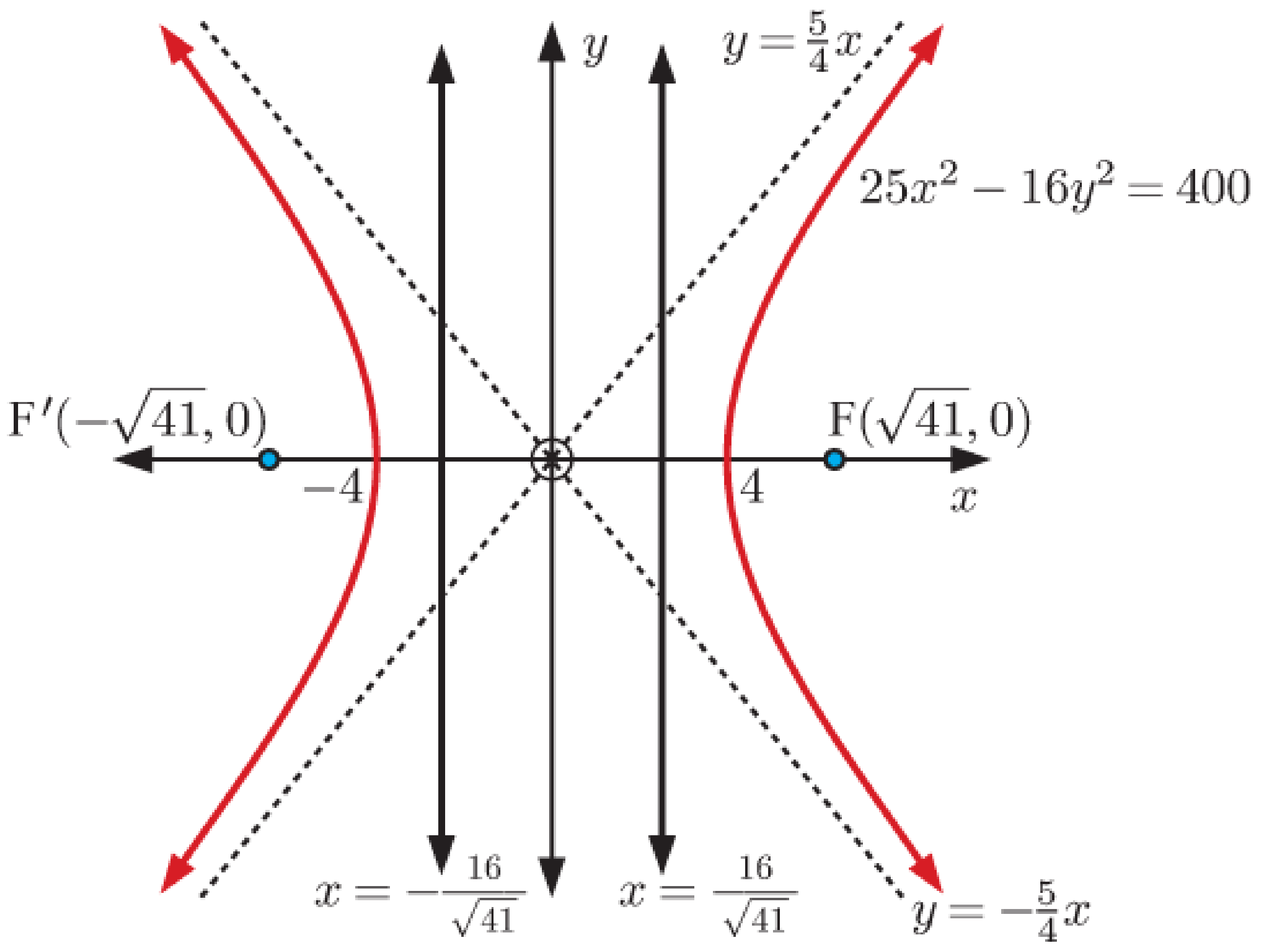
The focus $(\sqrt{41}, 0)$ has corresponding directrix $x = \tfrac{16}{\sqrt{41}}$.

The focus $(-\sqrt{41}, 0)$ has corresponding directrix $x = -\tfrac{16}{\sqrt{41}}$.

iii The asymptotes have equations $y = \pm\dfrac{b}{a}x$

$$\therefore y = \pm\tfrac{5}{4}x$$

iv



b i

$$4y^2 - x^2 = 16$$

$$\therefore \frac{y^2}{4} - \frac{x^2}{16} = 1$$

cuts the y-axis when $x = 0$

$$\therefore y^2 = 4$$

$$\therefore y = \pm 2$$

$\therefore$ the hyperbola cuts the y-axis at $(0, 2)$ and $(0, -2)$ but does not cut the x-axis.

ii $a^2 = 4$ and $b^2 = 16$

$$b^2 = a^2(e^2 - 1)$$

$$\therefore 16 = 4(e^2 - 1)$$

$$\therefore e^2 - 1 = 4$$

$$\therefore e^2 = 5$$

$$\therefore e = \sqrt{5} \quad \{\text{as } e > 0\}$$

$$\therefore ae = 2\sqrt{5} \quad \text{and} \quad \frac{a}{e} = \tfrac{2}{\sqrt{5}}$$

The focus $(0, 2\sqrt{5})$ has corresponding directrix $y = \tfrac{2}{\sqrt{5}}$.

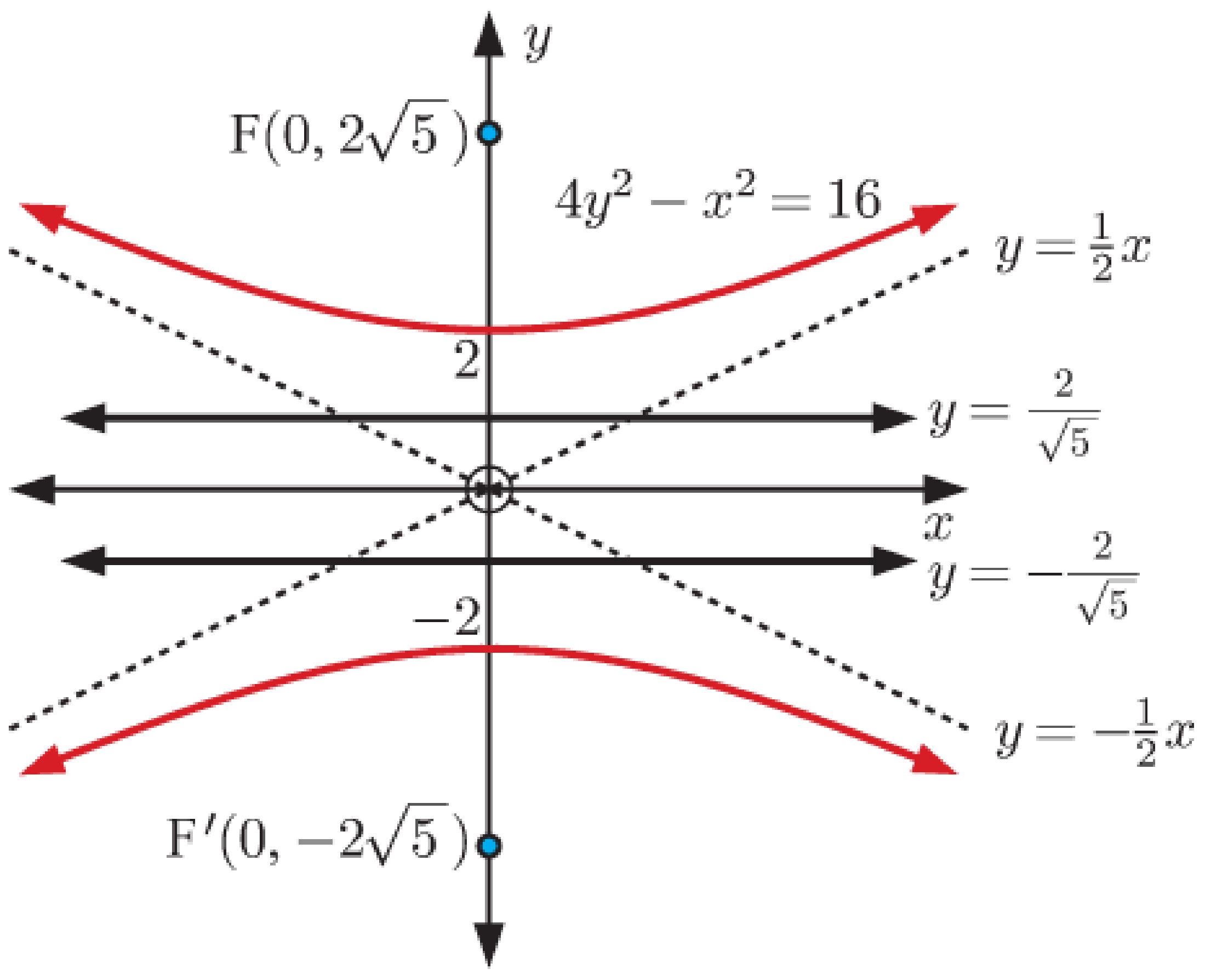
The focus $(0, -2\sqrt{5})$ has corresponding directrix $y = -\tfrac{2}{\sqrt{5}}$.

iii The asymptotes have equations $y = \pm\dfrac{a}{b}x$

$$\therefore y = \pm\tfrac{2}{4}x$$

$$\therefore y = \pm\tfrac{1}{2}x$$

iv



c i

$$x^2 - y^2 = 4$$

$$\therefore \frac{x^2}{4} - \frac{y^2}{4} = 1$$

cuts the x-axis when $y = 0$

$$\therefore x^2 = 4$$

$$\therefore x = \pm 2$$

$\therefore$ the hyperbola cuts the x-axis at $(2, 0)$ and $(-2, 0)$ but does not cut the y-axis.

ii $a^2 = 4$ and $b^2 = 4$

$$b^2 = a^2(e^2 - 1)$$

$$\therefore 4 = 4(e^2 - 1)$$

$$\therefore e^2 - 1 = 1$$

$$\therefore e^2 = 2$$

$$\therefore e = \sqrt{2} \quad \{\text{as } e > 0\}$$

$$\therefore ae = 2\sqrt{2}$$

$$\text{and} \quad \frac{a}{e} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

The focus $(2\sqrt{2}, 0)$ has corresponding directrix $x = \sqrt{2}$.

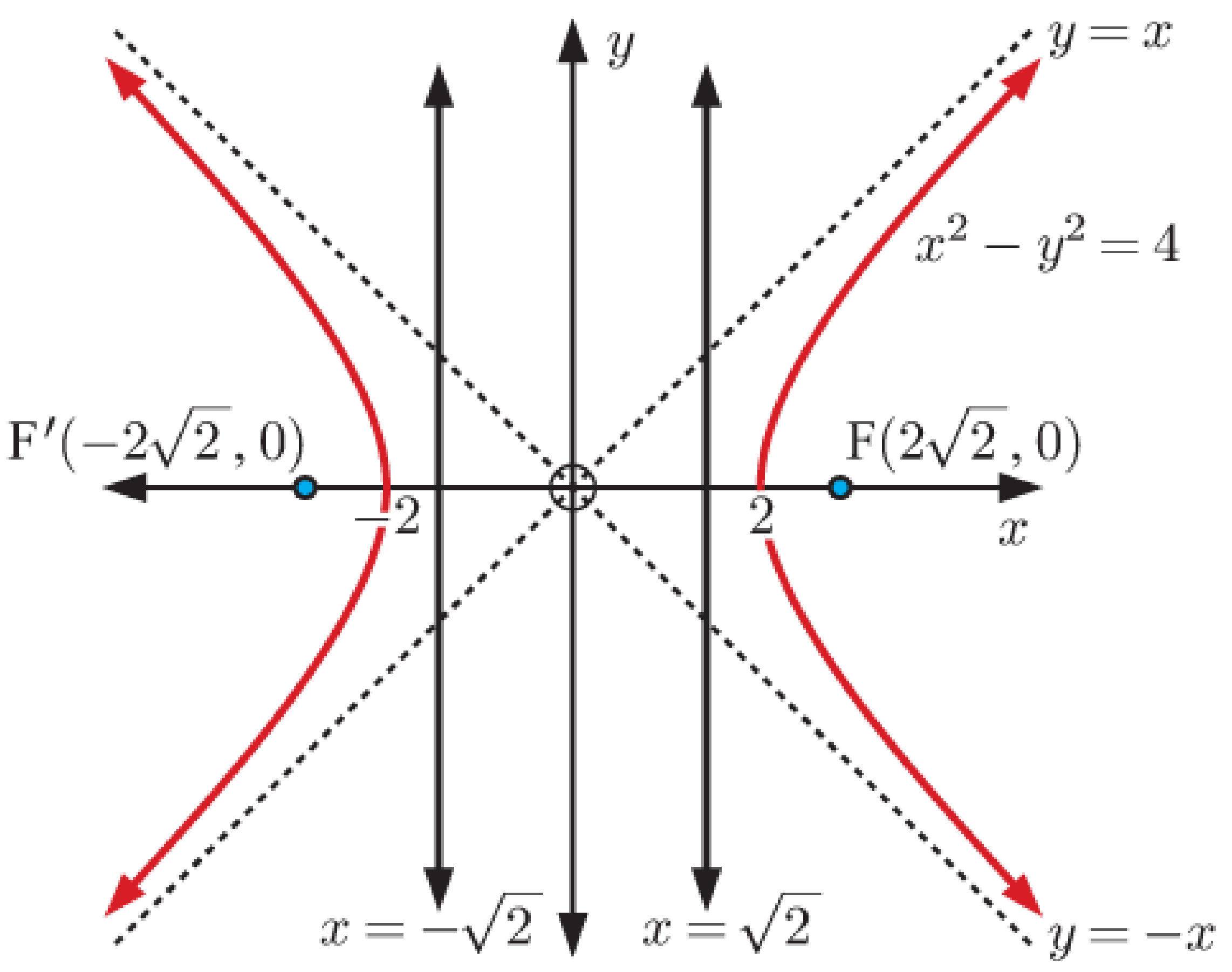
The focus $(-2\sqrt{2}, 0)$ has corresponding directrix $x = -\sqrt{2}$.

iii The asymptotes have equations $y = \pm\dfrac{b}{a}x$

$$\therefore y = \pm\tfrac{2}{2}x$$

$$\therefore y = \pm x$$

iv


y
y = x
$x^2 - y^2 = 4$
F′(−2√2, 0)
F(2√2, 0)
−2
2
x
x = −√2
x = √2
y = −x

d **i**

$$y^2 - x^2 = 9$$

$$\therefore \quad \frac{y^2}{9} - \frac{x^2}{9} = 1$$

cuts the y-axis when $x = 0$

$$\therefore \quad y^2 = 9$$

$$\therefore \quad y = \pm 3$$

$\therefore$ the hyperbola cuts the y-axis at $(0, 3)$ and $(0, -3)$ but does not cut the x-axis.

ii $a^2 = 9$ and $b^2 = 9$

$$b^2 = a^2(e^2 - 1)$$

$$\therefore \quad 9 = 9(e^2 - 1)$$

$$\therefore \quad e^2 - 1 = 1$$

$$\therefore \quad e^2 = 2$$

$$\therefore \quad e = \sqrt{2} \quad \{\text{as } e > 0\}$$

$$\therefore \quad ae = 3\sqrt{2} \quad \text{and} \quad \frac{a}{e} = \frac{3}{\sqrt{2}}$$

The focus $(0, 3\sqrt{2})$ has corresponding directrix $y = \frac{3}{\sqrt{2}}$.

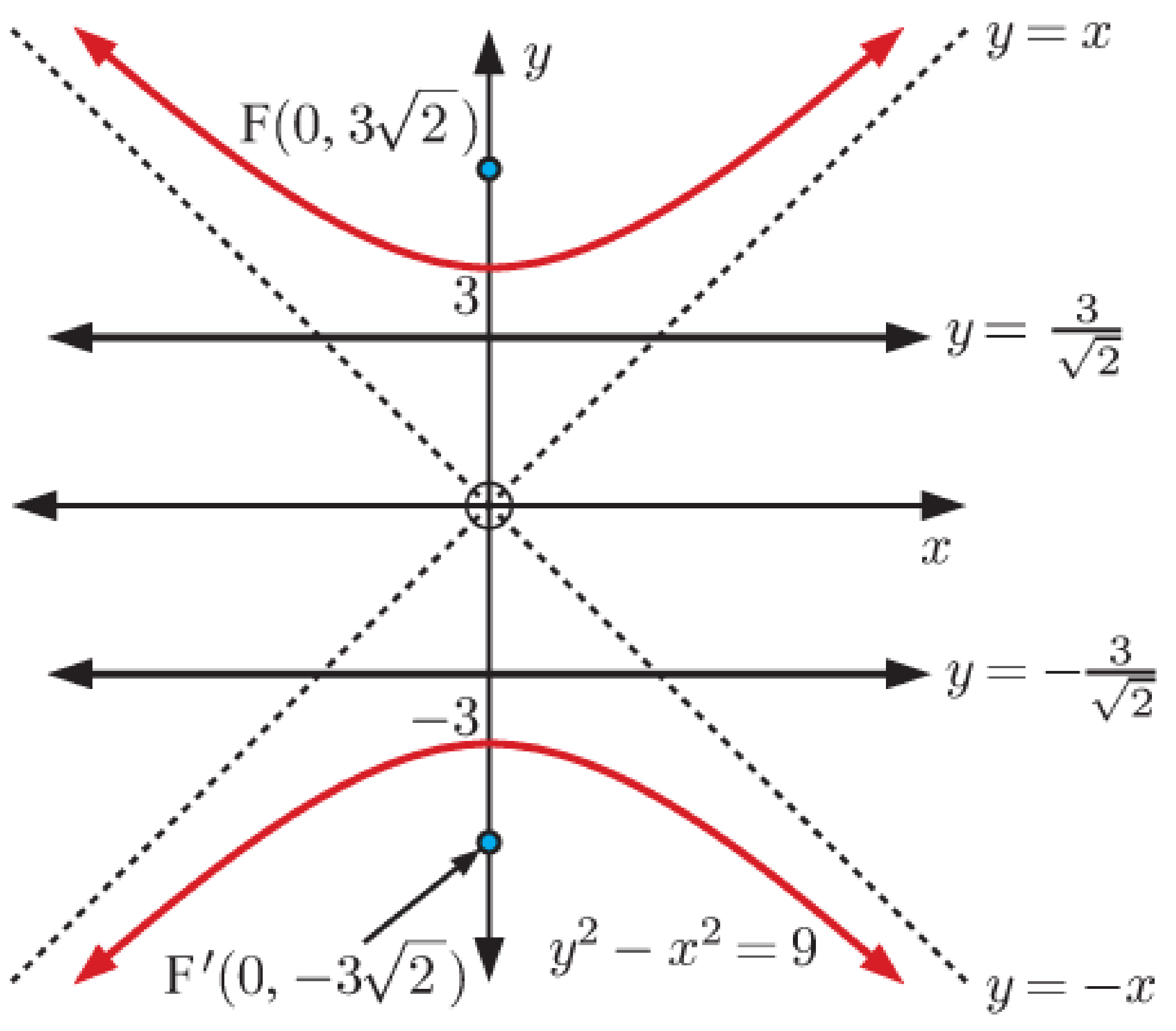
The focus $(0, -3\sqrt{2})$ has corresponding directrix $y = -\frac{3}{\sqrt{2}}$.

iii The asymptotes have equations $y = \pm\frac{a}{b}x$

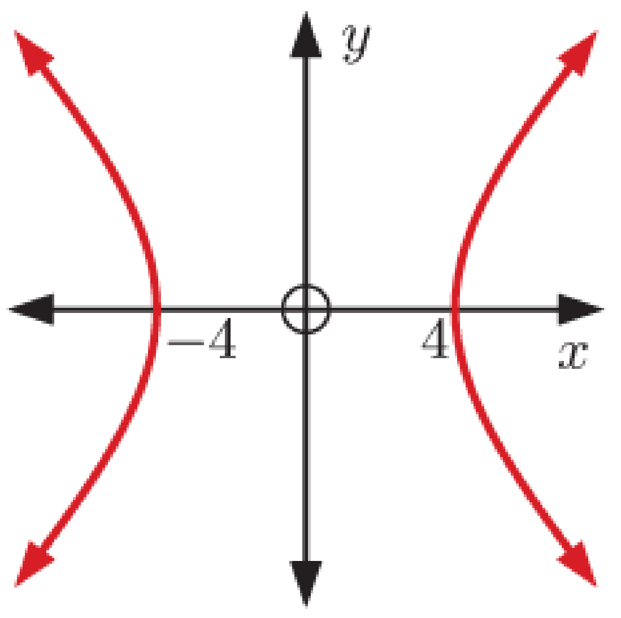
$$\therefore \quad y = \pm\tfrac{3}{3}x$$

$$\therefore \quad y = \pm x$$

iv


y
y = x
F(0, 3√2)
3
$y = \frac{3}{\sqrt{2}}$
x
$y = -\frac{3}{\sqrt{2}}$
−3
F′(0, −3√2)
$y^2 - x^2 = 9$
y = −x

2 **a**



$$a = 4 \quad \text{and} \quad e = 1\tfrac{1}{2}$$

$$b^2 = a^2(e^2 - 1)$$

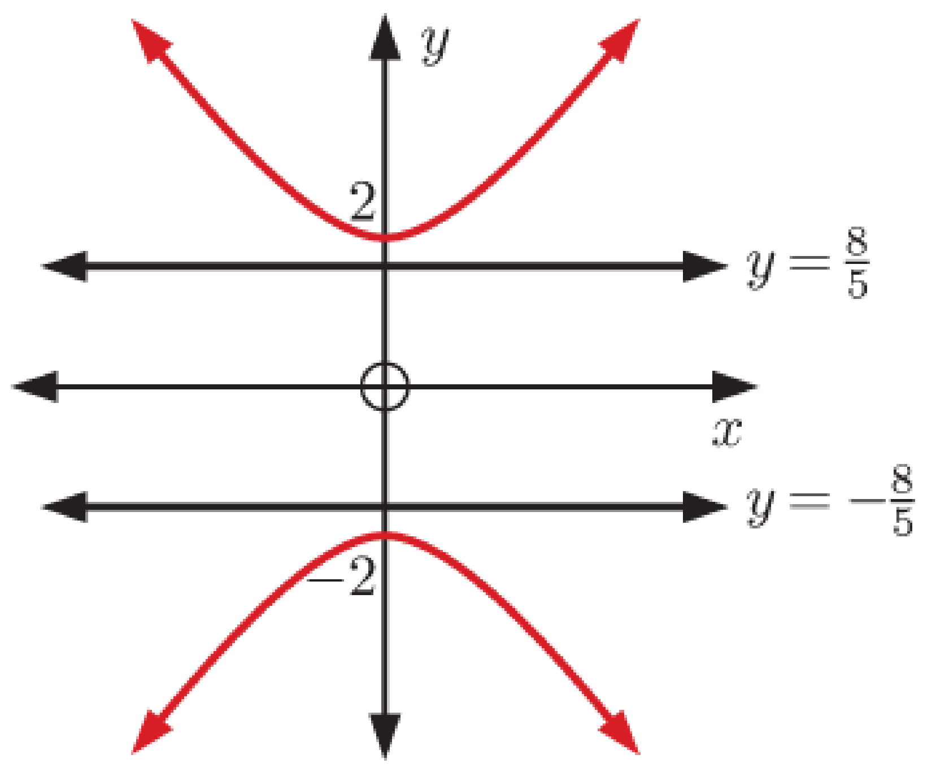
$$\therefore \quad b^2 = 16(\tfrac{9}{4} - 1)$$

$$\therefore \quad b^2 = 36 - 16$$

$$\therefore \quad b^2 = 20$$

$\therefore$ the equation of the hyperbola is $\dfrac{x^2}{16} - \dfrac{y^2}{20} = 1$.

b


y
2
$y = \frac{8}{5}$
x
$y = -\frac{8}{5}$
−2

$$a = 2 \quad \text{and} \quad \frac{a}{e} = \frac{8}{5}$$

$$\therefore \quad \frac{2}{e} = \frac{8}{5}$$

$$\therefore \quad e = \tfrac{5}{4}$$

$$b^2 = a^2(e^2 - 1)$$

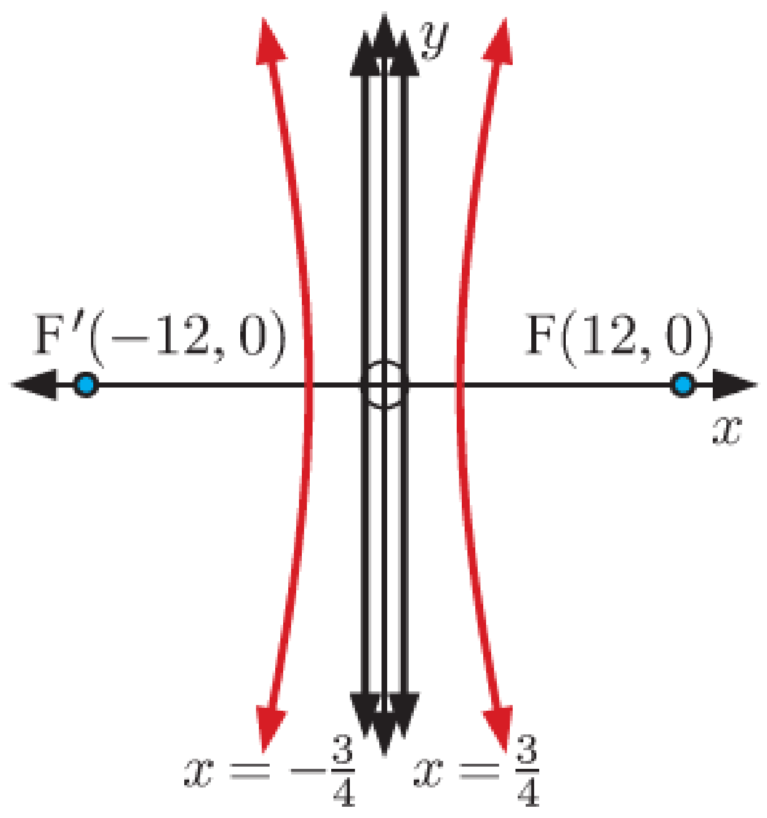
$$\therefore \quad b^2 = 4(\tfrac{25}{16} - 1)$$

$$\therefore \quad b^2 = 4 \times \tfrac{9}{16}$$

$$\therefore \quad b^2 = \tfrac{9}{4}$$

$\therefore$ the equation of the hyperbola is $\dfrac{y^2}{4} - \dfrac{4x^2}{9} = 1$.

c


y
F′(−12, 0)
F(12, 0)
x
$x = -\frac{3}{4}$
$x = \frac{3}{4}$

$$ae = 12 \quad \text{and} \quad \frac{a}{e} = \frac{3}{4}$$

$$\therefore \quad ae \times \frac{a}{e} = 12 \times \tfrac{3}{4}$$

$$\therefore \quad a^2 = 9$$

$$\text{and} \quad e = 4$$

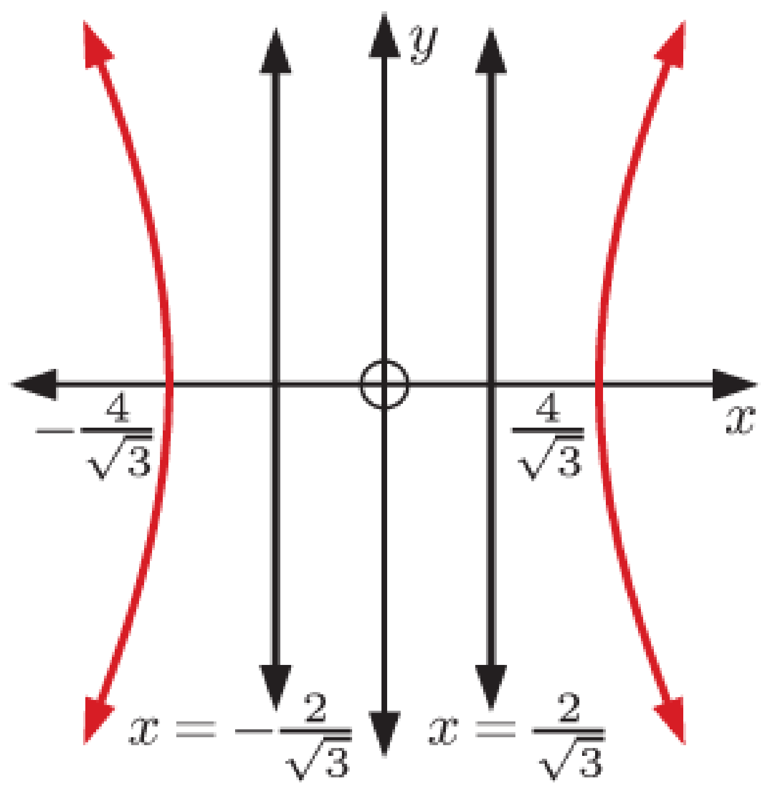
$$b^2 = a^2(e^2 - 1)$$

$$\therefore \quad b^2 = 9(16 - 1)$$

$$\therefore \quad b^2 = 135$$

$\therefore$ the equation of the hyperbola is $\dfrac{x^2}{9} - \dfrac{y^2}{135} = 1$.

d


y
$-\frac{4}{\sqrt{3}}$
$\frac{4}{\sqrt{3}}$
x
$x = -\frac{2}{\sqrt{3}}$
$x = \frac{2}{\sqrt{3}}$

$$a = \frac{4}{\sqrt{3}}$$

$$\text{and} \quad \frac{a}{e} = \frac{2}{\sqrt{3}}$$

$$\therefore \quad e = 2$$

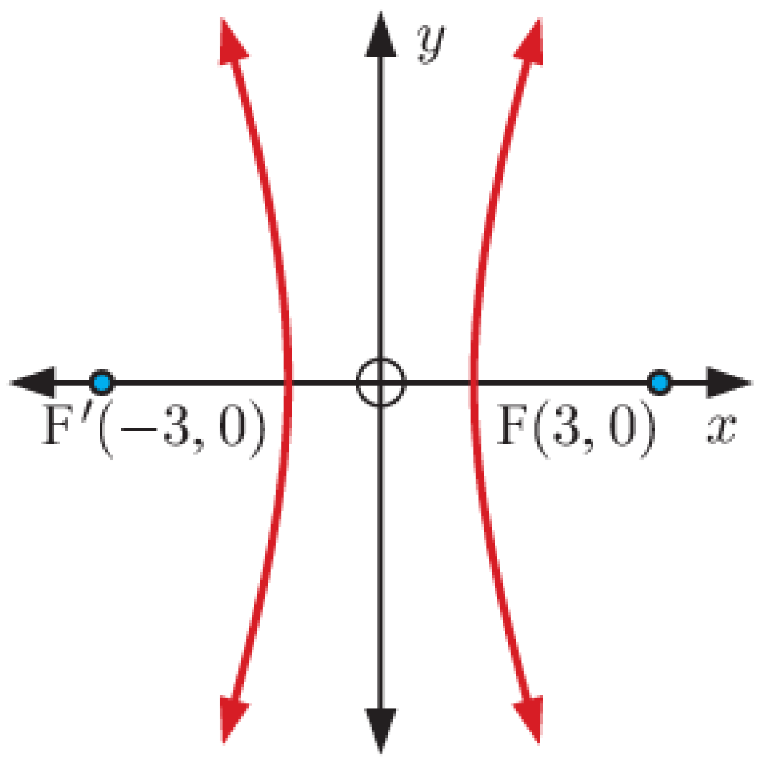
$$b^2 = a^2(e^2 - 1)$$

$$\therefore \quad b^2 = \tfrac{16}{3}(4 - 1)$$

$$\therefore \quad b^2 = 16$$

$\therefore$ the equation of the hyperbola is $\dfrac{3x^2}{16} - \dfrac{y^2}{16} = 1$.

e



$$|\mathrm{PF} - \mathrm{PF}'| = 2$$

$$\therefore \quad 2a = 2$$

$$\therefore \quad a = 1$$

$$\text{and} \quad ae = 3$$

$$\therefore \quad e = 3$$

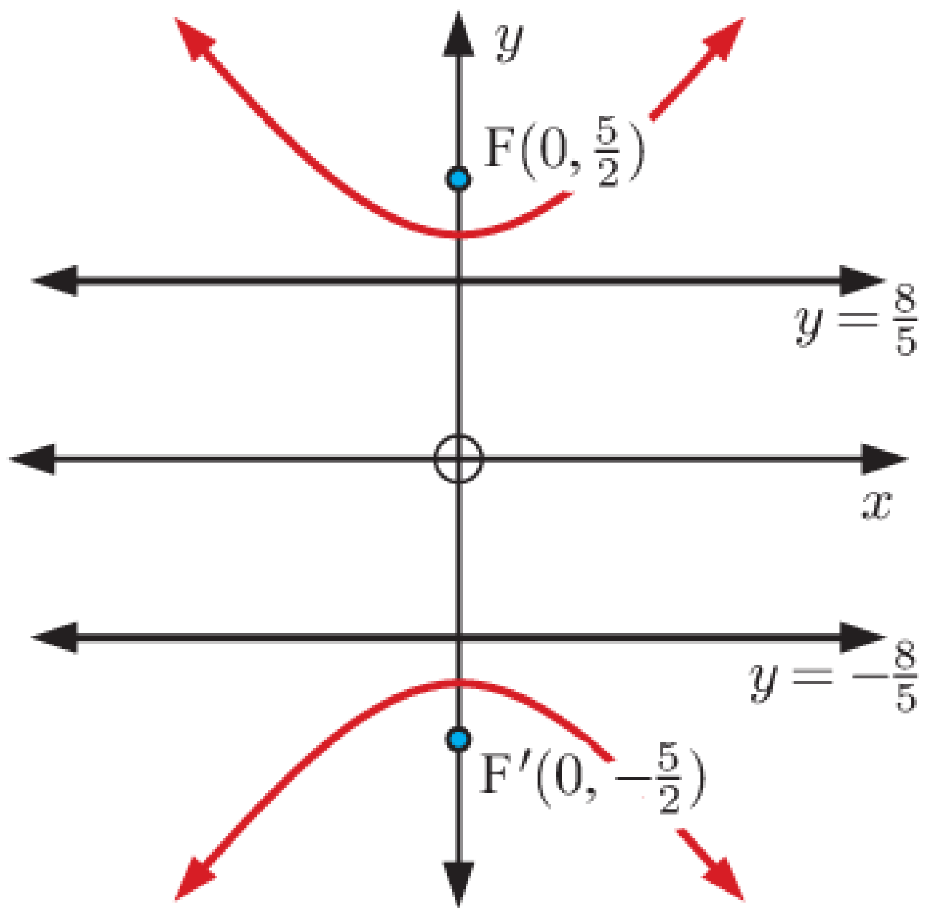
$$b^2 = a^2(e^2 - 1)$$

$$\therefore \quad b^2 = 1(9 - 1)$$

$$\therefore \quad b^2 = 8$$

$\therefore$ the equation of the hyperbola is $x^2 - \dfrac{y^2}{8} = 1$.

f


y
F(0, 5/2)
$y = \frac{8}{5}$
x
$y = -\frac{8}{5}$
F′(0, −5/2)

$$ae = \tfrac{5}{2} \quad \text{and} \quad \frac{a}{e} = \tfrac{8}{5}$$

$$\therefore \quad ae \times \frac{a}{e} = \tfrac{5}{2} \times \tfrac{8}{5}$$

$$\therefore \quad a^2 = 4$$

$$\text{and} \quad e = \tfrac{5}{4}$$

$$b^2 = a^2(e^2 - 1)$$

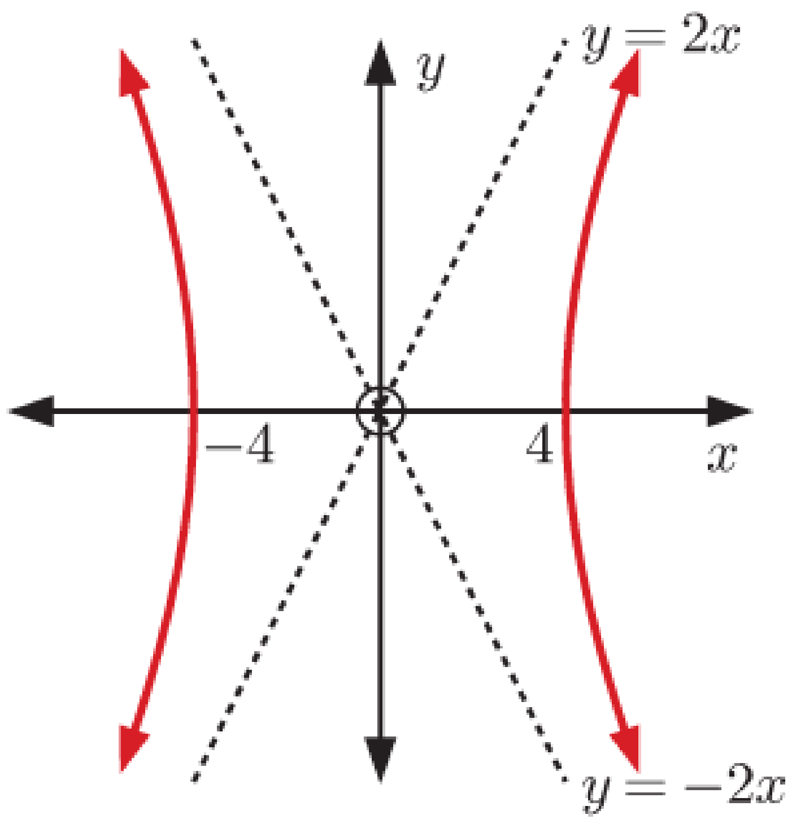
$$\therefore \quad b^2 = 4(\tfrac{25}{16} - 1)$$

$$\therefore \quad b^2 = 4(\tfrac{9}{16})$$

$$\therefore \quad b^2 = \tfrac{9}{4}$$

$\therefore$ the equation of the hyperbola is $\dfrac{y^2}{4} - \dfrac{4x^2}{9} = 1$.

g



$$a = 4$$

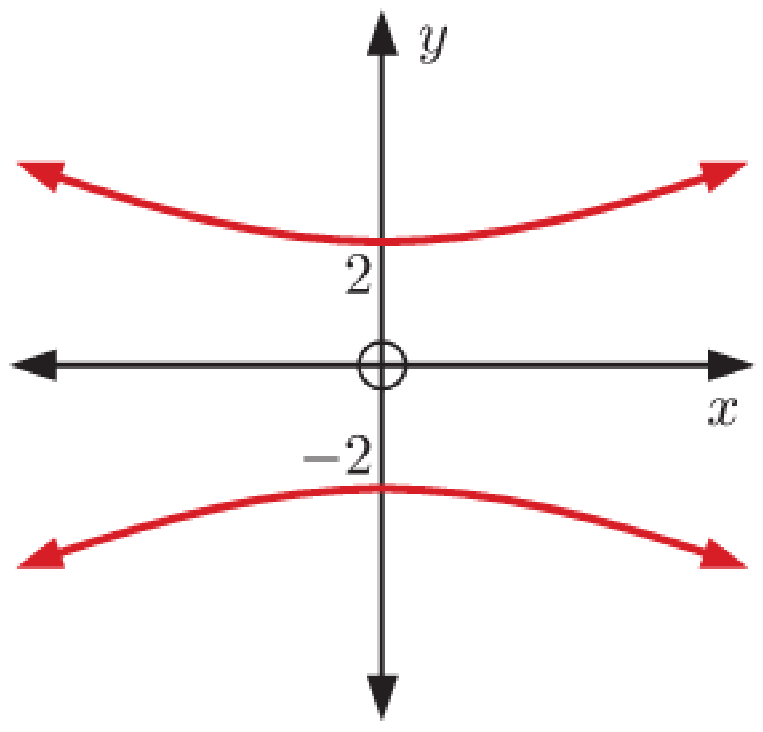
and $\pm\dfrac{b}{a}x = \pm 2x$

$$\therefore\ \frac{b}{a} = 2$$

$$\therefore\ b = 8$$

$\therefore$ the equation of the hyperbola is $\dfrac{x^2}{16} - \dfrac{y^2}{64} = 1.$

h

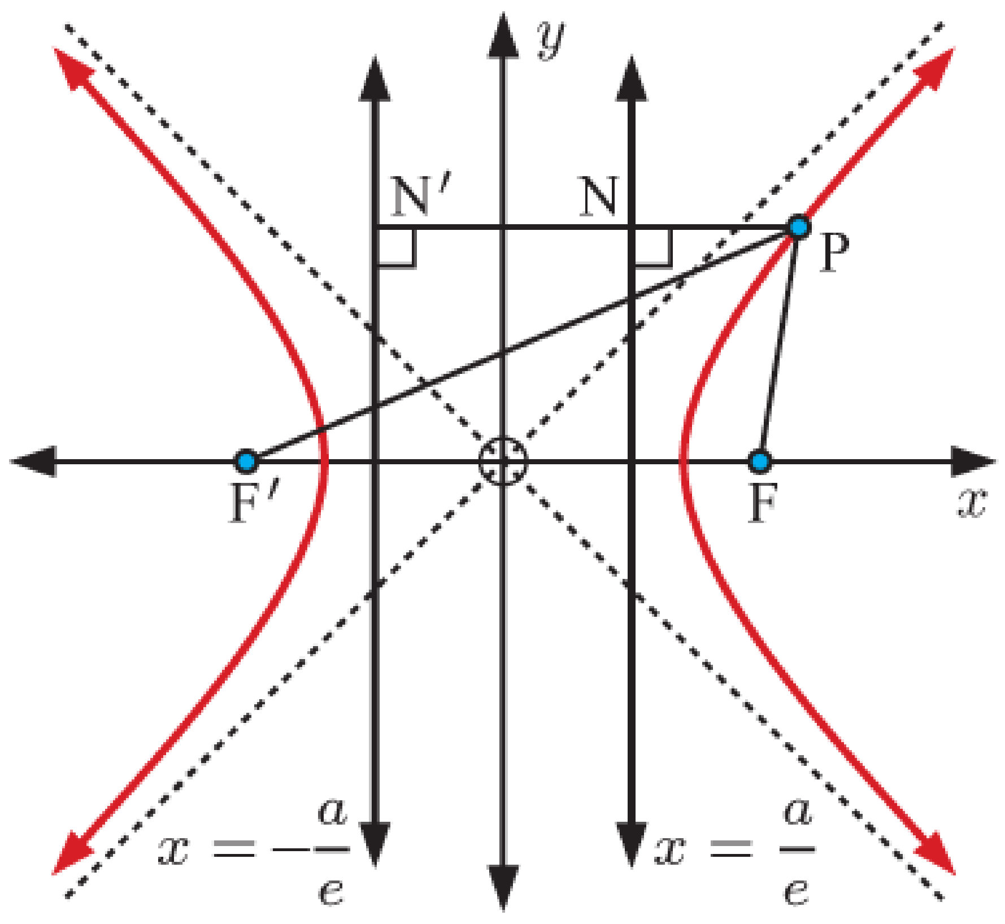


$a = 2$ and $e = \frac{5}{2}$

$$\begin{aligned} b^2 &= a^2(e^2 - 1) \\ \therefore\ b^2 &= 4(\tfrac{25}{4} - 1) \\ \therefore\ b^2 &= 25 - 4 \\ \therefore\ b^2 &= 21 \end{aligned}$$

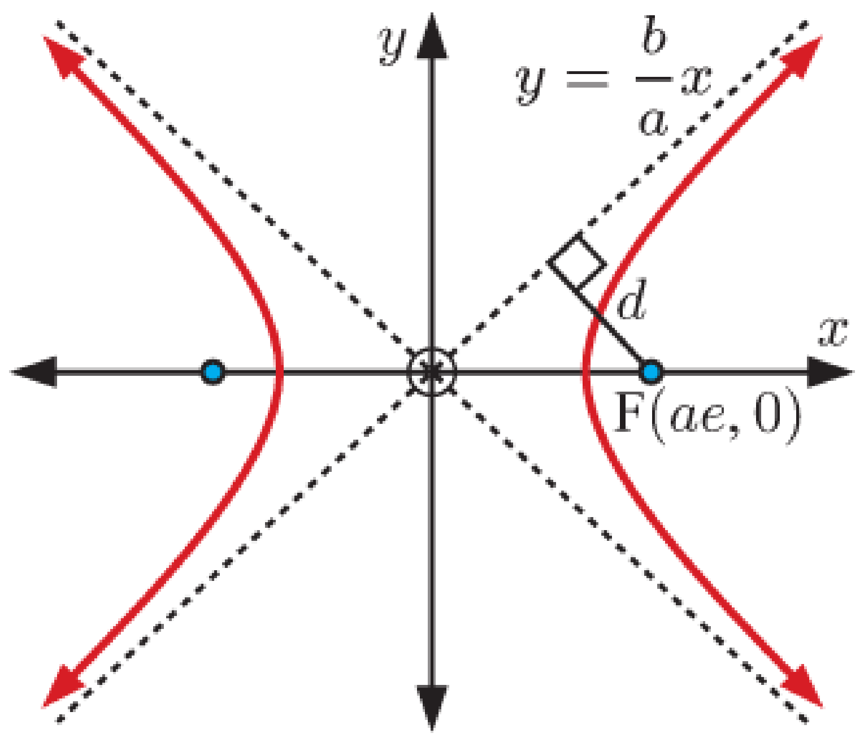
$\therefore$ the equation of the hyperbola is $\dfrac{y^2}{4} - \dfrac{x^2}{21} = 1.$

3 **a**



$$\begin{aligned} |\mathrm{PF} - \mathrm{PF'}| &= |e\,\mathrm{PN} - e\,\mathrm{PN'}| \\ &= |e(\mathrm{PN} - \mathrm{PN'})| \\ &= e\,|\mathrm{PN} - \mathrm{PN'}| \quad \{\text{as } e > 0\} \\ &= e\,|\mathrm{NN'}| \\ &= e\left(\frac{2a}{e}\right) \\ &= 2a \end{aligned}$$

b



The asymptote is

$y = \dfrac{b}{a}x$ or

$bx - ay = 0.$

$$\begin{aligned} \therefore\ d &= \frac{|b(ae) - a(0)|}{\sqrt{b^2 + a^2}} \\ &= \frac{|bae|}{\sqrt{a^2(e^2-1) + a^2}} \\ &= \frac{bae}{\sqrt{a^2e^2}} \quad \{\text{as } a,\ b, \text{ and } e > 0\} \\ &= \frac{bae}{ae} \\ &= b \end{aligned}$$

4 **a**

$$4x^2 - 9y^2 = 36$$

$$\therefore\ 8x - 18y\frac{dy}{dx} = 0 \quad \{\text{implicit differentiation}\}$$

$$\therefore\ \frac{dy}{dx} = \frac{4x}{9y}$$

At $(3, 0)$, $\dfrac{dy}{dx} = \dfrac{4(3)}{9(0)} = \dfrac{12}{0}$ which is undefined.

$\therefore$ the tangent is vertical

$\therefore$ the tangent has equation $x = 3$

and the normal has equation $y = 0$.

b At $(3\sqrt{2}, -2)$, $\dfrac{dy}{dx} = \dfrac{4(3\sqrt{2})}{9(-2)} = -\dfrac{2\sqrt{2}}{3}$

$\therefore$ the tangent has equation

$$2\sqrt{2}x + 3y = 2\sqrt{2}(3\sqrt{2}) + 3(-2)$$

which is $2\sqrt{2}x + 3y = 6$

The gradient of the normal at $(3\sqrt{2}, -2)$ is $\frac{3}{2\sqrt{2}}$.

$\therefore$ the normal has equation

$$3x - 2\sqrt{2}y = 3(3\sqrt{2}) - 2\sqrt{2}(-2)$$

which is $3x - 2\sqrt{2}y = 13\sqrt{2}$

5

$$x^2 - y^2 = 9$$

$$\therefore\ 2x - 2y\frac{dy}{dx} = 0 \quad \{\text{implicit differentiation}\}$$

$$\therefore\ \frac{dy}{dx} = \frac{x}{y}$$

When $x = 5$, $25 - y^2 = 9$

$$\therefore\ y^2 = 16$$

$$\therefore\ y = \pm 4$$

$\therefore$ the points are $(5, 4)$ and $(5, -4)$.

At $(5, 4)$, $\dfrac{dy}{dx} = \frac{5}{4}$

$\therefore$ the tangent has equation $5x - 4y = 5(5) - 4(4)$

which is $5x - 4y = 9$

The normal has gradient $-\frac{4}{5}$

$\therefore$ the normal has equation $4x + 5y = 4(5) + 5(4)$

which is $4x + 5y = 40$

At $(5, -4)$, $\dfrac{dy}{dx} = -\frac{5}{4}$

$\therefore$ the tangent has equation $5x + 4y = 5(5) + 4(-4)$

which is $5x + 4y = 9$

The normal has gradient $\frac{4}{5}$

$\therefore$ the normal has equation $4x - 5y = 4(5) - 5(-4)$

which is $4x - 5y = 40$

6 **a**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore\ \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \{\text{implicit differentiation}\}$$

$$\therefore\ \frac{y}{b^2}\frac{dy}{dx} = \frac{x}{a^2}$$

$$\therefore\ \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$\therefore$ the gradient of the normal at (x_1, y_1) is $-\dfrac{a^2y_1}{b^2x_1}$.

$\therefore$ the normal has equation

$$a^2y_1x + b^2x_1y = a^2y_1(x_1) + b^2x_1(y_1)$$

which is $a^2y_1x + b^2x_1y = (a^2 + b^2)x_1y_1$

b At (x_1, y_1), $\dfrac{dy}{dx} = \dfrac{b^2x_1}{a^2y_1}$

∴ the tangent has equation

$$b^2x_1x - a^2y_1y = b^2x_1(x_1) - a^2y_1(y_1)$$

which is $b^2x_1x - a^2y_1y = b^2{x_1}^2 - a^2{y_1}^2$

c The asymptote with positive gradient is $y = \dfrac{b}{a}x$.

The tangent meets the asymptote when

$$b^2x_1x - a^2y_1\left(\frac{b}{a}x\right) = b^2{x_1}^2 - a^2{y_1}^2$$

$$\therefore\ b^2x_1x - aby_1x = (bx_1+ay_1)(bx_1-ay_1)$$

$$\therefore\ bx\cancel{(bx_1-ay_1)} = (bx_1+ay_1)\cancel{(bx_1-ay_1)}$$

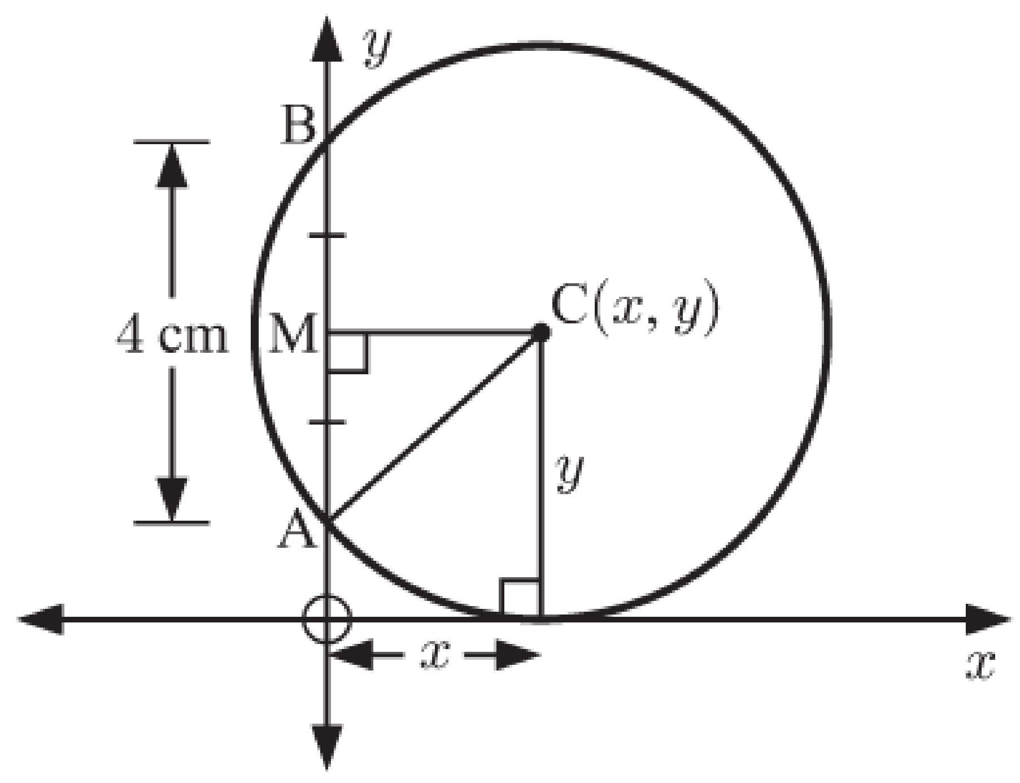
$$\therefore\ x = \frac{bx_1+ay_1}{b} \quad \{bx_1 - ay_1 \neq 0\}$$

$$\therefore\ y = \frac{b}{a}\left(\frac{bx_1+ay_1}{b_1}\right)$$

$$\therefore\ y = \frac{bx_1+ay_1}{a}$$

∴ T is $\left(\dfrac{bx_1+ay_1}{b}, \dfrac{bx_1+ay_1}{a}\right)$.

7


y
B
4 cm
M
C(x, y)
y
A
x
x

Let the centre of one of the circles be $C(x, y)$, and let M be the midpoint of [AB].

Now $CM = x$ and $CA = y$ {radius of a circle}

∴ $y^2 = x^2 + 2^2$ {Pythagoras}

∴ the locus of C is $y^2 - x^2 = 4$

which is $\dfrac{y^2}{4} - \dfrac{x^2}{4} = 1$

which is the equation of a rectangular hyperbola.

∴ all centres lie on a rectangular hyperbola.

EXERCISE 2N.5

1 a $\dfrac{(x-1)^2}{16} + \dfrac{(y+3)^2}{9} = 1$ comes from $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

under the translation $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

Now $a^2 = 16$ and $b^2 = 9$

$b^2 = a^2(1-e^2)$

∴ $9 = 16(1-e^2)$

∴ $1 - e^2 = \frac{9}{16}$

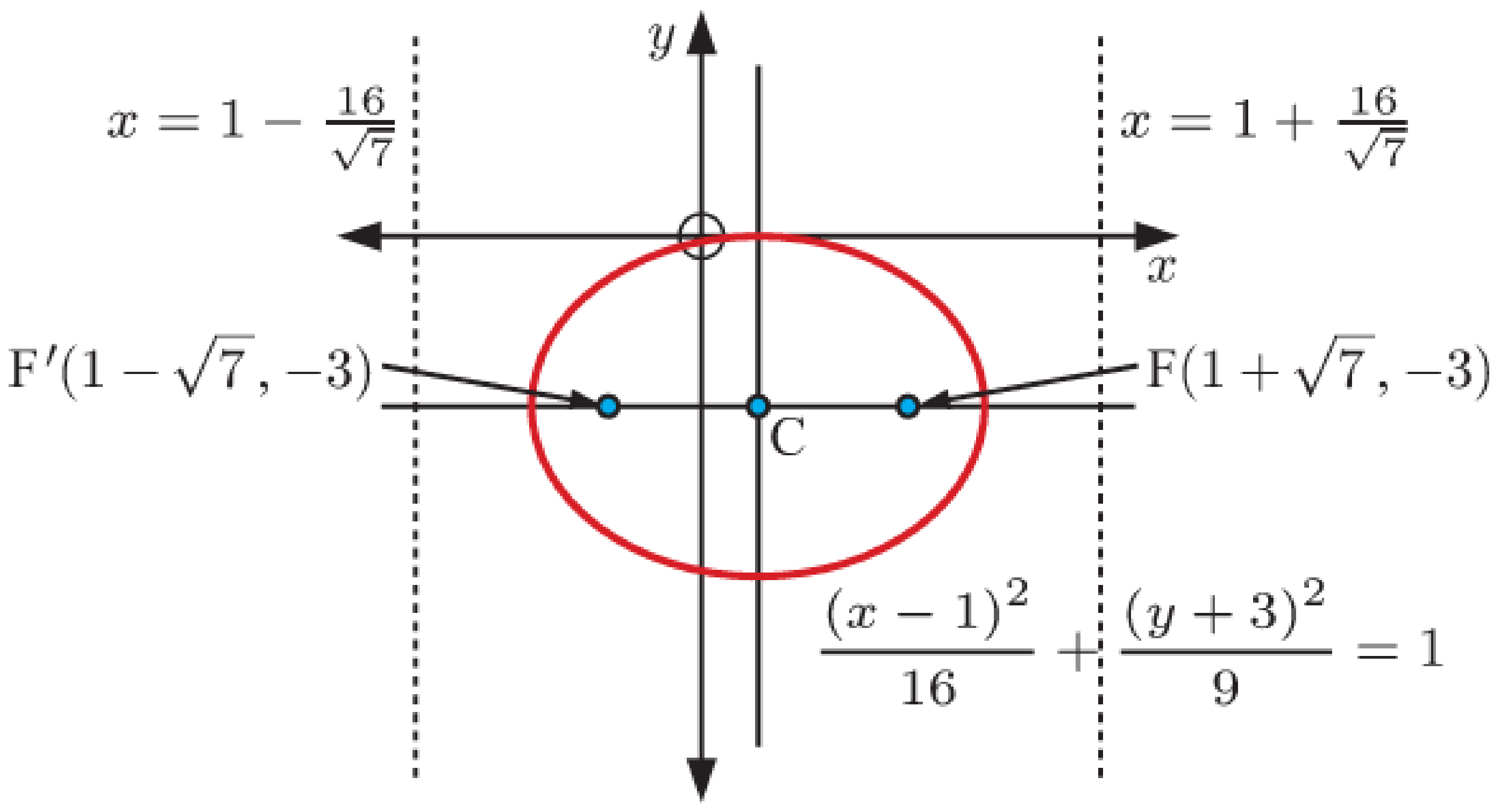
∴ $e^2 = \frac{7}{16}$

∴ $e = \frac{\sqrt{7}}{4}$ {as $e > 0$}

Now $ae = \sqrt{7}$ and $\dfrac{a}{e} = \frac{16}{\sqrt{7}}$

∴ $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ has foci $(\pm\sqrt{7}, 0)$ and directrices $x = \pm\frac{16}{\sqrt{7}}$.

Hence $\dfrac{(x-1)^2}{16} + \dfrac{(y+3)^2}{9} = 1$ has foci $(1 \pm \sqrt{7}, -3)$ and directrices $x = 1 \pm \frac{16}{\sqrt{7}}$.


$x = 1 - \frac{16}{\sqrt{7}}$
y
$x = 1 + \frac{16}{\sqrt{7}}$
x
$F'(1-\sqrt{7}, -3)$
$F(1+\sqrt{7}, -3)$
C
$\frac{(x-1)^2}{16} + \frac{(y+3)^2}{9} = 1$

b $(y+4)^2 = -8(x+2)$ comes from $y^2 = -8x$ under the translation $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$.

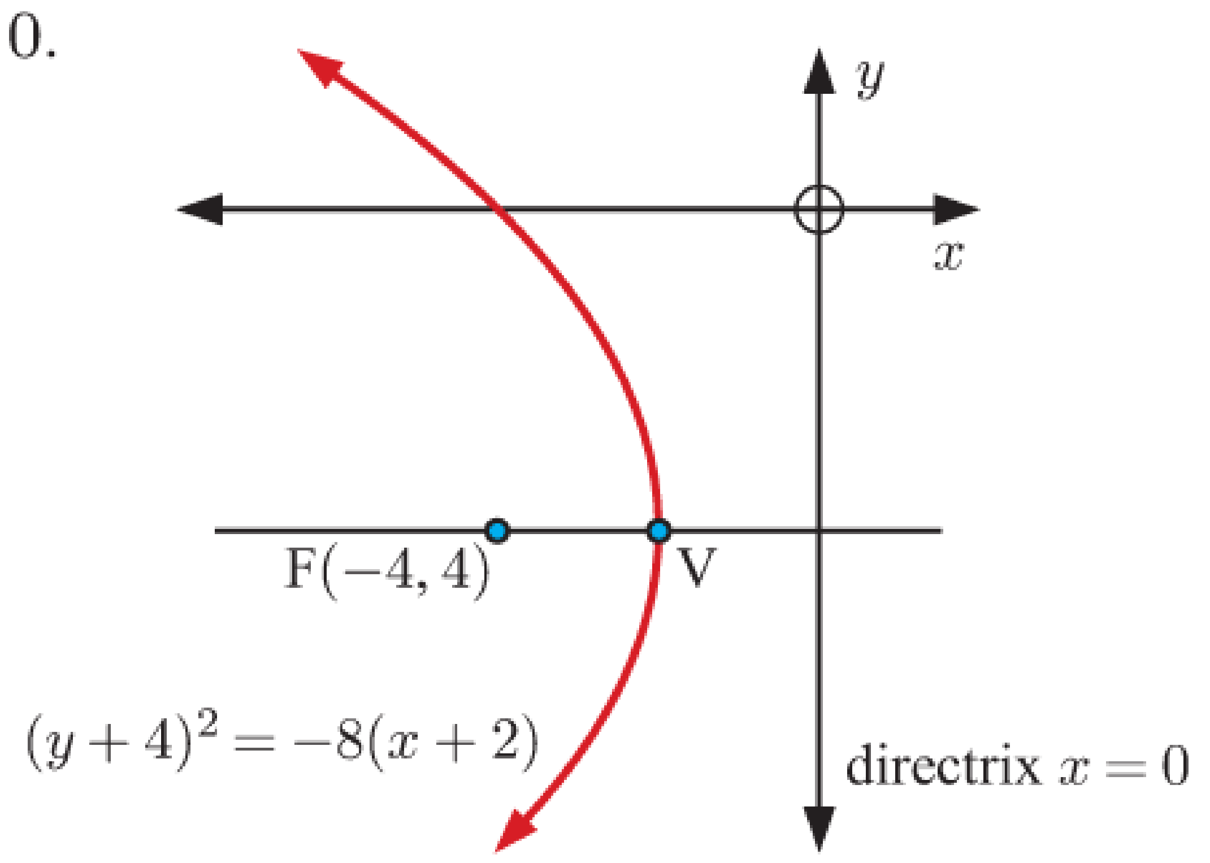
$y^2 = 4ax$

∴ $4a = -8$

∴ $a = -2$

∴ $y^2 = -8x$ has focus $(-2, 0)$ and directrix $x = 2$.

Hence $(y+4)^2 = -8(x+2)$ has focus $(-4, -4)$ and directrix $x = 0$.


y
x
F(−4, 4)
V
$(y+4)^2 = -8(x+2)$
directrix $x = 0$

c $(x+2)^2 - \dfrac{(y-1)^2}{4} = 1$ comes from $x^2 - \dfrac{y^2}{4} = 1$

under the translation $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

$a^2 = 1$ and $b^2 = 4$

$b^2 = a^2(e^2-1)$

∴ $4 = 1(e^2-1)$

∴ $e^2 - 1 = 4$

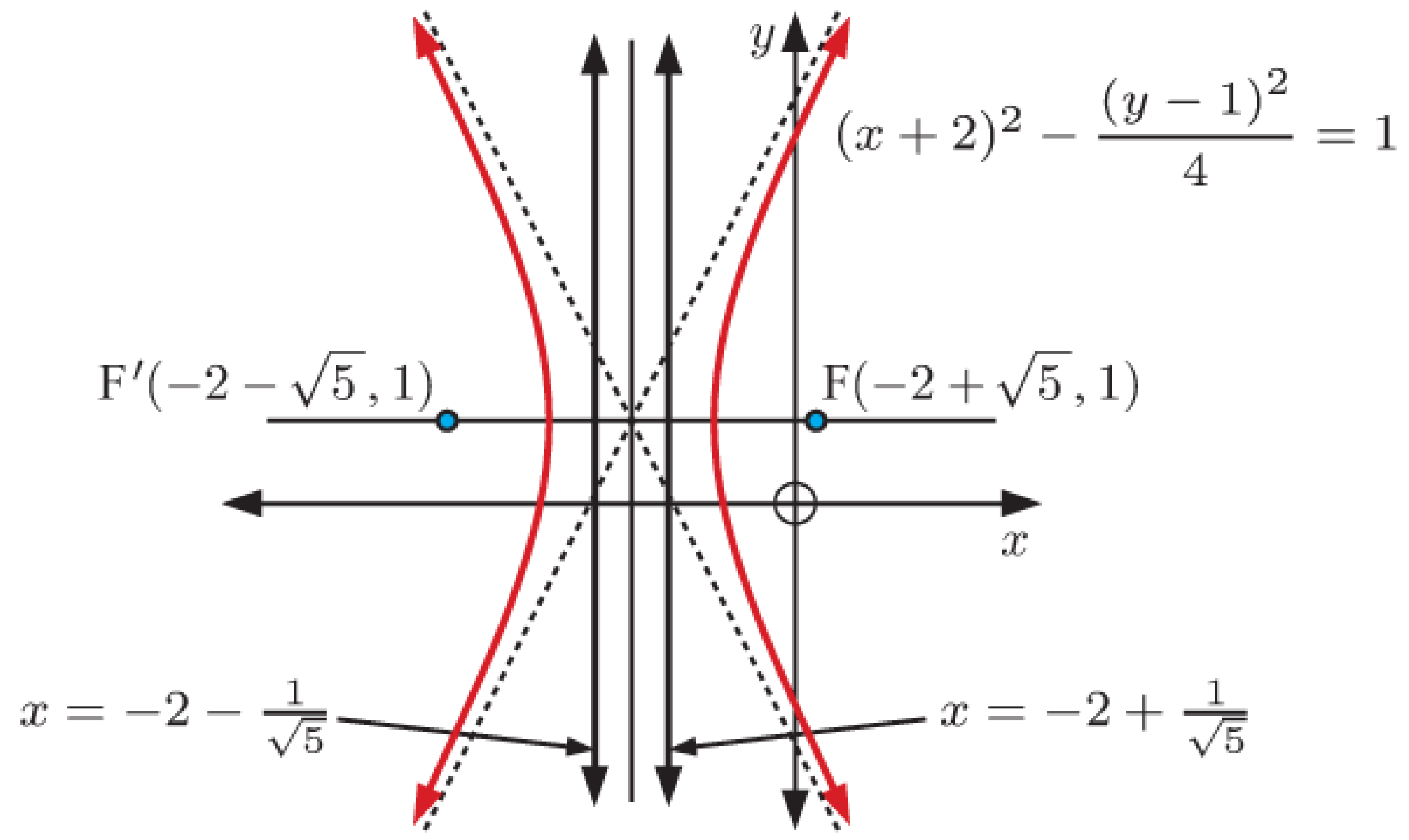
∴ $e^2 = 5$

∴ $e = \sqrt{5}$ {as $e > 0$}

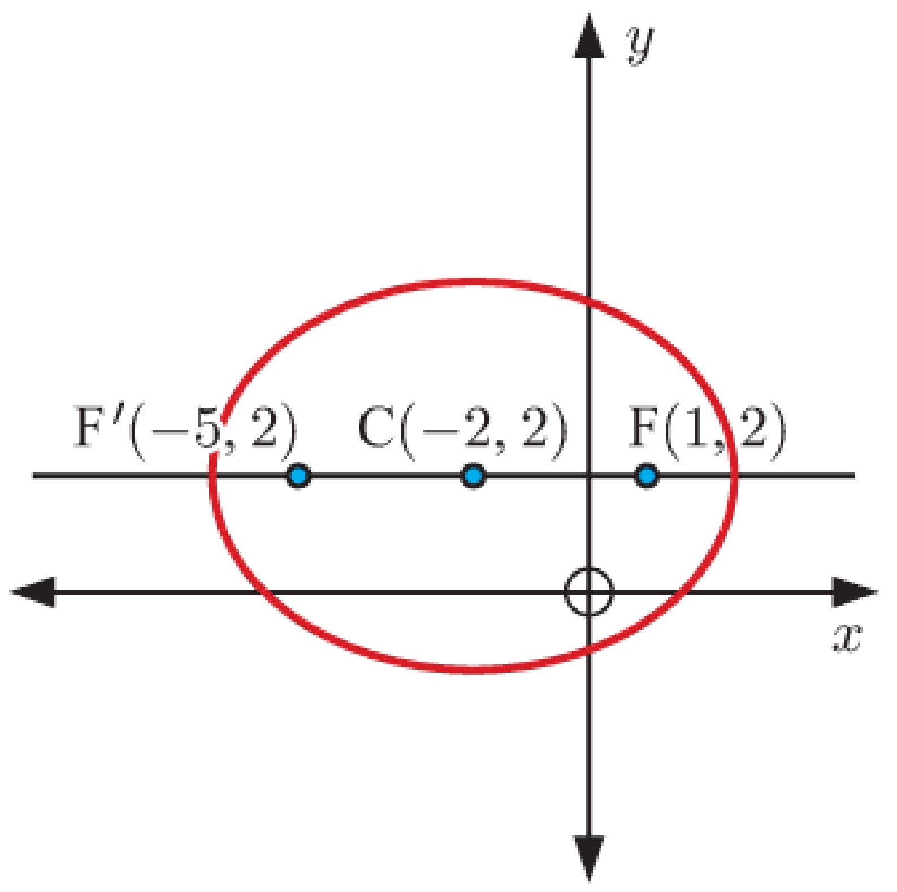
∴ $ae = \sqrt{5}$ and $\dfrac{a}{e} = \frac{1}{\sqrt{5}}$

∴ $x^2 - \dfrac{y^2}{4} = 1$ has foci $(\pm\sqrt{5}, 0)$ and directrices $x = \pm\frac{1}{\sqrt{5}}$.

Hence $(x+2)^2 - \dfrac{(y-1)^2}{4} = 1$ has foci $(-2 \pm \sqrt{5}, 1)$ and directrices $x = -2 \pm \frac{1}{\sqrt{5}}$.


y
$(x+2)^2 - \frac{(y-1)^2}{4} = 1$
$F'(-2-\sqrt{5}, 1)$
$F(-2+\sqrt{5}, 1)$
x
$x = -2 - \frac{1}{\sqrt{5}}$
$x = -2 + \frac{1}{\sqrt{5}}$

2 **a**



C is $\left(\frac{-5+1}{2}, \frac{2+2}{2}\right)$

$\therefore$ C is $(-2, 2)$

$\text{FF}' = 2ae = 6$

$\therefore \quad ae = 3$

and $\quad e = \frac{2}{3}$

$\therefore \quad a = \frac{9}{2}$

$b^2 = a^2(1 - e^2)$

$\therefore \quad b^2 = \frac{81}{4}(1 - \frac{4}{9})$

$\therefore \quad b^2 = \frac{81}{4} \times \frac{5}{9}$

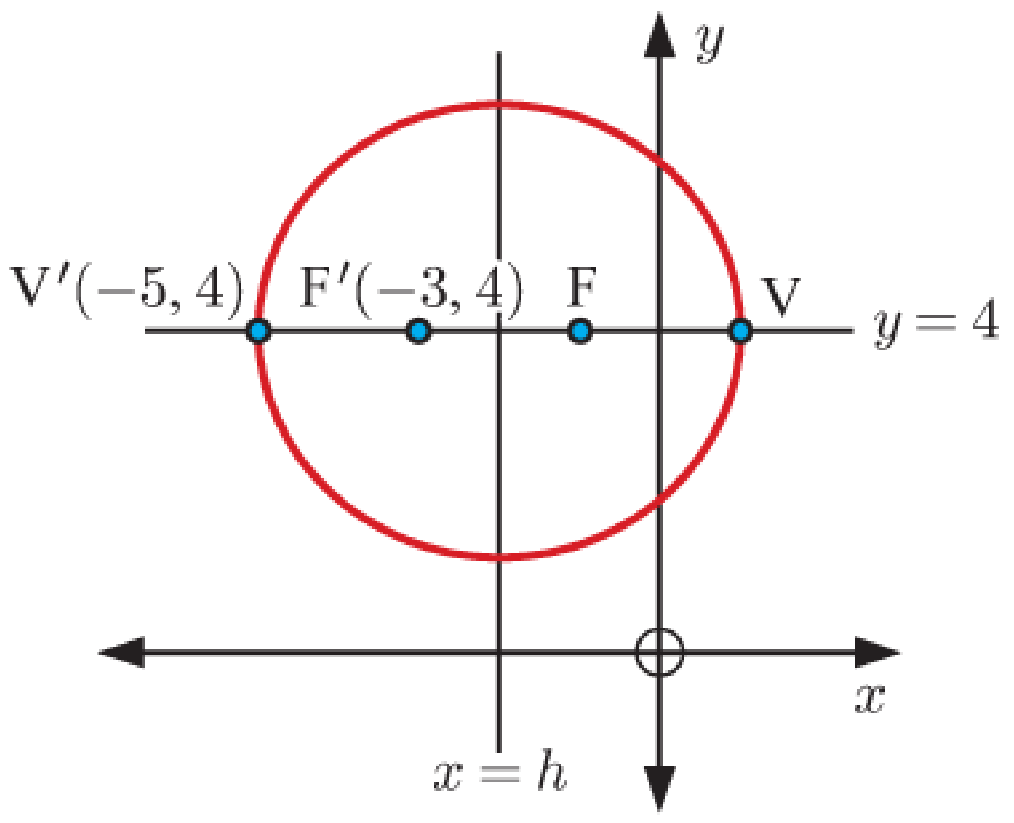
$\therefore \quad b^2 = \frac{45}{4}$

$\therefore$ the ellipse comes from $\dfrac{4x^2}{81} + \dfrac{4y^2}{45} = 1$

under the translation $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$, which is

$$\frac{4(x+2)^2}{81} + \frac{4(y-2)^2}{45} = 1.$$

b



Let the minor axis be $x = h$. $\quad e = \frac{1}{3}$

$\therefore \quad h - (-5) = a \quad$ and $\quad h - (-3) = ae$

$\therefore \quad h + 5 = a \quad$ and $\quad h + 3 = ae = \frac{1}{3}a$

$\therefore \quad h = a - 5 \quad$ and $\quad h = \frac{1}{3}a - 3$

$\therefore \quad a - 5 = \frac{1}{3}a - 3$

$\therefore \quad 3a - 15 = a - 9$

$\therefore \quad 2a = 6$

$\therefore \quad a = 3 \quad$ and $\quad h = -2$

$b^2 = a^2(1 - e^2)$

$\therefore \quad b^2 = 9(1 - \frac{1}{9})$

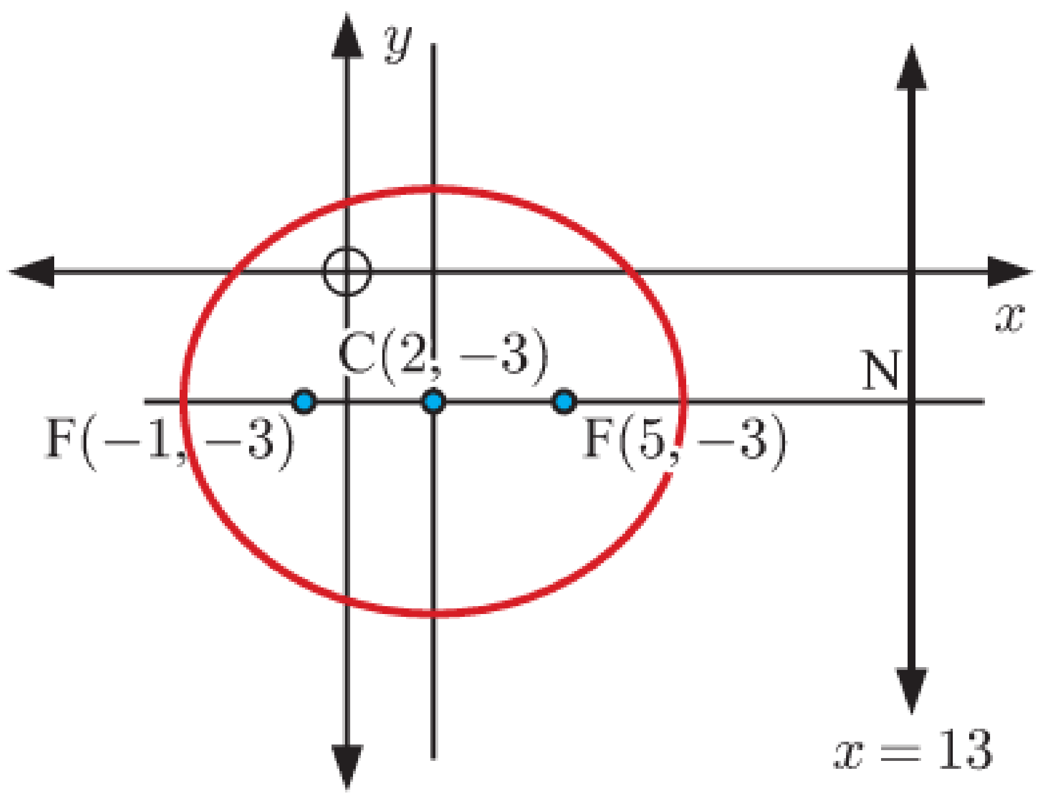
$\therefore \quad b^2 = 8$

$\therefore$ the ellipse comes from $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$

under the translation $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$, which is

$$\frac{(x+2)^2}{9} + \frac{(y-4)^2}{8} = 1.$$

c



C is $\left(\frac{-1+5}{2}, -3\right) \quad \therefore \quad$ C is $(2, -3)$

$\text{FF}' = 2ae = 6 \quad$ and $\quad \text{CN} = \dfrac{a}{e} = 11$

$\therefore \quad ae = 3$

$\therefore \quad ae \times \dfrac{a}{e} = 3 \times 11$

$\therefore \quad a^2 = 33 \quad$ and $\quad e = \frac{3}{\sqrt{33}}$

$b^2 = a^2(1 - e^2)$

$\therefore \quad b^2 = 33(1 - \frac{9}{33})$

$\therefore \quad b^2 = 33 - 9$

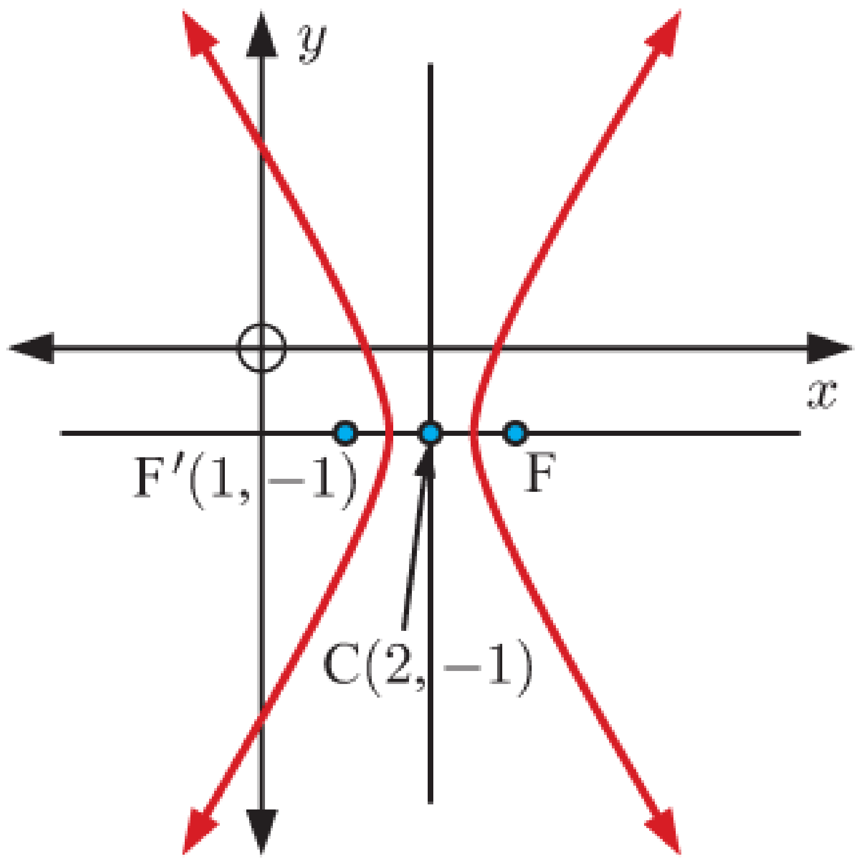
$\therefore \quad b^2 = 24$

$\therefore$ the ellipse comes from $\dfrac{x^2}{33} + \dfrac{y^2}{24} = 1$

under the translation $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, which is

$$\frac{(x-2)^2}{33} + \frac{(y+3)^2}{24} = 1.$$

3 **a**



$\text{CF} = ae = 1 \quad$ and $\quad e = 2$

$\therefore \quad a = \frac{1}{2}$

$b^2 = a^2(e^2 - 1)$

$\therefore \quad b^2 = \frac{1}{4}(4 - 1)$

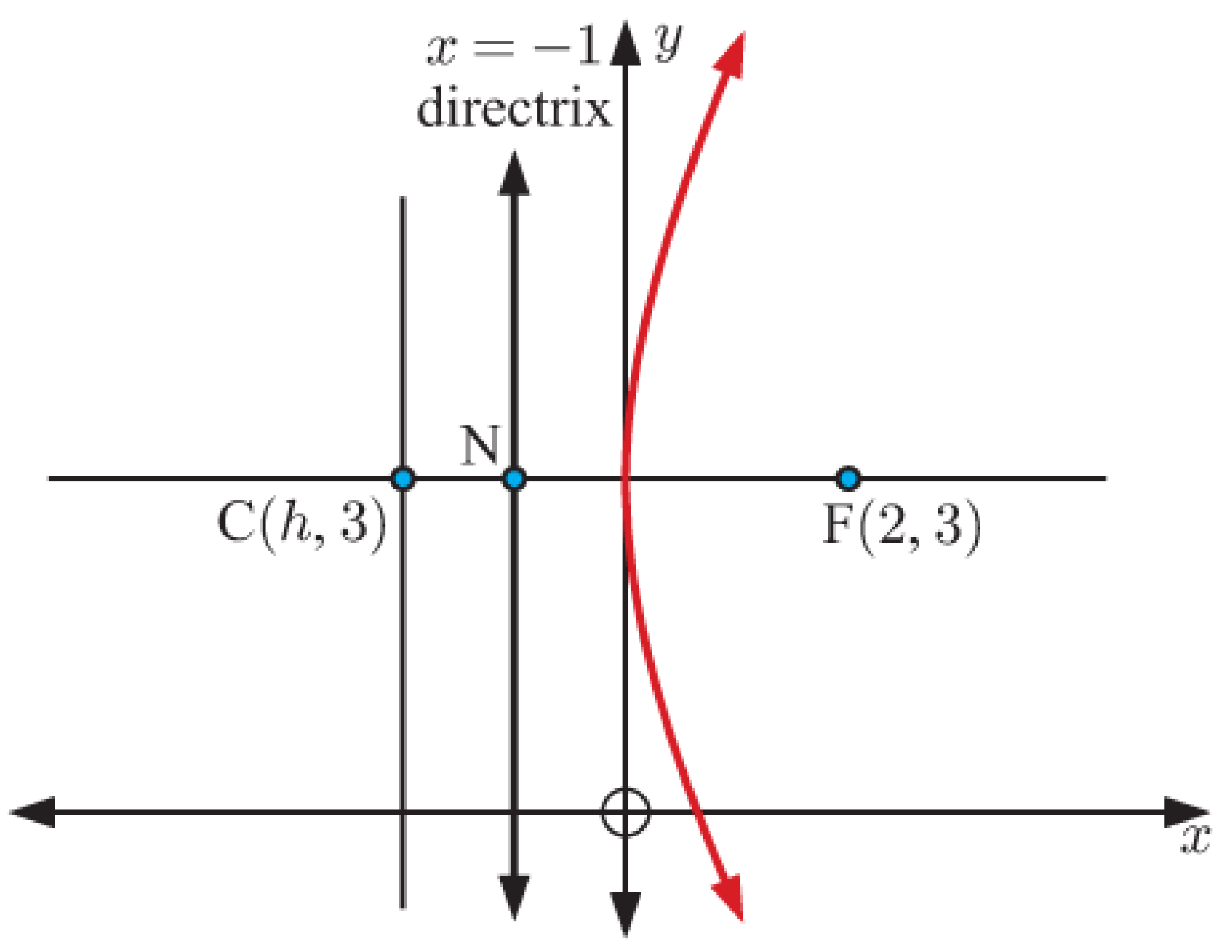
$\therefore \quad b^2 = \frac{3}{4}$

$\therefore$ the hyperbola comes from $\dfrac{x^2}{\frac{1}{4}} - \dfrac{y^2}{\frac{3}{4}} = 1$

under the translation $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, which is

$$4(x-2)^2 - \frac{4(y+1)^2}{3} = 1.$$

b



Let C be the point $(h, 3)$.

$e = 2$

$\text{CF} = ae = 2 - h \quad$ and $\quad \text{CN} = \dfrac{a}{e} = -1 - h$

$\therefore \quad 2a = 2 - h \quad$ and $\quad \dfrac{a}{2} = -1 - h$

$\therefore \quad h = 2 - 2a \quad$ and $\quad h = -1 - \dfrac{a}{2}$

$\therefore\ 2-2a=-1-\frac{a}{2}$

$\therefore\ \frac{3a}{2}=3$

$\therefore\ a=2$ and $h=-2$

$b^2=a^2(e^2-1)$

$\therefore\ b^2=4(4-1)$

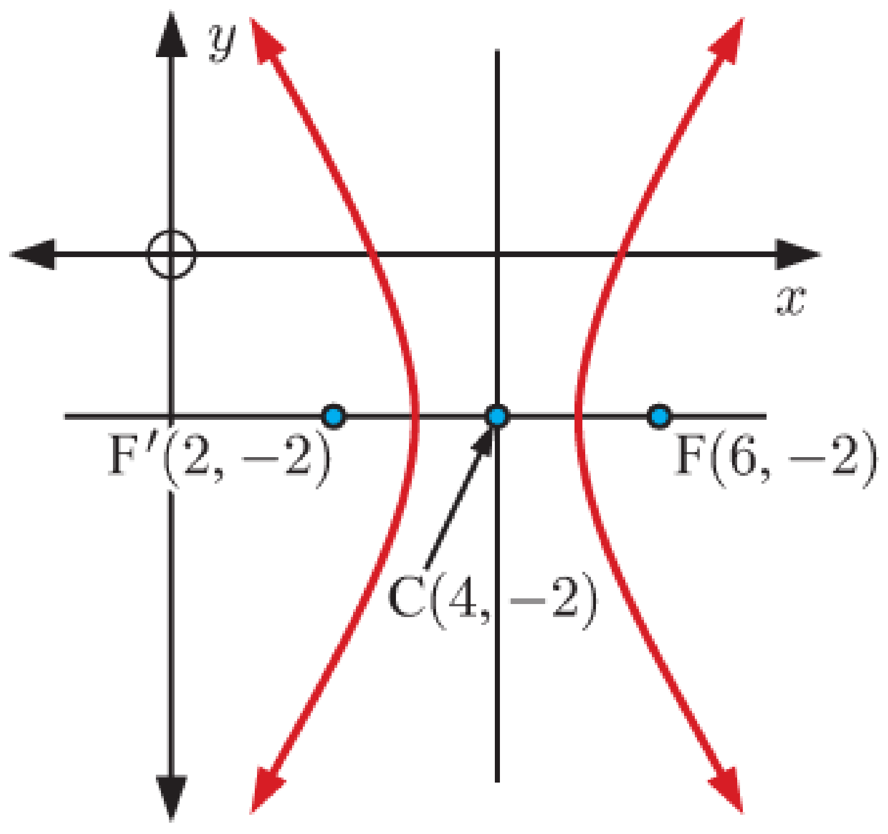
$\therefore\ b^2=12$

$\therefore$ the hyperbola comes from $\frac{x^2}{4}-\frac{y^2}{12}=1$

under the translation $\begin{pmatrix}-2\\3\end{pmatrix}$, which is

$$\frac{(x+2)^2}{4}-\frac{(y-3)^2}{12}=1.$$

c



C is $(4,-2)$. Let $P(x,y)$ be a point on the hyperbola.

$|\text{PF}-\text{PF}'|=2a$ {focal distance property}

But $|\text{PF}-\text{PF}'|=2$

$\therefore\ a=1$

$\text{FF}'=2ae=4$

$\therefore\ e=2$

$b^2=a^2(e^2-1)$

$\therefore\ b^2=1(4-1)$

$\therefore\ b^2=3$

$\therefore$ the hyperbola comes from $x^2-\frac{y^2}{3}=1$

under the translation $\begin{pmatrix}4\\-2\end{pmatrix}$, which is

$$(x-4)^2-\frac{(y+2)^2}{3}=1.$$

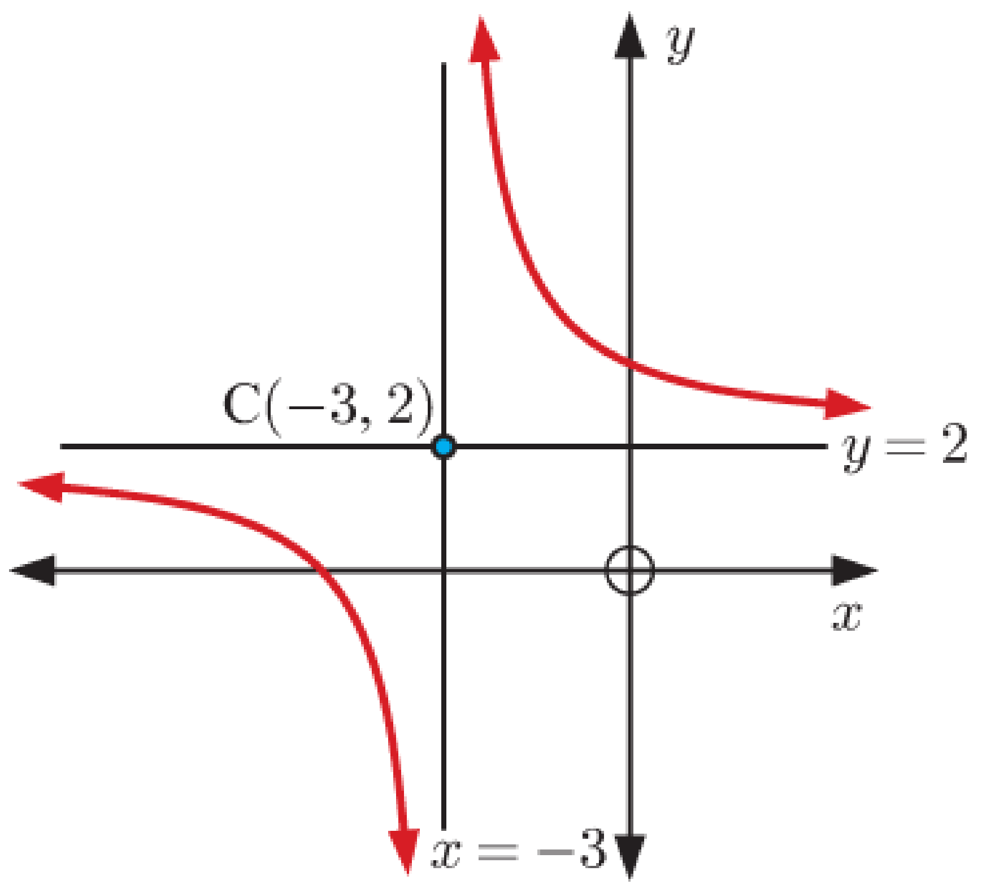
4 **a** $xy-2x+3y-10=0$

$\therefore\ (x+3)(y-2)=10-6$

$\therefore\ (x+3)(y-2)=4$

b The curve comes from the rectangular hyperbola $xy=4$ under the translation $\begin{pmatrix}-3\\2\end{pmatrix}$.

$\therefore$ the curve is a rectangular hyperbola.



c When $y=0$, $-2x=10$

$\therefore\ x=-5$

$\therefore$ the x-intercept is -5. ✓

When $x=0$, $3y=10$

$\therefore\ y=3\frac{1}{3}$

$\therefore$ the y-intercept is $3\frac{1}{3}$. ✓

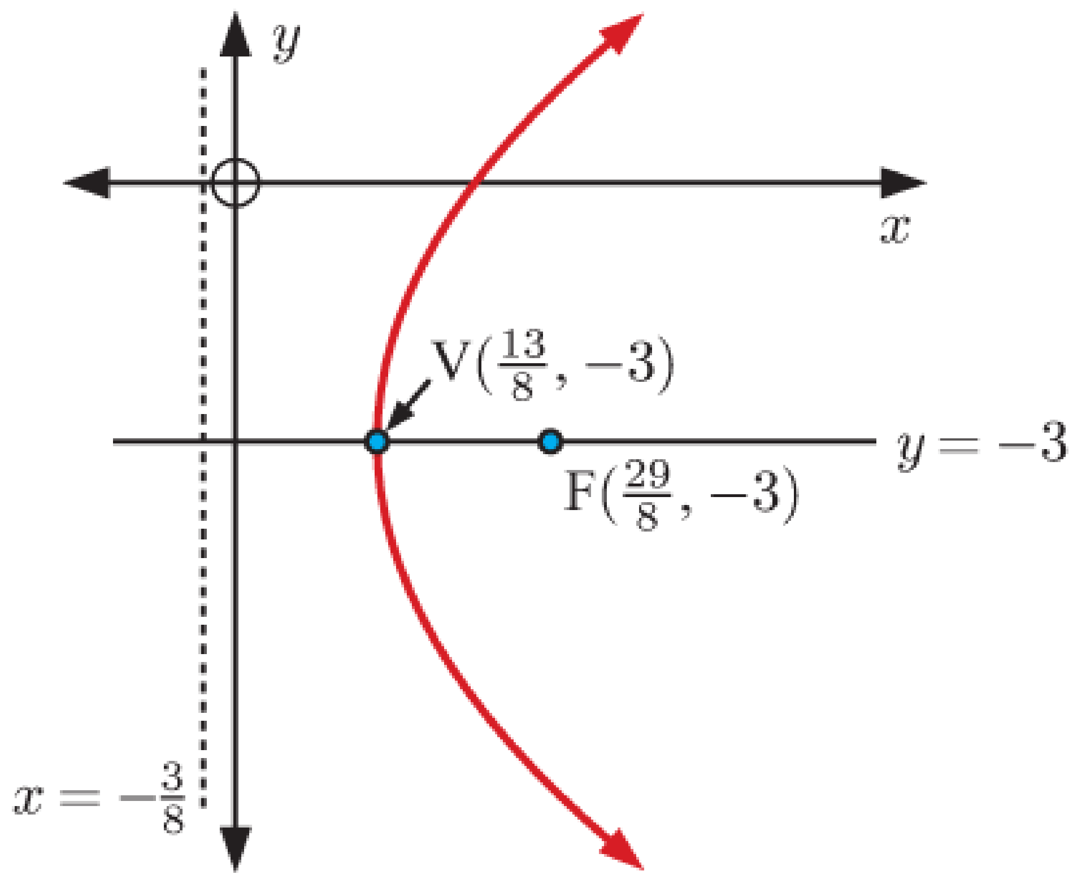
5 **a** $y^2-8x+6y+22=0$

$\therefore\ y^2+6y+3^2=8x-22+9$

$\therefore\ (y+3)^2=8x-13$

$\therefore\ (y+3)^2=8(x-\frac{13}{8})$

b The curve comes from the parabola $y^2=8x$ under the translation $\begin{pmatrix}\frac{13}{8}\\-3\end{pmatrix}$. $\therefore$ the curve is a parabola.



c When $y=0$, $8x=22$

$\therefore\ x=\frac{11}{4}$

$\therefore$ the x-intercept is $2\frac{3}{4}$.

When $x=0$, $y^2+6y+22=0$

with $\Delta=(6)^2-4(1)(22)=-52$

$\therefore\ \Delta<0$

$\therefore$ the graph does not cut the y-axis.

d $y^2=8x$ has focus $(2,0)$ and directrix $x=-2$

$\therefore\ (y+3)^2=8(x-\frac{13}{8})$ has focus $(\frac{29}{8},-3)$ and directrix $x=-\frac{3}{8}$.

6 **a** **i** $x^2+4y^2-6x+32y+69=0$

$\therefore\ x^2-6x+3^2+4(y^2+8y+4^2)=-69+9+64$

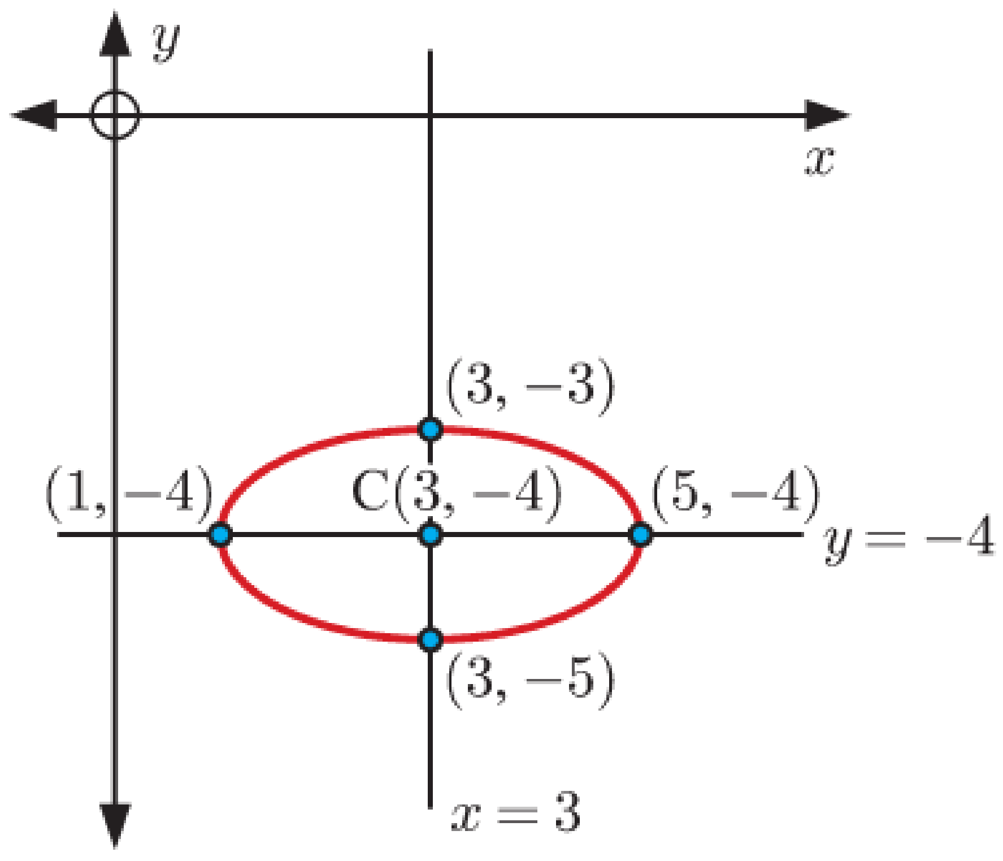
$\therefore\ (x-3)^2+4(y+4)^2=4$

$\therefore\ \frac{(x-3)^2}{4}+(y+4)^2=1$

The curve comes from the ellipse $\frac{x^2}{4}+y^2=1$ under the translation $\begin{pmatrix}3\\-4\end{pmatrix}$.

$\therefore$ the curve is an ellipse.

ii



iii $a^2 = 4$ and $b^2 = 1$

$b^2 = a^2(1-e^2)$

∴ $1 = 4(1-e^2)$

∴ $1 - e^2 = \frac{1}{4}$

∴ $e^2 = \frac{3}{4}$

∴ $e = \frac{\sqrt{3}}{2}$ {as $e > 0$}

∴ $ae = \sqrt{3}$ and $\frac{a}{e} = \frac{4}{\sqrt{3}}$

∴ $\frac{x^2}{4} + y^2 = 1$ has foci $(\pm\sqrt{3}, 0)$ and directrices $x = \pm\frac{4}{\sqrt{3}}$.

Hence $\frac{(x-3)^2}{4} + (y+4)^2 = 1$ has foci $(3 \pm \sqrt{3}, -4)$ and directrices $x = -4 \pm \frac{4}{\sqrt{3}}$.

b **i** $4x^2 - 9y^2 + 16x + 18y = 9$

∴ $4(x^2 + 4x + 2^2) - 9(y^2 - 2y + 1^2) = 9 + 16 - 9$

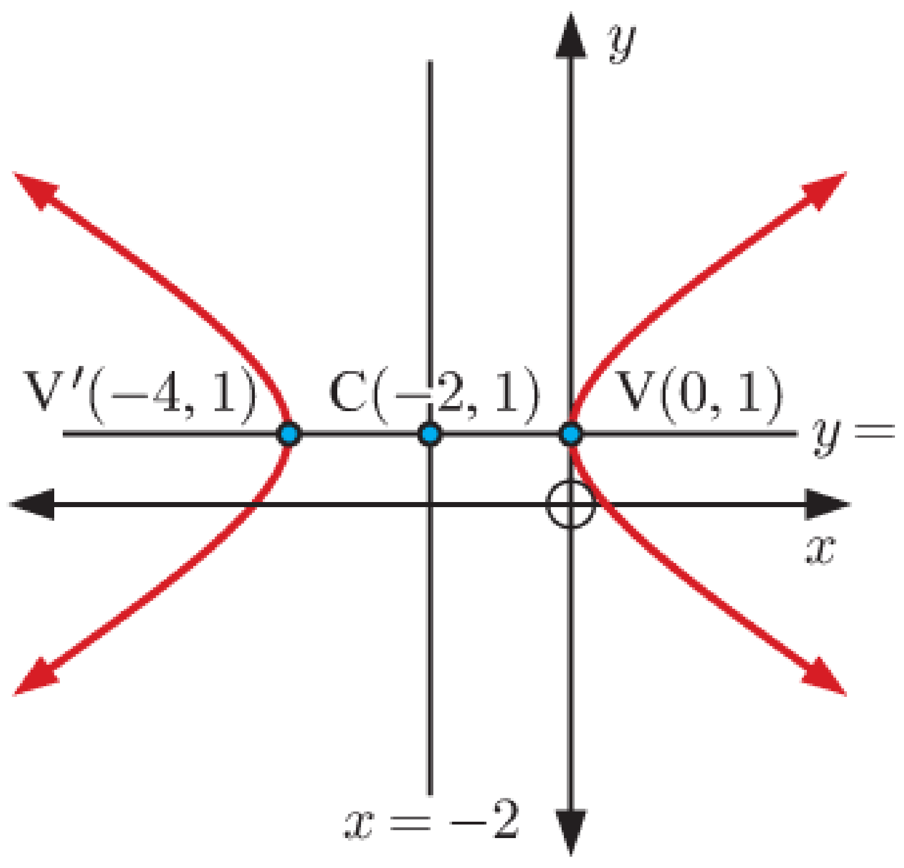
∴ $4(x+2)^2 - 9(y-1)^2 = 16$

∴ $\frac{(x+2)^2}{4} - \frac{9(y-1)^2}{16} = 1$

The curve comes from the hyperbola $\frac{x^2}{4} - \frac{9y^2}{16} = 1$ under the translation $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

∴ the curve is a hyperbola.

ii



iii $a^2 = 4$ and $b^2 = \frac{16}{9}$

$b^2 = a^2(e^2 - 1)$

∴ $\frac{16}{9} = 4(e^2 - 1)$

∴ $e^2 - 1 = \frac{4}{9}$

∴ $e^2 = \frac{13}{9}$

∴ $e = \frac{\sqrt{13}}{3}$ {as $e > 0$}

∴ $ae = \frac{2\sqrt{13}}{3}$ and $\frac{a}{e} = \frac{6}{\sqrt{13}}$

∴ $\frac{x^2}{4} - \frac{9y^2}{16} = 1$ has foci $(\pm\frac{2\sqrt{13}}{3}, 0)$ and directrices $x = \pm\frac{6}{\sqrt{13}}$.

Hence $\frac{(x+2)^2}{4} - \frac{9(y-1)^2}{16} = 1$ has foci $(-2 \pm \frac{2\sqrt{13}}{3}, 1)$ and directrices $x = -2 \pm \frac{6}{\sqrt{13}}$.

7 $3x^2 + y^2 - 6x - 4y + 40 = 0$

∴ $3(x^2 - 2x + 1^2) + y^2 - 4y + 2^2 = -40 + 3 + 4$

∴ $3(x-1)^2 + (y-2)^2 = -33$

which is not possible as the LHS is always positive.

∴ the equation does not have a graph.

EXERCISE 20

1 **a** Since $x = t$ and $y = \frac{9}{t}$ we can eliminate t by multiplying.

$xy = t\left(\frac{9}{t}\right)$ $\{t \neq 0\}$

∴ $xy = 9$

b If $x = t$ then $y = 1 - 5t = 1 - 5x$

∴ $y = 1 - 5x$

c $x = 1 + 2t$, $y = 3 - t$

∴ $t = 3 - y$

∴ $x = 1 + 2(3 - y)$

∴ $x = 1 + 6 - 2y$

∴ $x + 2y = 7$

d If $x = t$ then $y = t^2 - 1 = x^2 - 1$

∴ $y = x^2 - 1$

e $x = t^2$, $y = t^3$

∴ $x^3 = (t^2)^3 = t^6$

and $y^2 = (t^3)^2 = t^6$

∴ $y^2 = x^3$

f $x = t^2$, $y = 4t$

∴ $t = \frac{y}{4}$

∴ $x = \left(\frac{y}{4}\right)^2$

∴ $x = \frac{y^2}{16}$

∴ $y^2 = 16x$

2 **a** $x = 2\cos\theta$, $y = 3\sin\theta$

$\cos^2\theta + \sin^2\theta = 1$, ∴ $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

∴ $\frac{x^2}{4} + \frac{y^2}{9} = 1$

b $x = 2 + \cos\theta$, $y = \sin\theta$

$\cos^2\theta + \sin^2\theta = 1$, ∴ $(x-2)^2 + y^2 = 1$

c $x = \cos\theta$, $y = \cos 2\theta$

We use the identity $\cos 2\theta = 2\cos^2\theta - 1$

∴ $y = 2\cos^2\theta - 1$

∴ $y = 2x^2 - 1$

d $x = \sin\theta$, $y = \cos 2\theta$

We use the identity $\cos 2\theta = 1 - 2\sin^2\theta$

∴ $y = 1 - 2\sin^2\theta$

∴ $y = 1 - 2x^2$

e $x = \tan\theta$, $y = 2\sec\theta$

We use the identity $\sec^2\theta = 1 + \tan^2\theta$

∴ $\left(\frac{y}{2}\right)^2 = \sec^2\theta$

∴ $\frac{y^2}{4} = 1 + \tan^2\theta$

∴ $\frac{y^2}{4} = 1 + x^2$

∴ $4x^2 - y^2 = -4$

f $x = \cos\theta$, $y = \sin 2\theta$

We use the identities $\sin 2\theta = 2\cos\theta\sin\theta$

and $\sin^2\theta = 1 - \cos^2\theta$

Now $y^2 = \sin^2 2\theta$

$= 4\cos^2\theta\sin^2\theta$

$= 4\cos^2\theta(1 - \cos^2\theta)$

∴ $y^2 = 4x^2(1 - x^2)$

3 **a** $x + 4y = 5$

If $y = t$, then $x + 4t = 5$

$\therefore$ $x = 5 - 4t$, $y = t$ are the parametric equations.

b $xy = -8$

If $x = t$, then $ty = -8$

$\therefore$ $x = t$, $y = -\dfrac{8}{t}$ $(t \neq 0)$ are the parametric equations.

c $y^2 = 9x$

If $x = t^2$, then $y^2 = 9t^2$

$\therefore$ $y = 3t$

(we do not need $y = \pm 3t$, as t could be positive or negative)

$\therefore$ $x = t^2$, $y = 3t$ are the parametric equations.

d
$$x^2 + y^2 = 9$$
$$\therefore \frac{x^2}{9} + \frac{y^2}{9} = 1$$
$$\therefore \left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

But $\cos^2\theta + \sin^2\theta = 1$ for all θ

$\therefore$ we let $\dfrac{x}{3} = \cos\theta$ and $\dfrac{y}{3} = \sin\theta$.

$\therefore$ $x = 3\cos\theta$, $y = 3\sin\theta$ are the parametric equations.

e
$$4x^2 + y^2 = 16$$
$$\therefore \frac{x^2}{4} + \frac{y^2}{16} = 1$$
$$\therefore \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

But $\cos^2\theta + \sin^2\theta = 1$ for all θ

$\therefore$ we let $\dfrac{x}{2} = \cos\theta$ and $\dfrac{y}{4} = \sin\theta$.

$\therefore$ $x = 2\cos\theta$, $y = 4\sin\theta$ are the parametric equations.

f $x^2 = -4y$

If we let $y = -t^2$, $x^2 = -4(-t^2) = 4t^2$

$\therefore$ $x = 2t$

$\therefore$ $x = 2t$, $y = -t^2$ are the parametric equations.

g
$$3x^2 + 5y^2 = 15$$
$$\therefore \frac{x^2}{5} + \frac{y^2}{3} = 1$$
$$\therefore \left(\frac{x}{\sqrt{5}}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1$$

But $\cos^2\theta + \sin^2\theta = 1$ for all θ

$\therefore$ we let $\dfrac{x}{\sqrt{5}} = \cos\theta$ and $\dfrac{y}{\sqrt{3}} = \sin\theta$.

$\therefore$ $x = \sqrt{5}\cos\theta$, $y = \sqrt{3}\sin\theta$ are the parametric equations.

h
$$\frac{x^2}{4} = 1 + \frac{y^2}{9}$$
$$\therefore \left(\frac{x}{2}\right)^2 = 1 + \left(\frac{y}{3}\right)^2$$

But $\sec^2\theta = 1 + \tan^2\theta$ for all θ

$\therefore$ we let $\dfrac{x}{2} = \sec\theta$ and $\dfrac{y}{3} = \tan\theta$.

$\therefore$ $x = 2\sec\theta$, $y = 3\tan\theta$ are the parametric equations.

i
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$
$$\therefore \left(\frac{x}{4}\right)^2 = 1 + \left(\frac{y}{3}\right)^2$$

But $\sec^2\theta = 1 + \tan^2\theta$ for all θ

$\therefore$ we let $\dfrac{x}{4} = \sec\theta$ and $\dfrac{y}{3} = \tan\theta$.

$\therefore$ $x = 4\sec\theta$, $y = 3\tan\theta$ are the parametric equations.

4 **a** The line meets the curve when $2t^2 + t = 3$
$$\therefore 2t^2 + t - 3 = 0$$
$$\therefore (2t+3)(t-1) = 0$$
$$\therefore t = -\tfrac{3}{2} \text{ or } 1$$

When $t = -\frac{3}{2}$, $x = 2(-\frac{3}{2})^2 = \frac{9}{2}$ and $y = -\frac{3}{2}$

When $t = 1$, $x = 2$ and $y = 1$

$\therefore$ they meet at $(\frac{9}{2}, -\frac{3}{2})$ and $(2, 1)$.

b $x = 2t^2$, $y = t$

$\therefore$ $x = 2y^2$ which meets $x + y = 3$

where $2y^2 = 3 - y$
$$\therefore 2y^2 + y - 3 = 0$$
$$\therefore (2y+3)(y-1) = 0$$
$$\therefore y = -\tfrac{3}{2} \text{ or } 1$$

When $y = -\frac{3}{2}$, $x = 3 - (-\frac{3}{2}) = \frac{9}{2}$ ✓

When $y = 1$, $x = 3 - 1 = 2$ ✓

5 **a** $x = 3t$ $\qquad$ $y = t^2 - 3t$

$\therefore \dfrac{dx}{dt} = 3$ $\qquad$ $\therefore \dfrac{dy}{dt} = 2t - 3$
$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-3}{3}$$

When $t = 2$, $x = 3(2) = 6$

and $y = 2^2 - 3(2) = -2$

$\therefore$ $(6, -2)$ is the point of contact and $\dfrac{dy}{dx} = \frac{1}{3}$.

Thus, the equation of the tangent is
$$x - 3y = (6) - 3(-2)$$

which is $x - 3y = 12$

b $x = 2\cos\theta$ $\qquad$ $y = 5\sin\theta$

$\therefore \dfrac{dx}{d\theta} = -2\sin\theta$ $\qquad$ $\therefore \dfrac{dy}{d\theta} = 5\cos\theta$
$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5\cos\theta}{-2\sin\theta}$$

When $\theta = \frac{\pi}{4}$, $x = 2\cos(\frac{\pi}{4}) = \frac{2}{\sqrt{2}}$

and $y = 5\sin(\frac{\pi}{4}) = \frac{5}{\sqrt{2}}$

$\therefore$ $\left(\frac{2}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$ is the point of contact.
$$\frac{dy}{dx} = \frac{5\left(\frac{1}{\sqrt{2}}\right)}{-2\left(\frac{1}{\sqrt{2}}\right)} = -\tfrac{5}{2}$$

Thus, the equation of the tangent is
$$5x + 2y = 5(\tfrac{2}{\sqrt{2}}) + 2(\tfrac{5}{\sqrt{2}})$$

which is $5x + 2y = \frac{20}{\sqrt{2}}$, or $5\sqrt{2}x + 2\sqrt{2}y = 20$

c $x = \sec\theta \qquad\qquad y = \tan\theta$

$$\therefore\ \frac{dx}{d\theta} = \sec\theta\tan\theta \qquad \therefore\ \frac{dy}{d\theta} = \sec^2\theta$$

$$\begin{aligned}\therefore\ \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sec^2\theta}{\sec\theta\tan\theta} &= \frac{\sec\theta}{\tan\theta}\\ &= \frac{1}{\cancel{\cos\theta}} \times \frac{\cancel{\cos\theta}}{\sin\theta}\\ &= \frac{1}{\sin\theta}\end{aligned}$$

When $\theta = \frac{\pi}{3}$, $x = \sec(\frac{\pi}{3}) = 2$

and $y = \tan(\frac{\pi}{3}) = \sqrt{3}$

$\therefore$ $(2, \sqrt{3})$ is the point of contact and $\dfrac{dy}{dx} = \frac{2}{\sqrt{3}}$.

Thus, the equation of the tangent is

$$2x - \sqrt{3}y = 2(2) - \sqrt{3}(\sqrt{3})$$

which is $2x - \sqrt{3}y = 1$

6 **a** $x = 1 - t^2 \qquad\qquad y = 4t$

$$\therefore\ \frac{dx}{dt} = -2t \qquad \therefore\ \frac{dy}{dt} = 4$$

$$\therefore\ \frac{dy}{dx} = \frac{4}{-2t} = -\frac{2}{t}$$

Since the gradient of the tangent is 4,

$$-\frac{2}{t} = 4$$

$$\therefore\ t = -\tfrac{1}{2}$$

When $t = -\frac{1}{2}$, $x = 1 - (-\frac{1}{2})^2 = \frac{3}{4}$

and $y = 4(-\frac{1}{2}) = -2$

$\therefore$ $(\frac{3}{4}, -2)$ is the point of contact.

Thus, the equation of the tangent is

$$4x - y = 4(\tfrac{3}{4}) - (-2)$$

which is $4x - y = 5$

b $x = 1 - t \qquad\qquad y = t^3$

$$\therefore\ \frac{dx}{dt} = -1 \qquad \therefore\ \frac{dy}{dt} = 3t^2$$

$$\therefore\ \frac{dy}{dx} = -3t^2$$

Since the point of contact is $(1, 0)$,

$$1 = 1 - t \quad\text{and}\quad 0 = t^3$$

$$\therefore\ t = 0$$

$$\therefore\ \frac{dy}{dx} = 0$$

Thus, the equation of the tangent is $y = 0$.

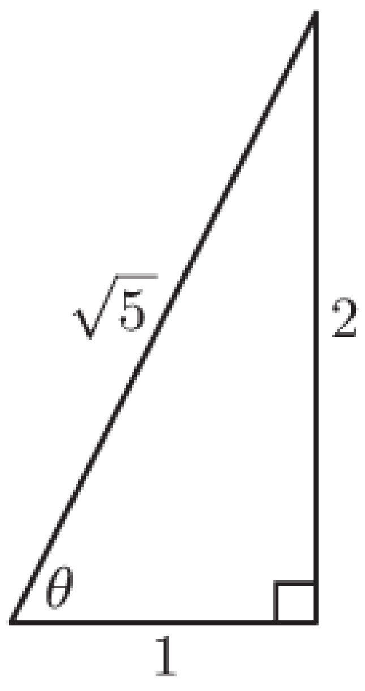
7 The line meets the curve when

$$(1 + \sin\theta) + 2(1 - \cos\theta) = 3$$

$$\therefore\ \cancel{1} + \sin\theta + \cancel{2} - 2\cos\theta = \cancel{3}\,0$$

$$\therefore\ \sin\theta = 2\cos\theta$$

$$\therefore\ \tan\theta = 2$$



$\therefore$ $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = \frac{2}{\sqrt{5}}$

or $\cos\theta = -\frac{1}{\sqrt{5}}$, $\sin\theta = -\frac{2}{\sqrt{5}}$

$\therefore$ they meet at $\left(1 + \frac{2}{\sqrt{5}},\ 1 - \frac{1}{\sqrt{5}}\right)$ and at $\left(1 - \frac{2}{\sqrt{5}},\ 1 + \frac{1}{\sqrt{5}}\right)$.

8 **a** $x = t + \dfrac{1}{t}$, $y = t - \dfrac{1}{t}$, $t \neq 0$

$\therefore$ $x + y = 2t$ and $x - y = \dfrac{2}{t}$

and so $(x + y)(x - y) = 2t\left(\dfrac{2}{t}\right)$

which is $x^2 - y^2 = 4$

b If $x^2 - y^2 = 4$

then $2x - 2y\dfrac{dy}{dx} = 0$

$$\therefore\ \frac{dy}{dx} = \frac{x}{y}$$

When $t = 2$, $x = 2 + \frac{1}{2} = 2\frac{1}{2}$

and $y = 2 - \frac{1}{2} = 1\frac{1}{2}$

$\therefore$ $(2\frac{1}{2}, 1\frac{1}{2})$ is the point of contact and the gradient of the tangent $\dfrac{dy}{dx} = \dfrac{2\frac{1}{2}}{1\frac{1}{2}} = \frac{5}{3}$.

$\therefore$ the gradient of the normal is $-\frac{3}{5}$.

Thus, the equation of the normal is

$$3x + 5y = 3(2\tfrac{1}{2}) + 5(1\tfrac{1}{2})$$

which is $3x + 5y = 15$

9 If $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$, $\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{4}\right)^2 = 1$

But $\cos^2\theta + \sin^2\theta = 1$ for all θ.

$\therefore$ we let $\dfrac{x}{2} = \cos\theta$ and $\dfrac{y}{4} = \sin\theta$

$$\therefore\ x = 2\cos\theta,\quad y = 4\sin\theta$$

If $\text{A} = (0, 4)$, $\text{M} = (X, Y)$, $\text{B} = (2\cos\theta, 4\sin\theta)$

then $X = \dfrac{0 + 2\cos\theta}{2}$,

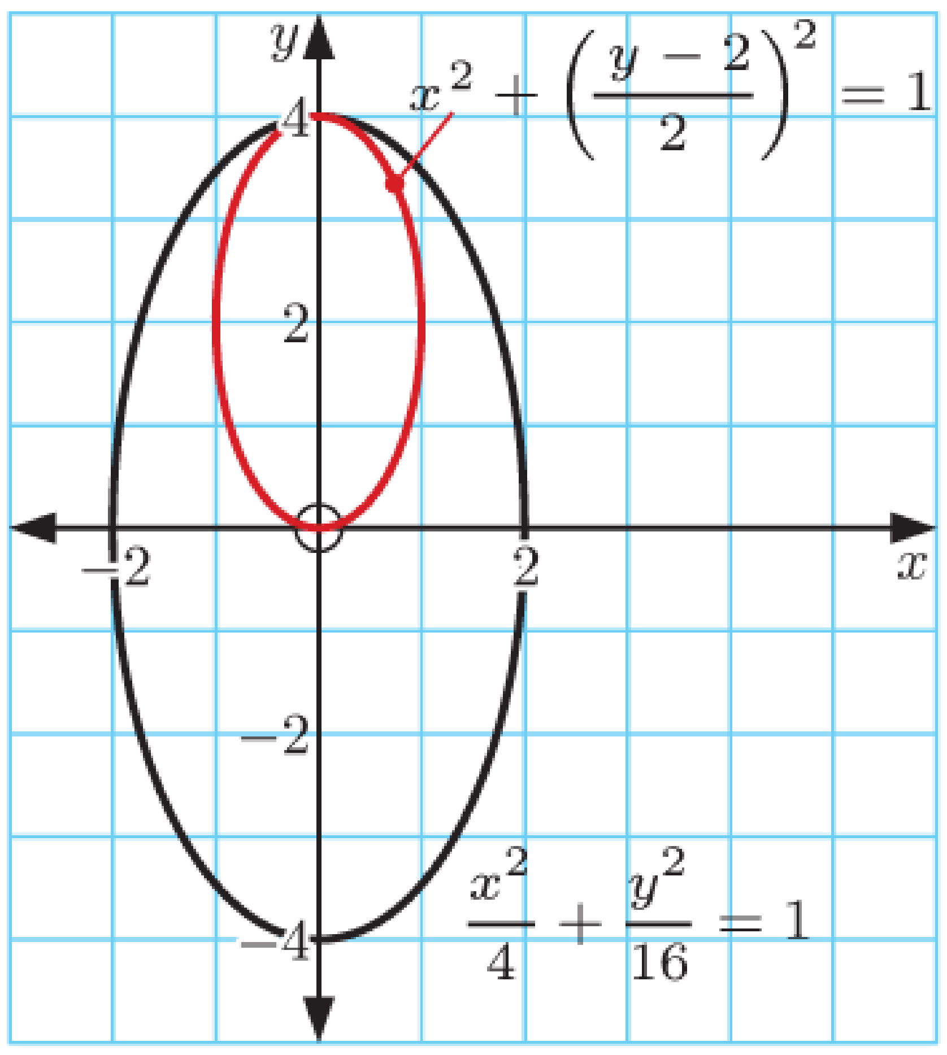
$Y = \dfrac{4 + 4\sin\theta}{2}$

$\therefore$ $X = \cos\theta$,

$Y = 2 + 2\sin\theta$

$$\therefore\ X^2 + \left(\frac{Y-2}{2}\right)^2 = 1$$

which is an ellipse
with graph shown:



EXERCISE 2P

1 **a** Let $x = a\cos\theta$ and $y = a\sin\theta$

$$\therefore\ \frac{dx}{d\theta} = -a\sin\theta \quad\text{and}\quad \frac{dy}{d\theta} = a\cos\theta$$

$$\therefore\ \frac{dy}{dx} = \frac{a\cos\theta}{-a\sin\theta} = -\frac{\cos\theta}{\sin\theta}$$

$\therefore$ the equation of the tangent is

$$\begin{aligned}(\cos\theta)x + (\sin\theta)y &= (\cos\theta)(a\cos\theta) + (\sin\theta)(a\sin\theta)\\ &= a\cos^2\theta + a\sin^2\theta\\ &= a(\cos^2\theta + \sin^2\theta)\\ &= a\end{aligned}$$

Thus, $(\cos\theta)x + (\sin\theta)y = a$.

b From **a**, the tangent has gradient $-\dfrac{\cos\theta}{\sin\theta}$.

$\therefore$ the normal has gradient $\dfrac{\sin\theta}{\cos\theta}$.

$\therefore$ the equation of the normal is

$$\begin{aligned}(\sin\theta)x-(\cos\theta)y &= (\sin\theta)(a\cos\theta)-(\cos\theta)(a\sin\theta)\\ &= a\sin\theta\cos\theta-a\sin\theta\cos\theta\\ &= 0\end{aligned}$$

$$\therefore\ (\cos\theta)y=(\sin\theta)x$$

$$\therefore\ y=\frac{\sin\theta}{\cos\theta}x$$

Thus, $y=(\tan\theta)x$.

2 When $\theta=\frac{\pi}{3}$, $\cos\theta=\frac{1}{2}$ and $\sin\theta=\frac{\sqrt{3}}{2}$.

Using **1 a**, the equation of the tangent is

$$\tfrac{1}{2}x+\tfrac{\sqrt{3}}{2}y=3$$

which is $x+\sqrt{3}y=6$

When $\theta=\frac{\pi}{3}$, $\tan\theta=\sqrt{3}$

Using **1 b**, the equation of the normal is $y=\sqrt{3}x$.

3 **a** Let $x=at^2$ and $y=2at$

$$\therefore\ \frac{dx}{dt}=2at \quad\text{and}\quad \frac{dy}{dt}=2a$$

$$\therefore\ \frac{dy}{dx}=\frac{2a}{2at}=\frac{1}{t}$$

$\therefore$ the equation of the tangent is

$$\begin{aligned}x-ty &= at^2-t(2at)\\ &= at^2-2at^2\\ &= -at^2\end{aligned}$$

Thus, $x-ty=-at^2$.

b From **a**, the gradient of the tangent is $\dfrac{1}{t}$.

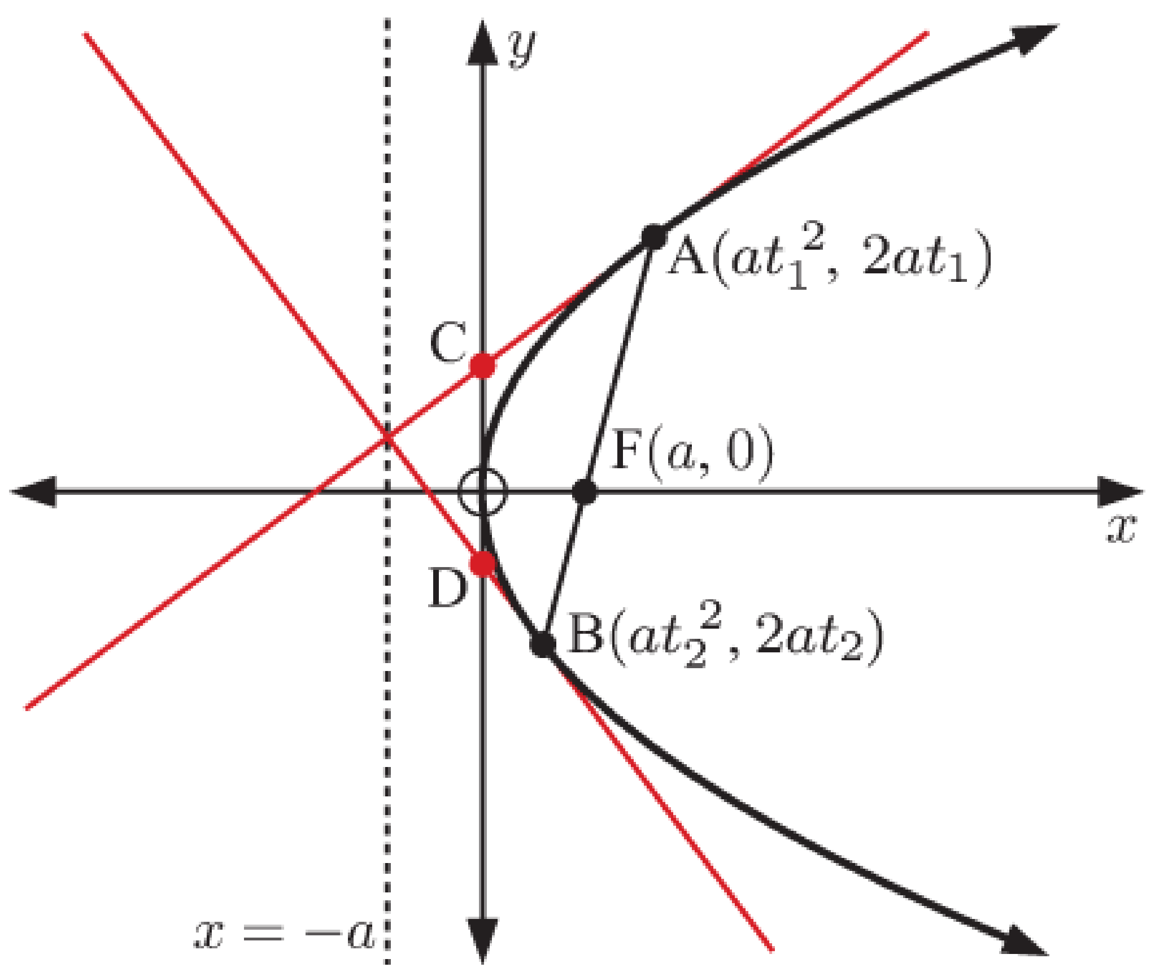
$\therefore$ the gradient of the normal is $-\dfrac{t}{1}$.

$\therefore$ the equation of the normal is

$$\begin{aligned}tx+y &= t(at^2)+2at\\ &= at^3+2at\end{aligned}$$

Thus, $tx+y=at^3+2at$.

4



a gradient of [AB] $=\dfrac{2at_2-2at_1}{at_2^2-at_1^2}$

$$\begin{aligned}&=\frac{2\cancel{a}\cancel{(t_2-t_1)}}{\cancel{a}(t_2+t_1)\cancel{(t_2-t_1)}}\\ &=\frac{2}{t_2+t_1}\qquad\left(=\frac{2}{t_1+t_2}\right)\end{aligned}$$

$\therefore$ the chord [AB] has equation

$$\begin{aligned}2x-(t_1+t_2)y &= 2(at_1{}^2)-(t_1+t_2)(2at_1)\\ &= \cancel{2at_1{}^2}-\cancel{2at_1{}^2}-2at_1t_2\end{aligned}$$

Thus, $2x-(t_1+t_2)y=-2at_1t_2$

b If [AB] is a focal chord, then $(a, 0)$ lies on [AB].

$$\begin{aligned}\therefore\ 2a-(t_1+t_2)(0)&=-2at_1t_2\\ \therefore\ 2\cancel{a}&=-2\cancel{a}t_1t_2\\ \therefore\ t_1t_2&=-1\quad\{\text{since } a\neq 0\}\end{aligned}$$

c Using **3 a**, the equation of the tangent at A is $x-t_1y=-at_1{}^2$ and the equation of the tangent at B is $x-t_2y=-at_2{}^2$.

If we multiply the first equation by t_2 and the second equation by $-t_1$ we get:

$$\begin{aligned}t_2x-t_1t_2y&=-at_1{}^2t_2\\ -t_1x+t_1t_2y&=at_1t_2{}^2\\ \hline \cancel{(t_2-t_1)}x&=at_1t_2\cancel{(t_2-t_1)}\\ \therefore\ x&=at_1t_2\qquad\{\text{since } t_2\neq t_1\}\\ \therefore\ x&=-a\end{aligned}$$

$\therefore$ these tangents always meet on the directrix.

Now, the gradient of the tangent at A is $\dfrac{1}{t_1}$, and the gradient of the tangent at B is $\dfrac{1}{t_2}$.

$$\begin{aligned}\frac{1}{t_1}\times\frac{1}{t_2}&=\frac{1}{t_1t_2}\\ &=-1\quad\{\text{using } \textbf{4 b}\}\end{aligned}$$

$\therefore$ these tangents are perpendicular.

Thus, the tangents at the extremities of the focal chord always intersect at right angles on the directrix.

d If $x=0$, then $x-t_1y=-at_1{}^2$ becomes
$y=at_1$

$\therefore$ C is $(0, at_1)$.

Likewise, D is $(0, at_2)$.

$\therefore$ the gradient of [CF] $=\dfrac{0-at_1}{a-0}=-t_1$

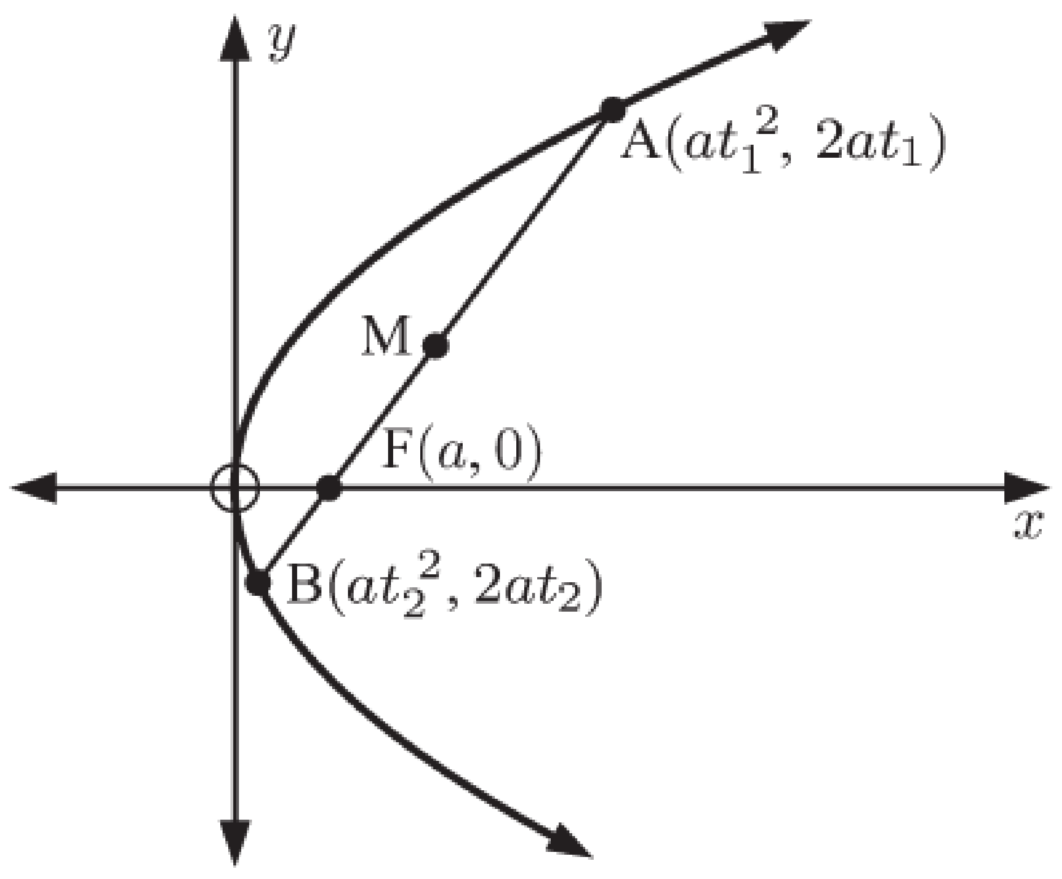
and the gradient of [DF] $=\dfrac{0-at_2}{a-0}=-t_2$

Multiplying these gradients together gives $t_1t_2=-1$.
{from **4 b**}

$\therefore$ [CF] and [DF] are perpendicular.

$\therefore$ [CD] subtends a right angle at the focus F.

e



Let the midpoint of [AB] be $M(X, Y)$.

This means $X=\dfrac{at_1{}^2+at_2{}^2}{2}$ and $Y=\dfrac{2at_1+2at_2}{2}$

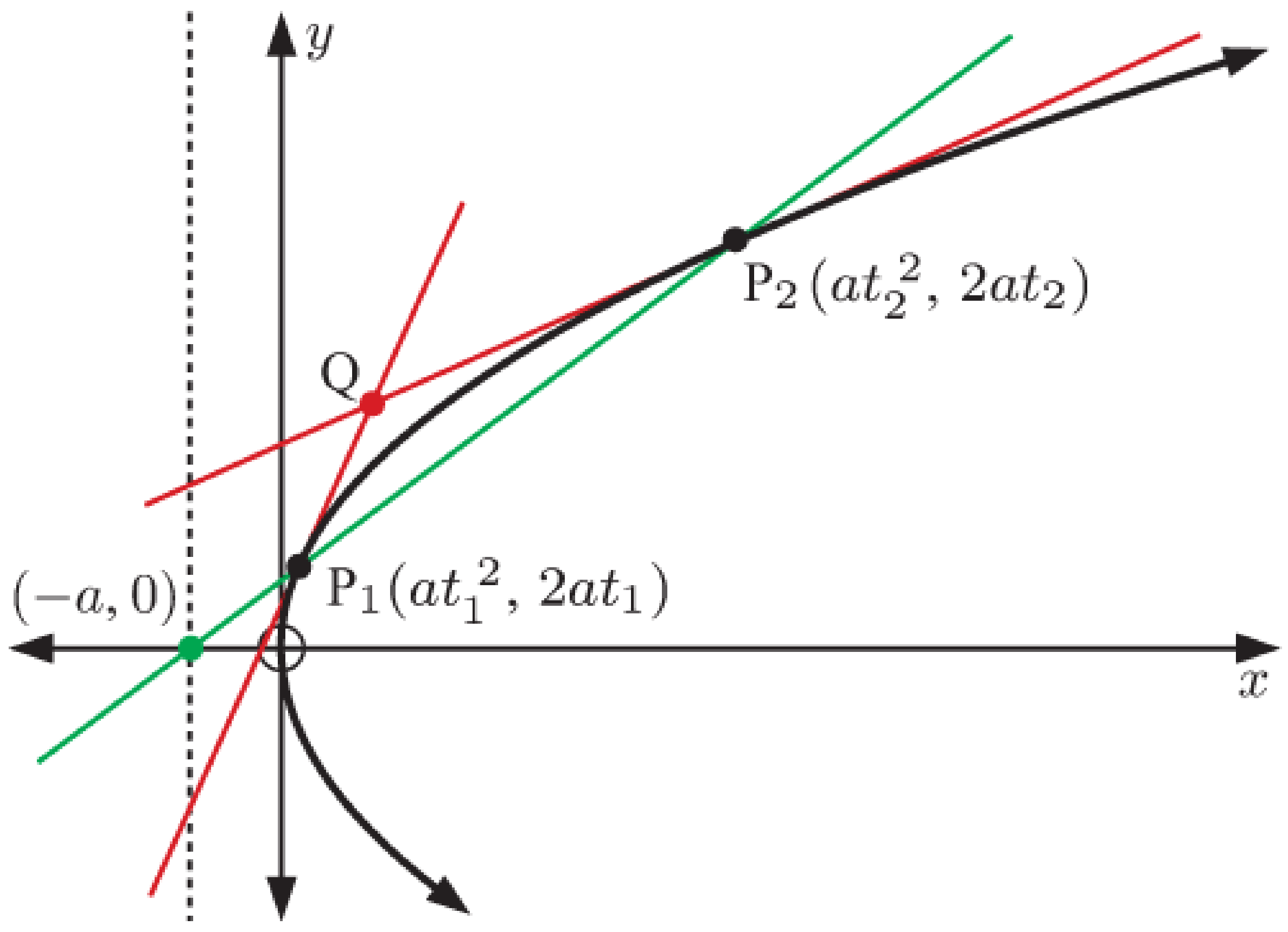
Thus $X=\dfrac{a}{2}\left(t_1{}^2+t_2^2\right)$ and $Y=a(t_1+t_2)$

$$\text{Now,}\quad (t_1+t_2)^2 = t_1^{\,2} + 2t_1t_2 + t_2^{\,2}$$
$$= t_1^{\,2} + t_2^{\,2} - 2 \quad \{\text{using } \mathbf{4\ b}\}$$
$$\therefore\quad \left(\frac{Y}{a}\right)^2 = \frac{2X}{a} - 2$$
$$\therefore\quad \frac{Y^2}{a^2} = \frac{2X}{a} - 2$$
$$\therefore\quad Y^2 = 2aX - 2a^2$$
$$\therefore\quad Y^2 = 2a(X-a)$$
$$\therefore\quad Y^2 = 4\left(\frac{a}{2}\right)(X-a)$$

So, the Cartesian equation of the locus of M is

$y^2 = 4\left(\frac{a}{2}\right)(x-a)$, which is another parabola with vertex $(a,\ 0)$ and focus $\left(\frac{3a}{2},\ 0\right)$.

5



Let P_1 be $(at_1^{\,2},\ 2at_1)$ and P_2 be $(at_2^{\,2},\ 2at_2)$.

Now $[P_1P_2]$ has equation

$$2x - (t_1+t_2)y = -2at_1t_2 \quad \{\text{using } \mathbf{4\ a}\}$$

But $(-a,\ 0)$ lies on the line

$$\therefore\quad 2(-a) - (t_1+t_2)(0) = -2at_1t_2$$
$$\therefore\quad -2a = -2at_1t_2$$
$$\therefore\quad t_1t_2 = 1 \qquad \ldots\ldots\ (1)$$

Using **3 a**, the equation of the tangent at P_1 is

$x - t_1y = -at_1^{\,2}$ .... (2)

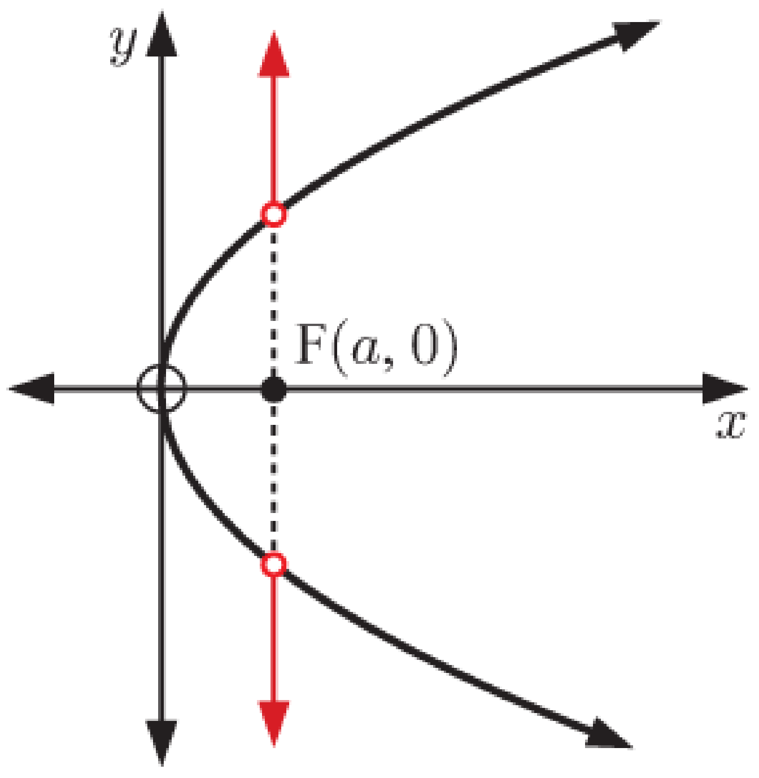
and the equation of the tangent at P_2 is

$x - t_2y = -at_2^{\,2}$ .... (3)

These tangents meet at Q:

$$\begin{array}{rll} t_2x - t_1t_2y &= -at_1^{\,2}t_2 & \{t_2 \times (2)\} \\ -t_1x + t_1t_2y &= at_1t_2^{\,2} & \{-t_1 \times (3)\} \\ \hline (t_2-t_1)x &= at_1t_2(t_2-t_1) & \\ \therefore\quad x &= at_1t_2 & \{\text{since } t_2 \neq t_1\} \\ \therefore\quad x &= a & \{\text{from } (1)\} \end{array}$$

$\therefore$ Q lies on the vertical line through the focus, but outside the parabola.



6 **a** Let $x = a\cos\theta$ and $y = b\sin\theta$

$$\therefore\quad \frac{dx}{d\theta} = -a\sin\theta \quad\text{and}\quad \frac{dy}{d\theta} = b\cos\theta$$
$$\therefore\quad \frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta}$$

$\therefore$ the equation of the tangent is

$$(b\cos\theta)x + (a\sin\theta)y$$
$$= (b\cos\theta)(a\cos\theta) + (a\sin\theta)(b\sin\theta)$$
$$= ab\cos^2\theta + ab\sin^2\theta$$
$$= ab(\cos^2\theta + \sin^2\theta)$$
$$= ab$$

Thus, $(b\cos\theta)x + (a\sin\theta)y = ab$.

b From **6 a**, the gradient of the tangent is $\dfrac{b\cos\theta}{-a\sin\theta}$.

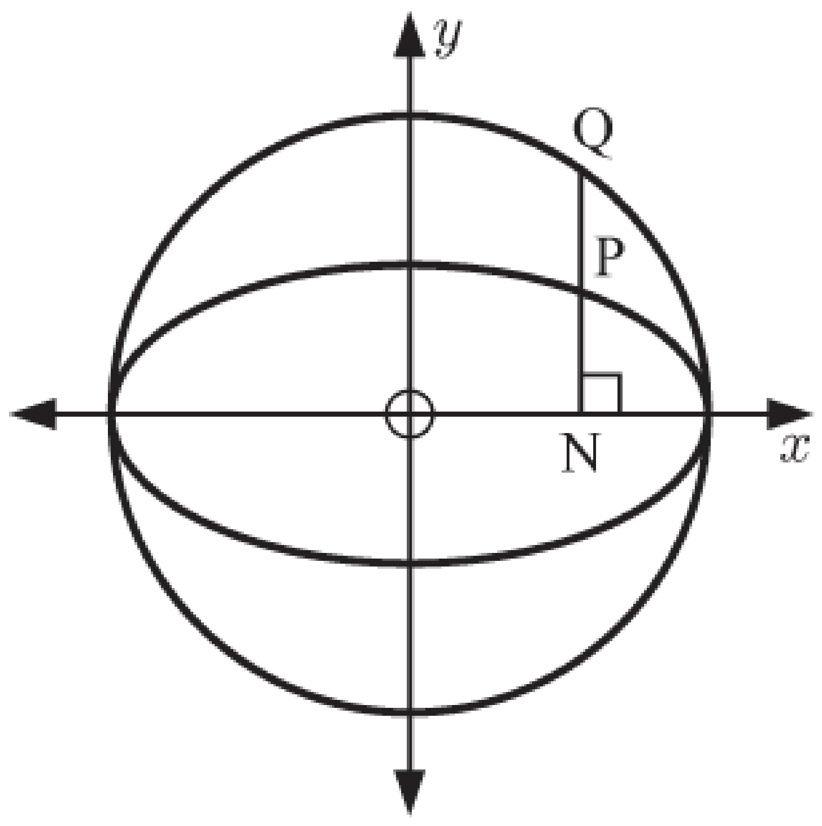
$\therefore$ the gradient of the normal is $\dfrac{a\sin\theta}{b\cos\theta}$.

$\therefore$ the equation of the normal is

$$(a\sin\theta)x - (b\cos\theta)y$$
$$= (a\sin\theta)(a\cos\theta) - (b\cos\theta)(b\sin\theta)$$
$$= a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta$$
$$= (a^2-b^2)\sin\theta\cos\theta$$

Thus, $(a\sin\theta)x - (b\cos\theta)y = (a^2-b^2)\sin\theta\cos\theta$.

7 **a**



Let P be $(a\cos\theta,\ b\sin\theta)$.

$\therefore$ Q has x-coordinate $a\cos\theta$ also.

But Q lies on $x^2 + y^2 = a^2$

$$\therefore\quad (a\cos\theta)^2 + y^2 = a^2$$
$$\therefore\quad a^2\cos^2\theta + y^2 = a^2$$
$$\therefore\quad y^2 = a^2 - a^2\cos^2\theta$$
$$\therefore\quad y^2 = a^2(1-\cos^2\theta)$$
$$\therefore\quad y^2 = a^2\sin^2\theta$$
$$\therefore\quad y = a\sin\theta$$

$\therefore$ Q is $(a\cos\theta,\ a\sin\theta)$.

$$\therefore\quad \text{PN} : \text{QN} = b\sin\theta : a\sin\theta$$
$$= b : a$$

b Using **6 a**, the equation of the tangent at P is $(b\cos\theta)x + (a\sin\theta)y = ab$.

This tangent cuts the x-axis when $y = 0$

$$\therefore\quad (b\cos\theta)x = ab$$
$$\therefore\quad x = \frac{a\cancel{b}}{\cancel{b}\cos\theta} = \frac{a}{\cos\theta}$$

The tangent at Q has the equation $(\cos\theta)x + (\sin\theta)y = a$.

This tangent cuts the x-axis when $y = 0$

$$\therefore\quad (\cos\theta)x = a$$
$$\therefore\quad x = \frac{a}{\cos\theta}$$

Thus, both tangents cut the x-axis at $\left(\dfrac{a}{\cos\theta},\ 0\right)$

provided that $\cos\theta \neq 0$, so $\theta \neq \frac{\pi}{2}$ (and so N is not $(0,\ 0)$).

8 **a** From **6 b**, the normal at P has equation

$$(a\sin\theta)x-(b\cos\theta)y=(a^2-b^2)\sin\theta\cos\theta$$

This cuts the x-axis when $y=0$.

$$\therefore\ (a\cancel{\sin\theta})x=(a^2-b^2)\cancel{\sin\theta}\cos\theta$$

$$\therefore\ x=\frac{(a^2-b^2)\cos\theta}{a}\quad\text{.... (1)}$$

But for an ellipse, $b^2=a^2(1-e^2)$

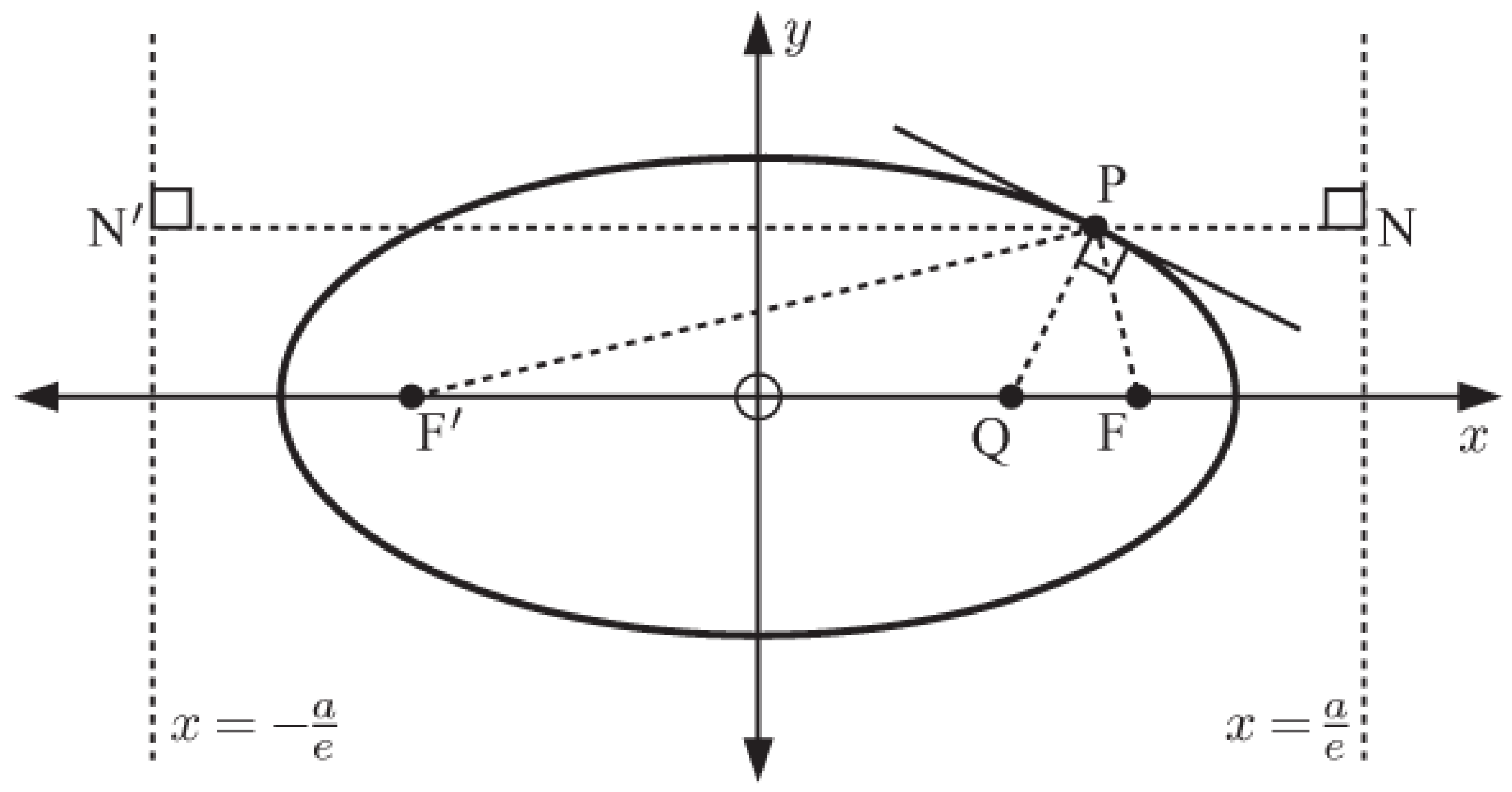
$$\therefore\ b^2=a^2-a^2e^2$$

$$\therefore\ a^2-b^2=a^2e^2$$

$$\therefore\ x=\frac{a^{\cancel{2}}e^2\cos\theta}{\cancel{a}}=ae^2\cos\theta$$

Thus, the normal at P cuts the x-axis at $(ae^2\cos\theta,\ 0)$.

b



$$\begin{aligned}\text{PF}&=e\text{PN}\\&=e\left(\frac{a}{e}-a\cos\theta\right)\\&=a-ae\cos\theta\\&=a(1-e\cos\theta)\end{aligned}$$

Likewise,

$$\begin{aligned}\text{PF}'&=e\text{PN}'\\&=e\left(a\cos\theta-\left(-\frac{a}{e}\right)\right)\\&=ae\cos\theta+a\\&=a(1+e\cos\theta)\end{aligned}$$

c

$$\begin{aligned}\text{QF}&=ae-ae^2\cos\theta\quad\{\text{using } \mathbf{a}\}\\&=ae(1-e\cos\theta)\end{aligned}$$

and

$$\begin{aligned}\text{QF}'&=ae^2\cos\theta-(-ae)\\&=ae(e\cos\theta+1)\end{aligned}$$

Now $\dfrac{\text{PF}}{\text{PF}'}=\dfrac{\cancel{a}(1-e\cos\theta)}{\cancel{a}(1+e\cos\theta)}=\dfrac{1-e\cos\theta}{1+e\cos\theta}$

and $\dfrac{\text{QF}}{\text{QF}'}=\dfrac{\cancel{ae}(1-e\cos\theta)}{\cancel{ae}(1+e\cos\theta)}=\dfrac{1-e\cos\theta}{1+e\cos\theta}$

So, $\dfrac{\text{PF}}{\text{PF}'}=\dfrac{\text{QF}}{\text{QF}'}$

$\therefore$ by the converse of the angle bisector theorem, $\text{F}\widehat{\text{P}}\text{Q}=\text{F}'\widehat{\text{P}}\text{Q}$.

$\therefore$ [PQ] bisects $\text{F}\widehat{\text{P}}\text{F}'$.

d As P varies, sound or light emanating from one focus is concentrated at the other focus. {Equal angles that [PQ] makes with the normal – reflection principle.}

9 The tangent at P has the equation $(b\cos\theta)x+(a\sin\theta)y=ab$.

This tangent meets the directrix $x=\dfrac{a}{e}$ when

$$(b\cos\theta)\frac{\cancel{a}}{e}+(\cancel{a}\sin\theta)y=\cancel{a}b$$

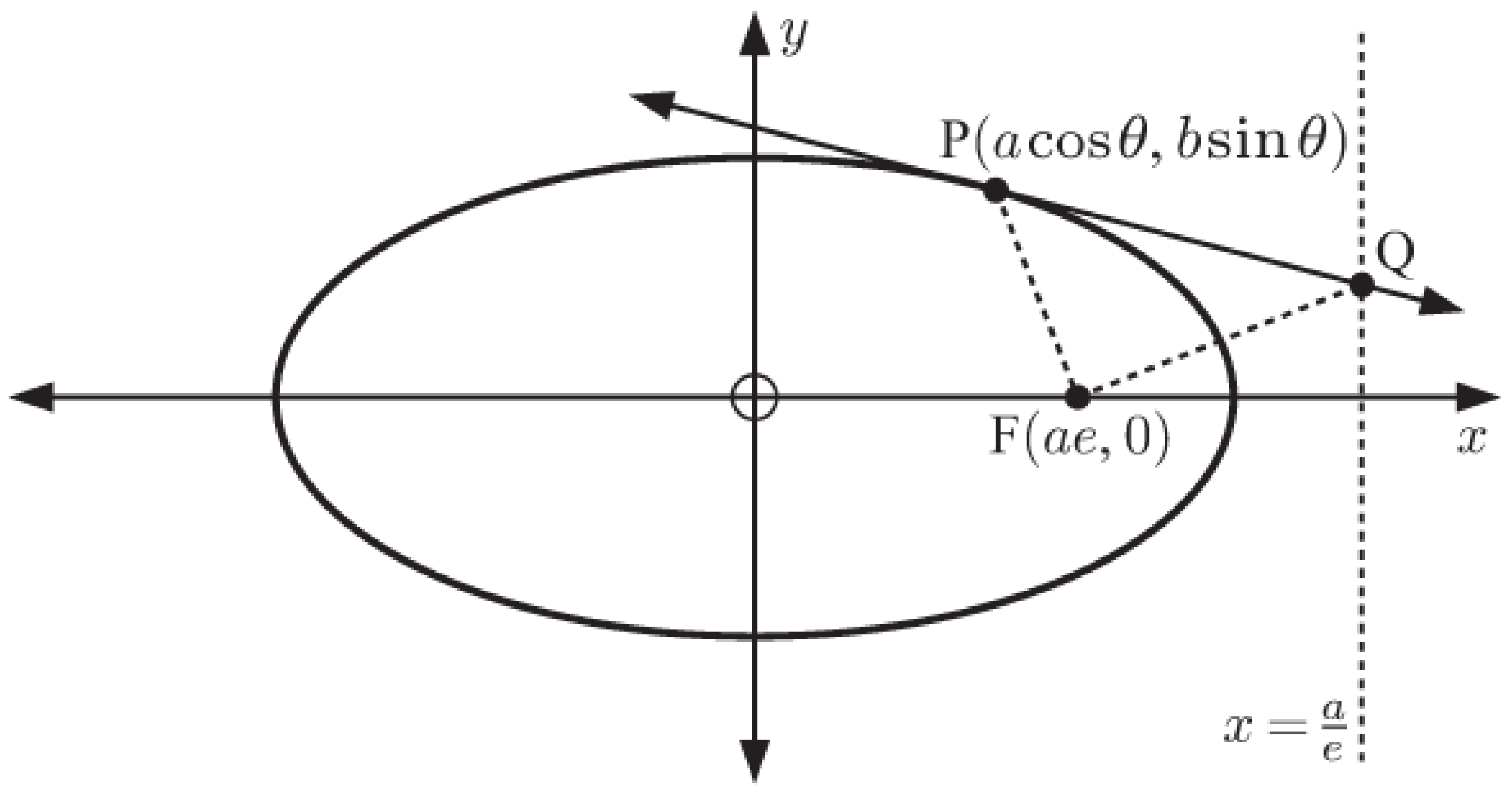
$$\therefore\ \frac{b\cos\theta}{e}+y\sin\theta=b$$

$$\therefore\ y\sin\theta=b-\frac{b\cos\theta}{e}$$

$$\therefore\ y\sin\theta=\frac{be-b\cos\theta}{e}$$

$$\therefore\ y=\frac{be-b\cos\theta}{e\sin\theta}$$

$\therefore$ Q is $\left(\dfrac{a}{e},\ \dfrac{b(e-\cos\theta)}{e\sin\theta}\right)$.



$$\begin{aligned}\text{gradient of [PF]}&=\frac{b\sin\theta-0}{a\cos\theta-ae}\\&=\frac{b\sin\theta}{a(\cos\theta-e)}\end{aligned}$$

$$\begin{aligned}\text{gradient of [QF]}&=\frac{\frac{b(e-\cos\theta)}{e\sin\theta}-0}{\frac{a}{e}-ae}\\&=\frac{b(e-\cos\theta)}{e\sin\theta(\frac{a}{e}-ae)}\\&=\frac{b(e-\cos\theta)}{a\sin\theta-ae^2\sin\theta}\\&=\frac{b(e-\cos\theta)}{a\sin\theta(1-e^2)}\\&=-\frac{b}{a}\left(\frac{1}{1-e^2}\right)\frac{\cos\theta-e}{\sin\theta}\\&=-\frac{b}{a}\left(\frac{1}{\frac{b^2}{a^2}}\right)\frac{\cos\theta-e}{\sin\theta}\\&\qquad\{\text{using }\ b^2=a^2(1-e^2)\}\\&=-\frac{\cancel{b}}{\cancel{a}}\,\frac{a^{\cancel{2}}}{b^{\cancel{2}}}\,\frac{\cos\theta-e}{\sin\theta}\end{aligned}$$

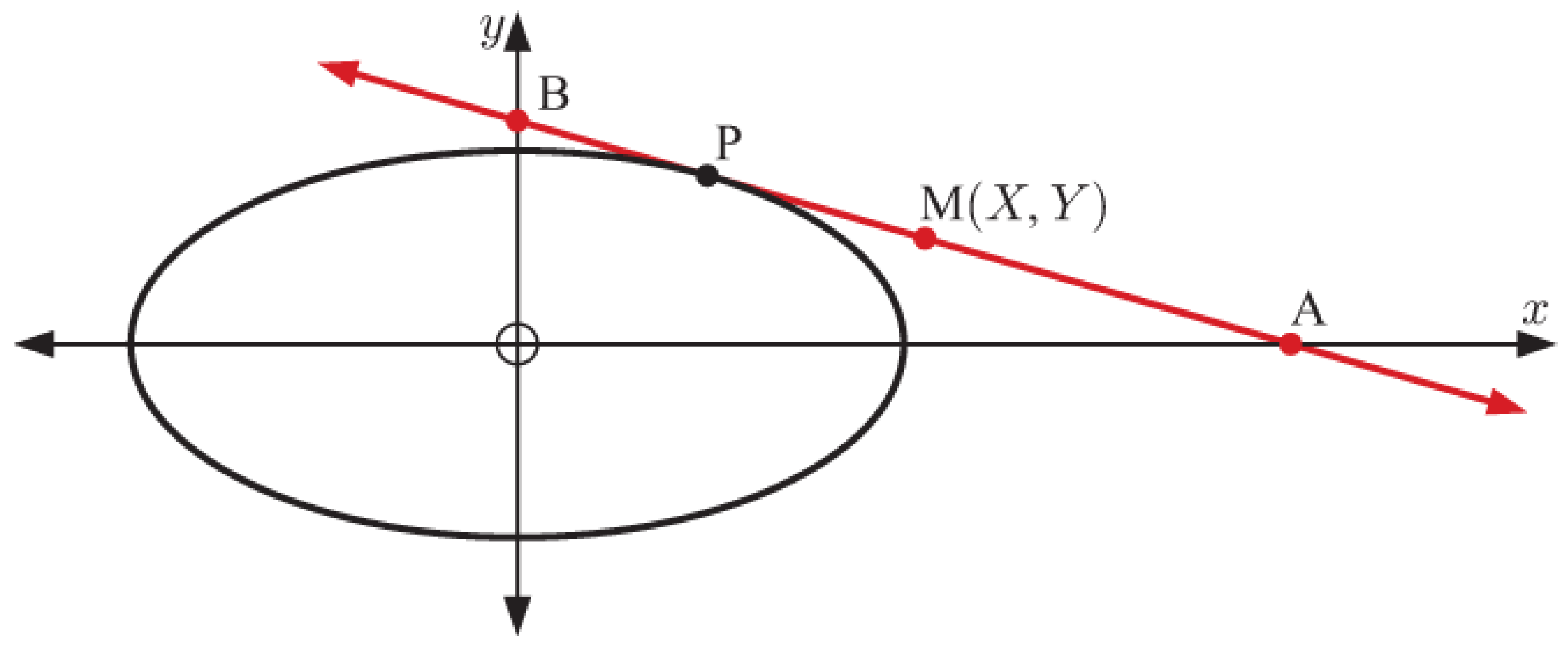
Thus, gradient of [QF] $=\dfrac{-a(\cos\theta-e)}{b\sin\theta}$

Now, m_{QF} is the negative reciprocal of m_{PF}.

$\therefore$ [PF] $\perp$ [QF]

$\therefore$ $\text{P}\widehat{\text{F}}\text{Q}$ is a right angle.

10



The tangent at P has the equation $(b\cos\theta)x+(a\sin\theta)y=ab$.

$\therefore$ A is $\left(\dfrac{a}{\cos\theta},\ 0\right)$ and B is $\left(0,\ \dfrac{b}{\sin\theta}\right)$.

Let M(X, Y) be the midpoint of [AB].

$$\therefore \quad X = \frac{\frac{a}{\cos\theta} + 0}{2} \quad \text{and} \quad Y = \frac{0 + \frac{b}{\sin\theta}}{2}$$

$$\therefore \quad X = \frac{a}{2\cos\theta} \quad \text{and} \quad Y = \frac{b}{2\sin\theta}$$

$$\therefore \quad \cos\theta = \frac{a}{2X} \quad \text{and} \quad \sin\theta = \frac{b}{2Y}$$

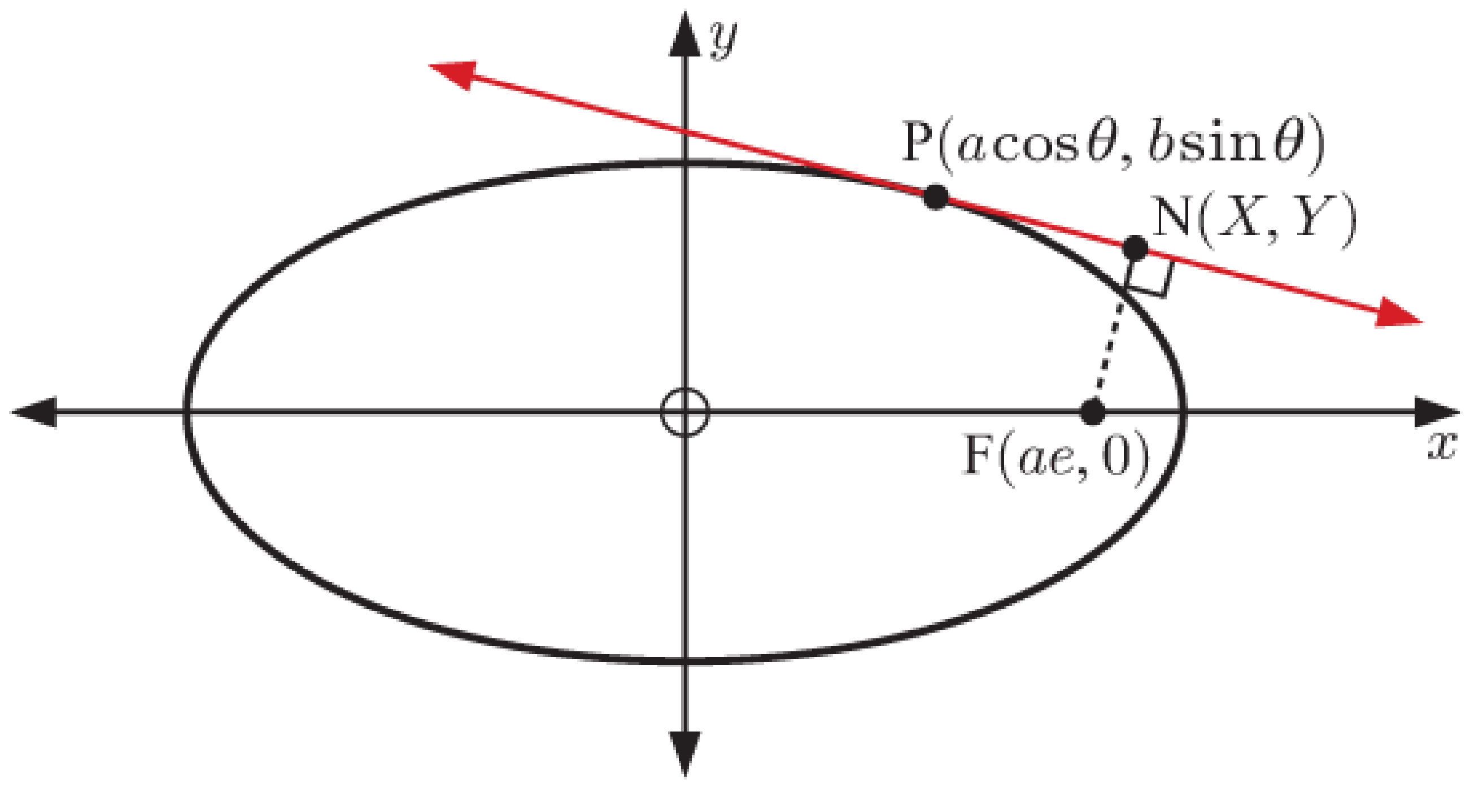
$$\therefore \quad \cos^2\theta = \frac{a^2}{4X^2} \quad \text{and} \quad \sin^2\theta = \frac{b^2}{4Y^2}$$

$$\therefore \quad \frac{a^2}{4X^2} + \frac{b^2}{4Y^2} = 1 \quad \{\text{since} \ \ \cos^2\theta + \sin^2\theta = 1\}$$

$$\therefore \quad a^2Y^2 + b^2X^2 = 4X^2Y^2$$

$\therefore$ the equation of the locus of M is $b^2x^2 + a^2y^2 = 4x^2y^2$.

11



The tangent at P has the equation

$(b\cos\theta)x + (a\sin\theta)y = ab$ .... (1)

and its gradient is $-\dfrac{b\cos\theta}{a\sin\theta}$

$\therefore$ the gradient of [FN] is $\dfrac{a\sin\theta}{b\cos\theta}$.

Now, [FN] has the equation

$$(a\sin\theta)x - (b\cos\theta)y = (a\sin\theta)ae - 0$$

$$\therefore \quad (a\sin\theta)x - (b\cos\theta)y = a^2e\sin\theta \quad .... (2)$$

[PN] and [FN] meet at N(X, Y).

We let $\sin\theta = S$ and $\cos\theta = C$

(1) and (2) then become $bCX + aSY = ab$

$$aSX - bCY = a^2eS$$

Squaring these, we get

$$b^2C^2X^2 + 2abSCXY + a^2S^2Y^2 = a^2b^2$$
$$a^2S^2X^2 - 2abSCXY + b^2C^2Y^2 = a^4e^2S^2$$

Adding:

$$a^2S^2X^2 + b^2C^2X^2 + a^2S^2Y^2 + b^2C^2Y^2 = a^2b^2 + a^4e^2S^2$$

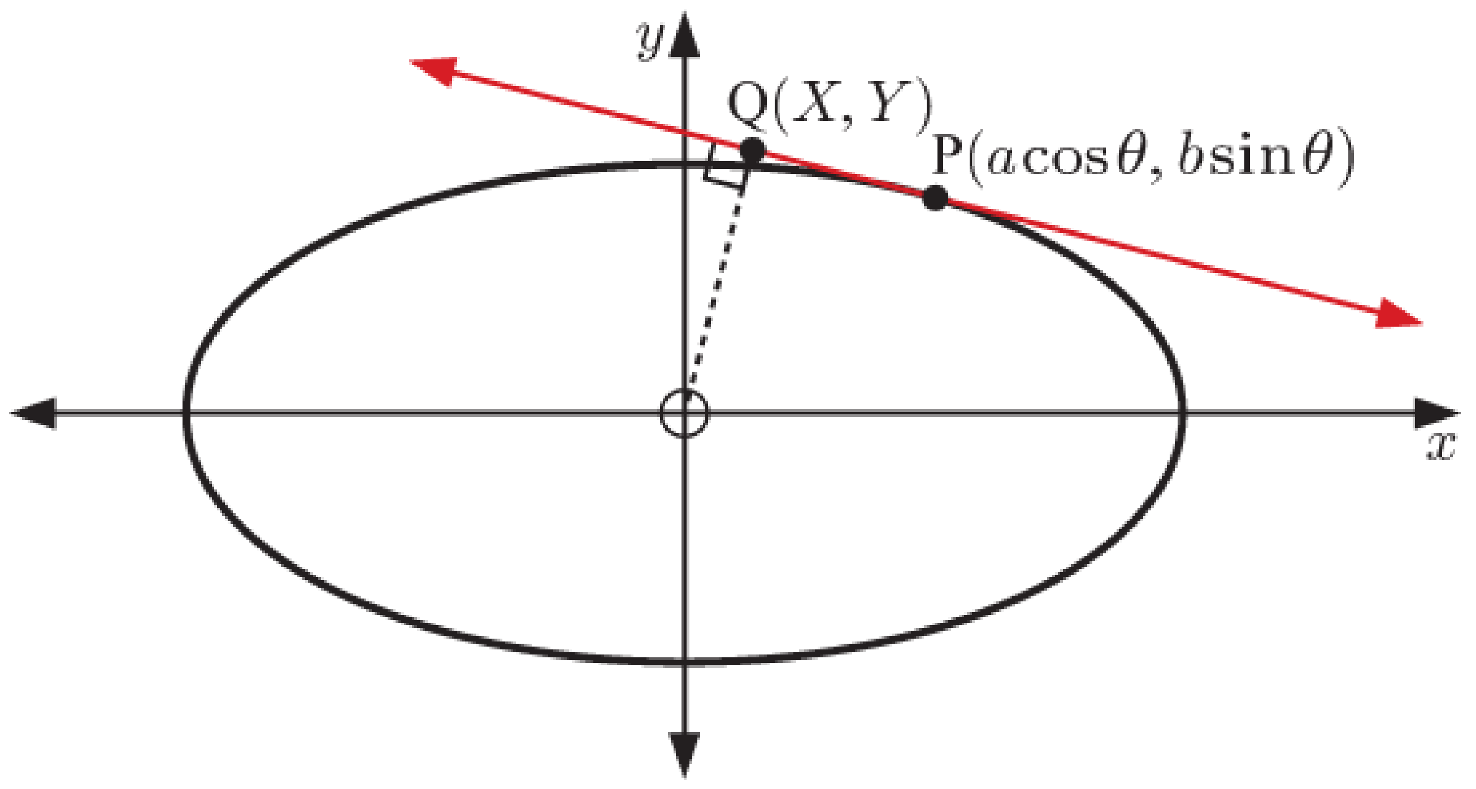
$$\begin{aligned}
\therefore \quad & (a^2S^2 + b^2C^2)X^2 + (a^2S^2 + b^2C^2)Y^2 \\
&= a^2(b^2 + a^2e^2S^2) \\
&= a^2(b^2 + [a^2 - b^2]S^2) \\
&\qquad \{\text{since} \ \ b^2 = a^2(1 - e^2) = a^2 - a^2e^2 \\
&\qquad\quad \therefore \ a^2e^2 = a^2 - b^2\} \\
&= a^2(b^2 + a^2S^2 - b^2S^2) \\
&= a^2(a^2S^2 + b^2[1 - S^2]) \\
&= a^2(a^2S^2 + b^2C^2)
\end{aligned}$$

Thus, $\cancel{(a^2S^2 + b^2C^2)}X^2 + \cancel{(a^2S^2 + b^2C^2)}Y^2$
$= a^2\cancel{(a^2S^2 + b^2C^2)}$

$$\therefore \quad X^2 + Y^2 = a^2$$

$\therefore$ the locus of N is the auxiliary circle $x^2 + y^2 = a^2$.

12



The tangent at P has equation $(b\cos\theta)x + (a\sin\theta)y = ab$

and its gradient is $-\dfrac{b\cos\theta}{a\sin\theta}$.

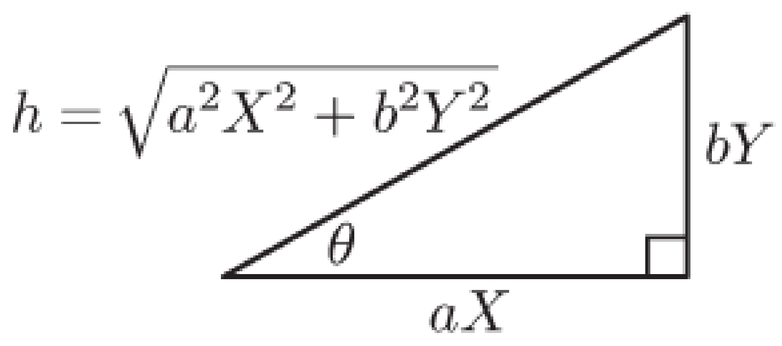
$\therefore$ [OQ] has gradient $\dfrac{a\sin\theta}{b\cos\theta}$.

The equation of [OQ] is $Y = \dfrac{a\sin\theta}{b\cos\theta}X$ .... (1)

Now, Q(X, Y) lies on the tangent.

$\therefore$ $(b\cos\theta)X + (a\sin\theta)Y = ab$ .... (2)

From (1), $Y = \left(\dfrac{a}{b}\tan\theta\right)X$



$$\therefore \quad \tan\theta = \frac{bY}{aX}$$

So, $\sin\theta = \dfrac{bY}{h}$ and $\cos\theta = \dfrac{aX}{h}$

and so (2) becomes

$$b\left(\frac{aX}{h}\right)X + a\left(\frac{bY}{h}\right)Y = ab$$

$$\therefore \quad \frac{\cancel{ab}X^2}{h} + \frac{\cancel{ab}Y^2}{h} = \cancel{ab}$$

$$\therefore \quad X^2 + Y^2 = h = \sqrt{a^2X^2 + b^2Y^2}$$

$$\therefore \quad (X^2 + Y^2)^2 = a^2X^2 + b^2Y^2$$

Thus, the equation of the locus of Q is

$(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

13 The tangent at P has equation $(b\cos\theta)x + (a\sin\theta)y = ab$

Using the distance from point to line formula,

$$\text{MF} = \frac{|b\cos\theta(ae) + (a\sin\theta)(0) - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\therefore \quad \text{MF} = \frac{|abe\cos\theta - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\text{and} \quad \text{M}'\text{F}' = \frac{|b\cos\theta(-ae) + a\sin\theta(0) - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\therefore \quad \text{M}'\text{F}' = \frac{|-abe\cos\theta - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\therefore \quad \text{MF.M}'\text{F}' = \frac{ab\,|e\cos\theta - 1| \times ab\,|e\cos\theta + 1|}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{a^2b^2\left|e^2\cos^2\theta - 1\right|}{b^2\cos^2\theta + a^2\sin^2\theta}$$

But $b^2\cos^2\theta + a^2\sin^2\theta$

$$\begin{aligned}
&= b^2\cos^2\theta + a^2(1 - \cos^2\theta) \\
&= b^2\cos^2\theta + a^2 - a^2\cos^2\theta \\
&= (b^2 - a^2)\cos^2\theta + a^2 \\
&= -a^2e^2\cos^2\theta + a^2 \qquad \{b^2 = a^2(1 - e^2) = a^2 - a^2e^2 \\
&\qquad\qquad\qquad\qquad\qquad\quad \therefore \ b^2 - a^2 = -a^2e^2\} \\
&= a^2(1 - e^2\cos^2\theta)
\end{aligned}$$

Thus $\quad \text{MF.M}'\text{F}' = \dfrac{a^2b^2\left|e^2\cos^2\theta - 1\right|}{a^2(1-e^2\cos^2\theta)}$

$$= b^2 \times \pm 1$$

$$= b^2 \quad \text{as} \quad \text{MF.M}'\text{F}' \quad \text{must be} \quad > 0$$

14 $x = a\sec\theta, \quad y = b\tan\theta$ represents any point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \quad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$$

$$\therefore \quad \frac{dy}{dx} = \frac{b\left(\frac{1}{\cos\theta}\right)}{a\left(\frac{\sin\theta}{\cos\theta}\right)} = \frac{b}{a\sin\theta}$$

$\therefore$ the tangent has gradient $\dfrac{b}{a\sin\theta}$.

$\therefore$ the equation of the tangent is

$$bx - (a\sin\theta)y = b(a\sec\theta) - (a\sin\theta)(b\tan\theta).$$

$$\therefore \quad bx - (a\sin\theta)y = \frac{ab}{\cos\theta} - \frac{ab\sin^2\theta}{\cos\theta}$$

$$= \frac{ab}{\cos\theta}(1-\sin^2\theta)$$

$$= \frac{ab}{\cos\theta}\cos^2\theta$$

$$= ab\cos\theta$$

Thus, the equation of the tangent is $\quad bx - (a\sin\theta)y = ab\cos\theta.$

15 The equation of the normal is

$$(a\sin\theta)x + by = (a^2+b^2)\tan\theta$$

When $y = 0$, we obtain $\quad (a\sin\theta)x = \dfrac{(a^2+b^2)\sin\theta}{\cos\theta}$

$$\therefore \quad x = \frac{a^2+b^2}{a\cos\theta}$$

When $x = 0$, we obtain $\quad by = (a^2+b^2)\tan\theta$

$$\therefore \quad y = \frac{(a^2+b^2)\tan\theta}{b}$$

So, A is $\left(\dfrac{a^2+b^2}{a\cos\theta}, 0\right)$ and B is $\left(0, \dfrac{(a^2+b^2)\tan\theta}{b}\right)$.

Let $\text{M}(X, Y)$ be the midpoint of [AB].

$$\therefore \quad X = \frac{a^2+b^2}{2a\cos\theta} \quad \text{and} \quad Y = \frac{(a^2+b^2)\tan\theta}{2b}$$

$$\therefore \quad \sec\theta = \frac{2aX}{a^2+b^2} \quad \text{and} \quad \tan\theta = \frac{2bY}{a^2+b^2}$$

$$\therefore \quad \left(\frac{2aX}{a^2+b^2}\right)^2 = \left(\frac{2bY}{a^2+b^2}\right)^2 + 1$$

$$\{\sec^2\theta = \tan^2\theta + 1\}$$

$$\therefore \quad \frac{4a^2X^2}{(a^2+b^2)^2} - \frac{4b^2Y^2}{(a^2+b^2)^2} = 1$$

$\therefore$ the locus of M is

$$\frac{4a^2x^2}{(a^2+b^2)^2} - \frac{4b^2y^2}{(a^2+b^2)^2} = 1$$

16 **a** The tangent at $\text{P}(a\sec\theta, b\tan\theta)$ has the equation $bx - (a\sin\theta)y = ab\cos\theta$.

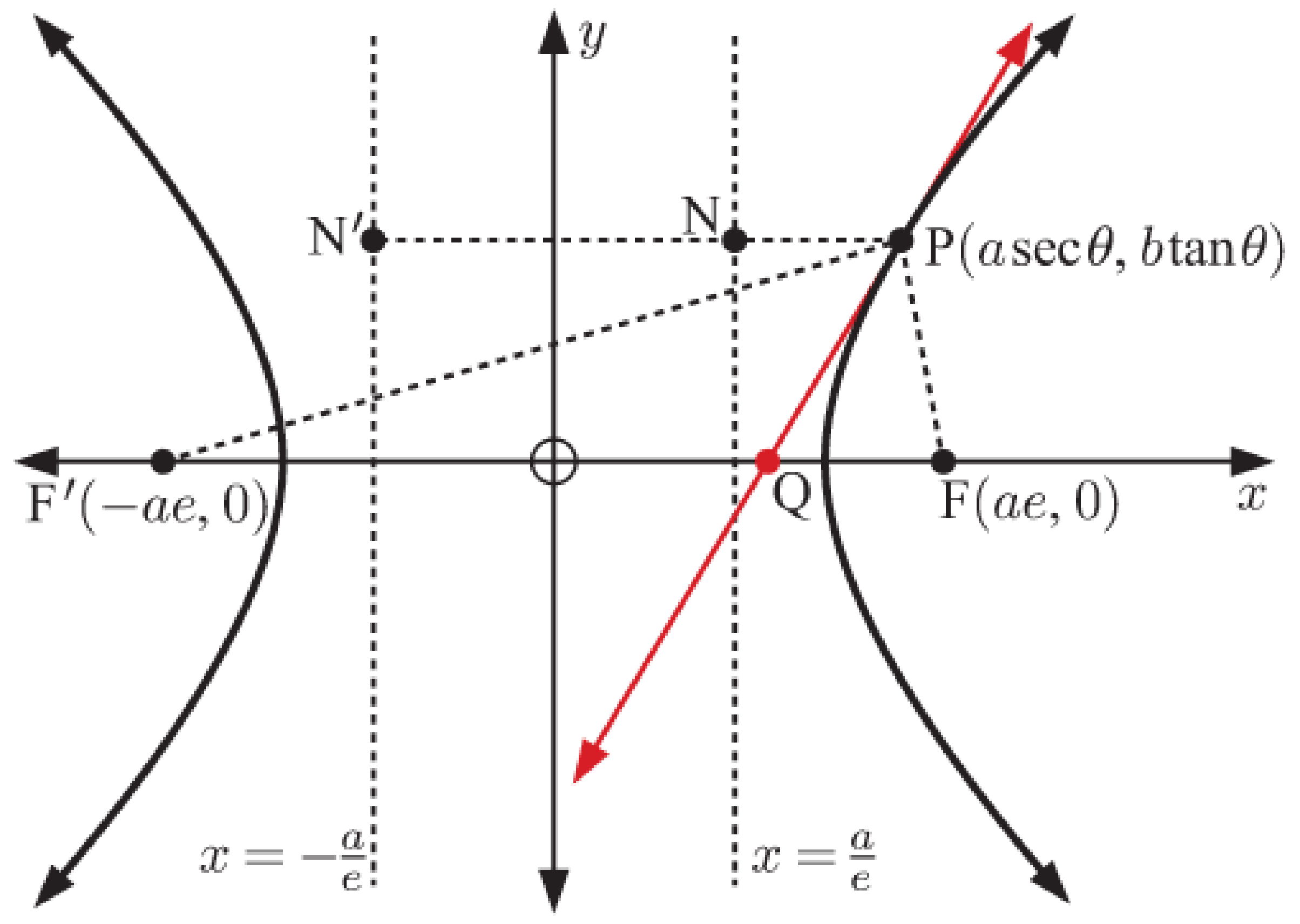
This tangent cuts the x-axis when $y = 0$.

$$\therefore \quad \cancel{b}x = a\cancel{b}\cos\theta$$

$$\therefore \quad x = a\cos\theta$$

$\therefore$ Q is $(a\cos\theta, 0)$.

b



$$\frac{\text{PF}'}{\text{PF}} = \frac{\cancel{e}\text{PN}'}{\cancel{e}\text{PN}} = \frac{a\sec\theta - \left(-\frac{a}{e}\right)}{a\sec\theta - \left(\frac{a}{e}\right)}$$

$$= \left(\frac{\frac{a}{\cos\theta} + \frac{a}{e}}{\frac{a}{\cos\theta} - \frac{a}{e}}\right) \times \frac{e\cos\theta}{e\cos\theta}$$

$$= \frac{ae + a\cos\theta}{ae - a\cos\theta}$$

$$= \frac{\cancel{a}(e+\cos\theta)}{\cancel{a}(e-\cos\theta)}$$

$$= \frac{e+\cos\theta}{e-\cos\theta}$$

Now, $\quad \dfrac{\text{QF}'}{\text{QF}} = \dfrac{a\cos\theta - (-ae)}{ae - a\cos\theta} \quad$ {using **a**}

$$= \frac{a\cos\theta + ae}{ae - a\cos\theta}$$

$$= \frac{\cancel{a}(e+\cos\theta)}{\cancel{a}(e-\cos\theta)}$$

$$= \frac{e+\cos\theta}{e-\cos\theta}$$

$$= \frac{\text{PF}'}{\text{PF}}$$

Thus, $\quad \dfrac{\text{PF}'}{\text{PF}} = \dfrac{\text{QF}'}{\text{QF}} = \dfrac{e+\cos\theta}{e-\cos\theta}.$

c By the converse of the angle bisector theorem, $\widehat{\text{QPF}} = \widehat{\text{QPF}'}$, no matter where P is located on the hyperbola.

17 **a** Let $x = ct$ and $y = \dfrac{c}{t} = ct^{-1}$

$$\therefore \quad \frac{dx}{dt} = c \quad \text{and} \quad \frac{dy}{dt} = -ct^{-2} = -\frac{c}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-\frac{c}{t^2}}{c} = -\frac{1}{t^2}$$

$\therefore$ the equation of the tangent is

$$x + t^2y = (ct) + t^2\left(\frac{c}{t}\right)$$

$$= ct + ct$$

Thus, $\quad x + t^2y = 2ct.$

b From **a**, the tangent has gradient $-\dfrac{1}{t^2}$.

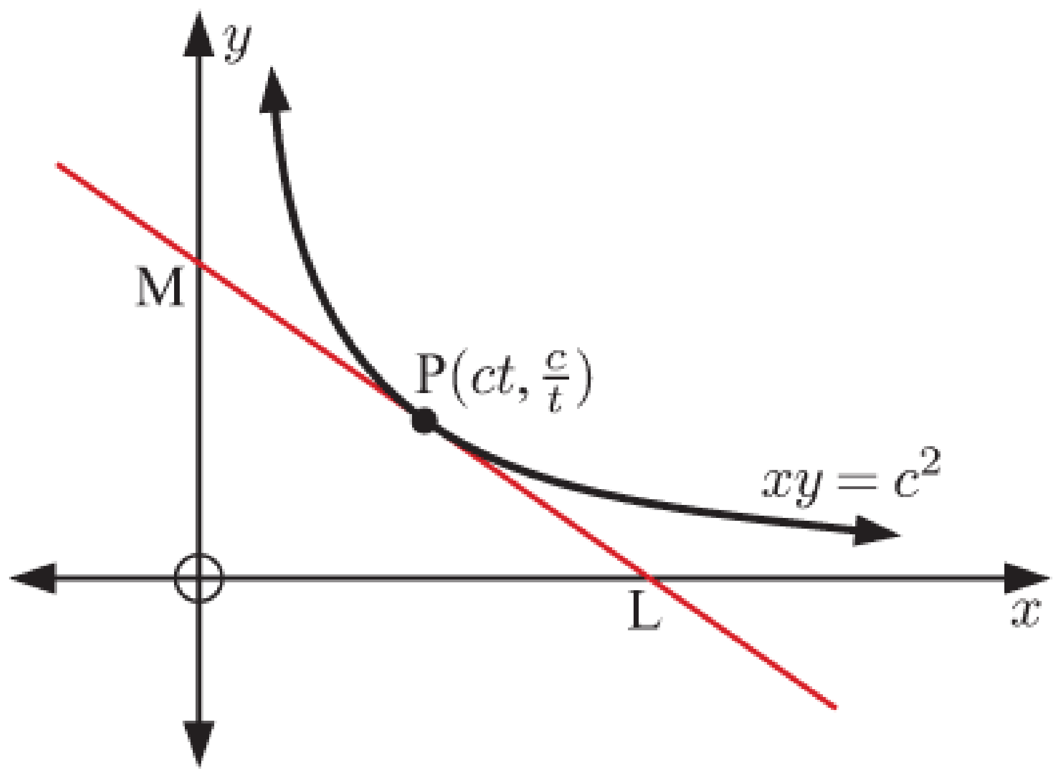
$\therefore$ the normal has gradient t^2.

$\therefore$ the equation of the normal is

$$t^2x - y = t^2(ct) - \left(\frac{c}{t}\right)$$

Thus, $t^2x - y = ct^3 - \dfrac{c}{t}$.

18



Let P be any point on $xy = c^2$.

L and M are the axes intercepts of the tangent at P {since the asymptotes of $xy = c^2$ are the axes}.

The equation of this tangent is $x + t^2y = 2ct$.

Now, when $y = 0$, $x = 2ct$

and when $x = 0$, $t^2y = 2ct$

$$\therefore \quad y = \frac{2c}{t}$$

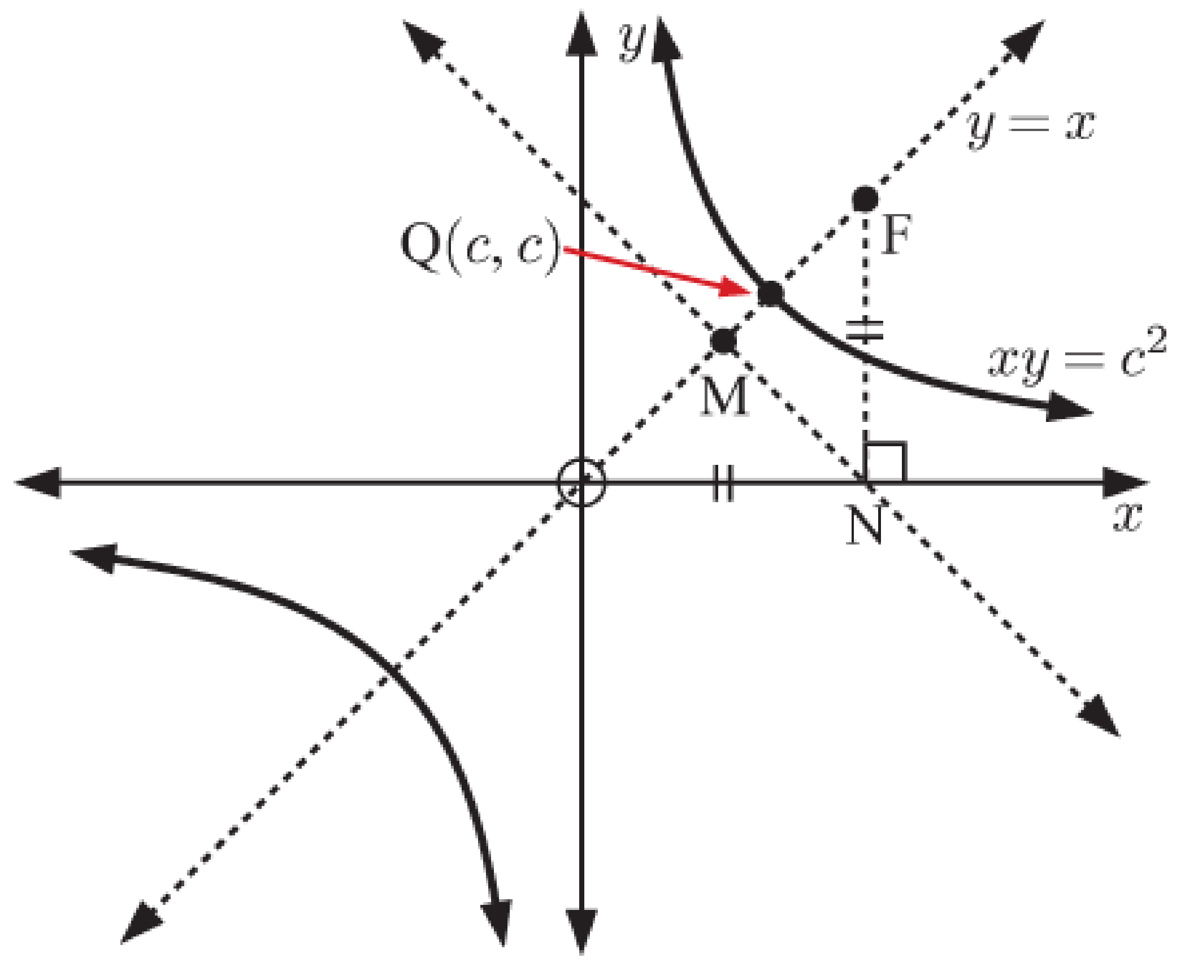
$\therefore$ L is $(2ct, 0)$ and M is $\left(0, \dfrac{2c}{t}\right)$.

The midpoint of [LM] is $\left(\dfrac{2ct + 0}{2}, \dfrac{0 + \frac{2c}{t}}{2}\right) = \left(ct, \dfrac{c}{t}\right)$.

So, the midpoint of [LM] is P.

So, irrespective of the position of point P on $xy = c^2$, P is the midpoint of the tangent [LM] to $xy = c^2$.

19



Since this is a rectangular hyperbola, $e = \sqrt{2}$.

$\therefore$ $OF = ae = a\sqrt{2}$

$\therefore$ in $\triangle ONF$, $ON = NF = a$ {Pythagoras}

$\therefore$ the focus is at (a, a).

But $OQ = a = \sqrt{c^2 + c^2}$ {since Q is (c, c)}

$$\therefore \quad a = c\sqrt{2}$$

$\therefore$ the foci are $(c\sqrt{2}, c\sqrt{2})$ and $(-c\sqrt{2}, -c\sqrt{2})$.

Let the equation of one directrix be $x + y + k = 0$ for some k.

$OM = \dfrac{|1(0) + 1(0) + k|}{\sqrt{1+1}}$ {distance from a point to a line}

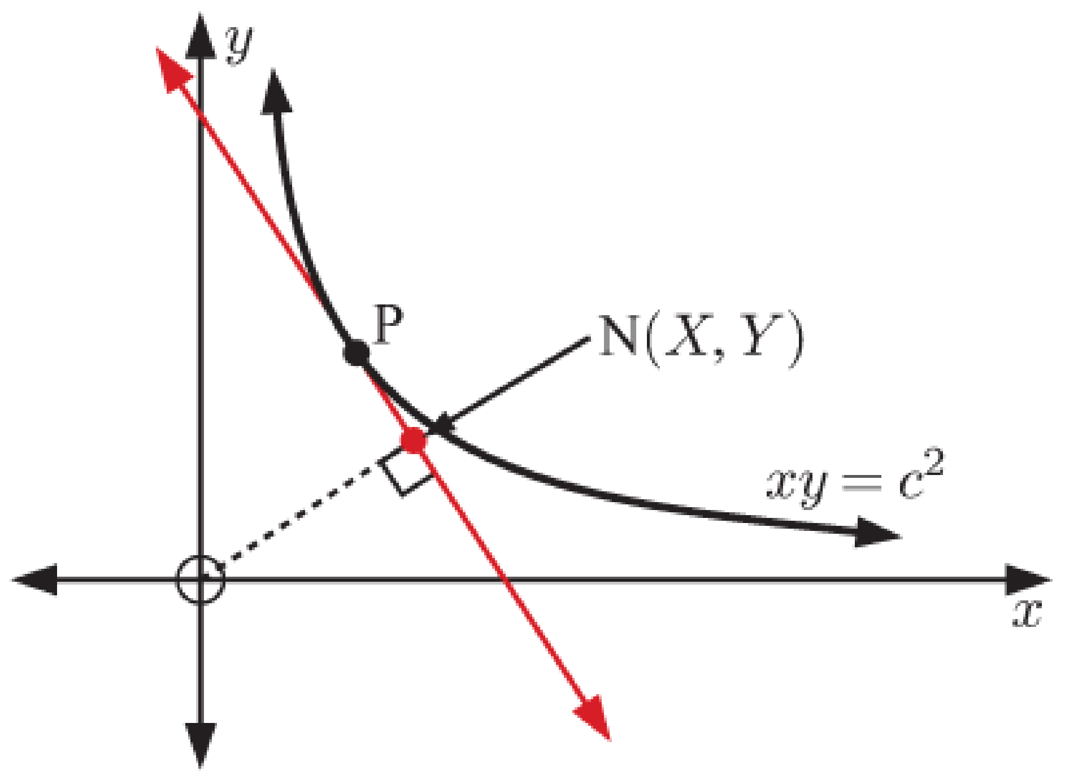
Since $OM = \dfrac{a}{e} = \dfrac{a}{\sqrt{2}}$, $\dfrac{|k|}{\sqrt{2}} = \dfrac{a}{\sqrt{2}}$

$$\therefore \quad |k| = a$$

$$\therefore \quad k = \pm a = \pm c\sqrt{2}$$

$\therefore$ the directrices are $x + y = \pm c\sqrt{2}$.

20



Consider the point P on the curve $xy = c^2$.

We assume $c > 0$.

The tangent at P has equation $x + t^2y = 2ct$ and has gradient $-\dfrac{1}{t^2}$.

$\therefore$ [ON] has gradient t^2

and its equation is $Y = t^2X$

$$\therefore \quad t^2 = \frac{Y}{X}$$

$\therefore$ at N, $X + \left(\dfrac{Y}{X}\right)Y = 2ct$ {since N lies on the tangent}

$$\therefore \quad X + \frac{Y^2}{X} = 2ct$$

$$\therefore \quad X^2 + Y^2 = 2cXt$$

We now consider Xt when t is positive or negative:

Case 1: If $t > 0$, P is in quadrant 1, and $X > 0$

$$\therefore \quad t = \sqrt{\frac{Y}{X}} \quad \text{and} \quad X = \sqrt{X^2}$$

$$\therefore \quad Xt = \sqrt{X^2}\sqrt{\frac{Y}{X}} = \sqrt{XY}$$

Case 2: If $t < 0$, P is in quadrant 3, and $X < 0$.

$$\therefore \quad t = -\sqrt{\frac{Y}{X}} \quad \text{and} \quad X = -\sqrt{X^2}$$

$$\therefore \quad Xt = -\sqrt{X^2}\left(-\sqrt{\frac{Y}{X}}\right) = \sqrt{XY}$$

In either case, we have $X^2 + Y^2 = 2c\sqrt{XY}$.

Thus, the equation of the locus of N is $x^2 + y^2 = 2c\sqrt{xy}$.

21 The normal at $\left(ct, \dfrac{c}{t}\right)$ has equation $t^2x - y = ct^3 - \dfrac{c}{t}$.

This normal cuts the x-axis at $A\left(ct - \dfrac{c}{t^3}, 0\right)$ and the y-axis at $B\left(0, -ct^3 + \dfrac{c}{t}\right)$.

If $M(X, Y)$ is the midpoint of [AB],

then $X = \dfrac{ct - \frac{c}{t^3}}{2}$, $Y = \dfrac{-ct^3 + \frac{c}{t}}{2}$

$\therefore$ $ct - \dfrac{c}{t^3} = 2X$, $-ct^3 + \dfrac{c}{t} = 2Y$

$$\therefore \quad ct^4 - c = 2t^3X \quad \text{and}$$
$$-ct^4 + c = 2tY$$
$$\therefore \quad 0 = 2t^3X + 2tY \quad \{\text{adding}\}$$
$$\therefore \quad -2tY = 2t^3X$$
$$\therefore \quad Y = -t^2X$$
$$\therefore \quad t^2 = -\frac{Y}{X} \quad .... \ (1)$$

But M is on $t^2x - y = ct^3 - \frac{c}{t}$

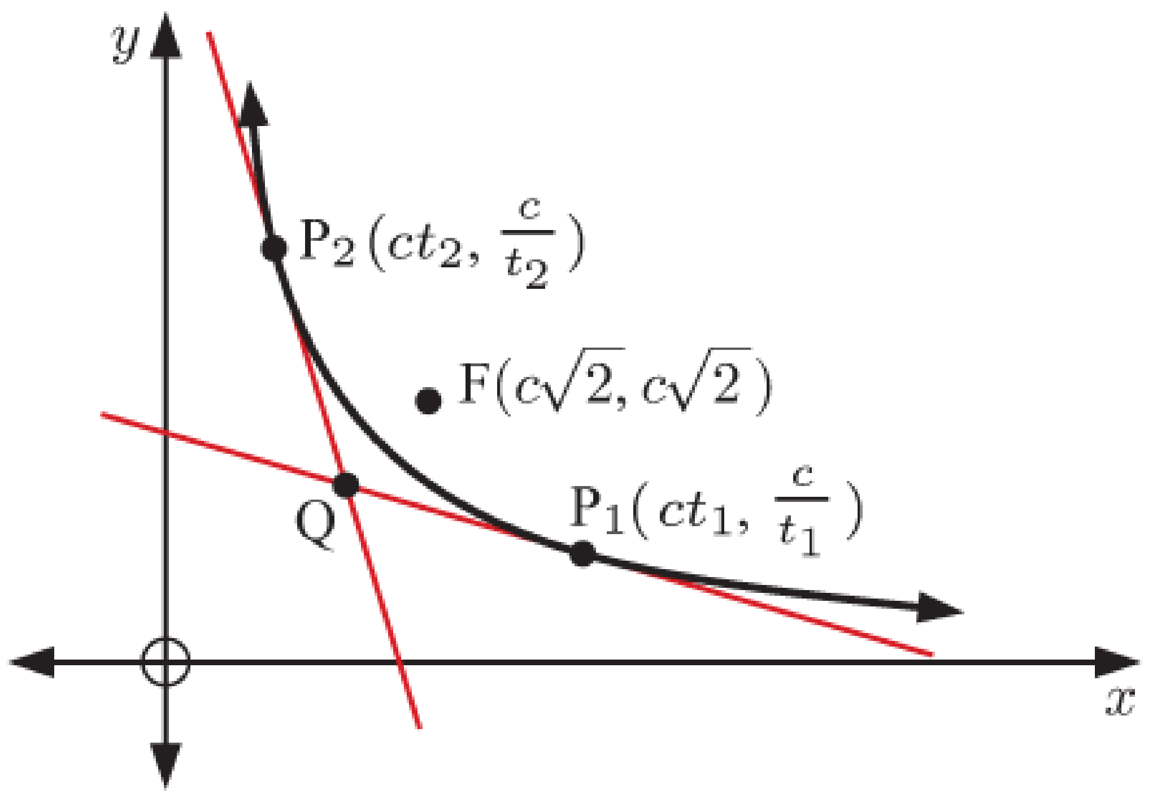
$$\therefore \quad t^2X - Y = \frac{c(t^4-1)}{t}$$

and from (1), $\left(-\frac{Y}{X}\right)X - Y = c\left(\frac{\frac{Y^2}{X^2}-1}{t}\right)$

$$\therefore \quad -2Y = \frac{c(Y^2-X^2)}{t \times X^2}$$
$$\therefore \quad t \times -2X^2Y = c(Y^2-X^2)$$
$$\therefore \quad t^2 \times 4X^4Y^2 = c^2(Y^2-X^2)^2$$
$$\therefore \quad -\frac{Y}{X} \times 4X^4Y^2 = c^2(Y^2-X^2)^2$$
$$\therefore \quad c^2(Y^2-X^2)^2 = -4X^3Y^3$$

$\therefore$ the Cartesian equation for the locus of M is $c^2(y^2-x^2)^2 = -4x^3y^3$.

22



y
P₂ $(ct_2, \frac{c}{t_2})$
F$(c\sqrt{2}, c\sqrt{2})$
Q
P₁ $(ct_1, \frac{c}{t_1})$
x

The tangents at P_1 and P_2 are

$x + {t_1}^2y = 2ct_1$ and $x + {t_2}^2y = 2ct_2$

If we let Q be (X, Y) then $X + {t_1}^2Y = 2ct_1$ .... (1)

and $X + {t_2}^2Y = 2ct_2$ .... (2)

We now solve (1) and (2):

$$2ct_1 - {t_1}^2Y = 2ct_2 - {t_2}^2Y \quad \{\text{equating } X\text{s}\}$$
$$\therefore \quad 2c(t_1-t_2) = Y({t_1}^2 - {t_2}^2)$$
$$\therefore \quad 2c\cancel{(t_1-t_2)} = Y(t_1+t_2)\cancel{(t_1-t_2)}$$
$$\therefore \quad Y = \frac{2c}{t_1+t_2}$$

So,
$$X = 2ct_1 - {t_1}^2Y$$
$$= 2ct_1 - {t_1}^2\left(\frac{2c}{t_1+t_2}\right)$$
$$= \frac{2ct_1(t_1+t_2) - {t_1}^2(2c)}{t_1+t_2}$$
$$= \frac{\cancel{2c{t_1}^2} + 2ct_1t_2 - \cancel{2c{t_1}^2}}{t_1+t_2}$$
$$\therefore \quad X = \frac{2ct_1t_2}{t_1+t_2}$$

Now, $[P_1P_2]$ has gradient
$$\frac{\frac{c}{t_1} - \frac{c}{t_2}}{ct_1 - ct_2}$$
$$= \frac{ct_2 - ct_1}{t_1t_2(ct_1-ct_2)}$$
$$= \frac{-\cancel{c}\cancel{(t_1-t_2)}}{\cancel{c}t_1t_2\cancel{(t_1-t_2)}}$$
$$= -\frac{1}{t_1t_2}$$

and $[P_1P_2]$ has the equation

$$x + t_1t_2y = (ct_1) + t_1t_2\left(\frac{c}{t_1}\right)$$
$$\therefore \quad x + t_1t_2y = ct_1 + ct_2$$
$$\therefore \quad x + t_1t_2y = c(t_1+t_2)$$

Since $[P_1P_2]$ is a focal chord, $F(c\sqrt{2}, c\sqrt{2})$ lies on $[P_1P_2]$.

$$\therefore \quad (c\sqrt{2}) + t_1t_2(c\sqrt{2}) = c(t_1+t_2)$$
$$\therefore \quad \frac{\cancel{c}\sqrt{2}}{\cancel{c}(t_1+t_2)} + \frac{\cancel{c}t_1t_2\sqrt{2}}{\cancel{c}(t_1+t_2)} = 1$$
$$\therefore \quad \left(\frac{Y}{2c}\right)\sqrt{2} + \left(\frac{X}{2c}\right)\sqrt{2} = 1 \quad \left\{Y = \frac{2c}{t_1+t_2} \text{ and } X = \frac{2ct_1t_2}{t_1+t_2}\right\}$$
$$\therefore \quad \frac{Y}{c\sqrt{2}} + \frac{X}{c\sqrt{2}} = 1$$
$$\therefore \quad X + Y = c\sqrt{2}$$

$\therefore$ the equation of the locus of Q is $x + y = c\sqrt{2}$, which is a straight line.

EXERCISE 2Q

1 **a** $6x^2 - 4xy + 9y^2 = 80$ can be written as

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 80$$

which has the form $\mathbf{x}^T\mathbf{A}\mathbf{x} = 80$

Now
$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda-6 & 2 \\ 2 & \lambda-9 \end{vmatrix}$$
$$= \lambda^2 - 15\lambda + 50$$
$$= (\lambda-5)(\lambda-10)$$

$\therefore$ $\lambda = 5, 10$ are the eigenvalues of $\mathbf{A}$.

When $\lambda = 5$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \quad \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \quad -x + 2y = 0$$
$$\therefore \quad \mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}t, \quad t \in \mathbb{R}$$

When $\lambda = 10$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \quad \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \quad 2x + y = 0$$
$$\therefore \quad \mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}t, \quad t \in \mathbb{R}$$

The normalised eigenvectors are $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Let $\mathbf{P} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ where $|\mathbf{P}| = 1$

{since $|\mathbf{P}| = 1$ for a rotation}

Now, $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$, so $\tan\theta = -2$.

From the column order we have $\lambda_1 = 10$ and $\lambda_2 = 5$.

Let $\mathbf{x} = \mathbf{P}\mathbf{x}'$

$$\therefore\ (\mathbf{P}\mathbf{x}')^T\mathbf{A}(\mathbf{P}\mathbf{x}') = 80$$

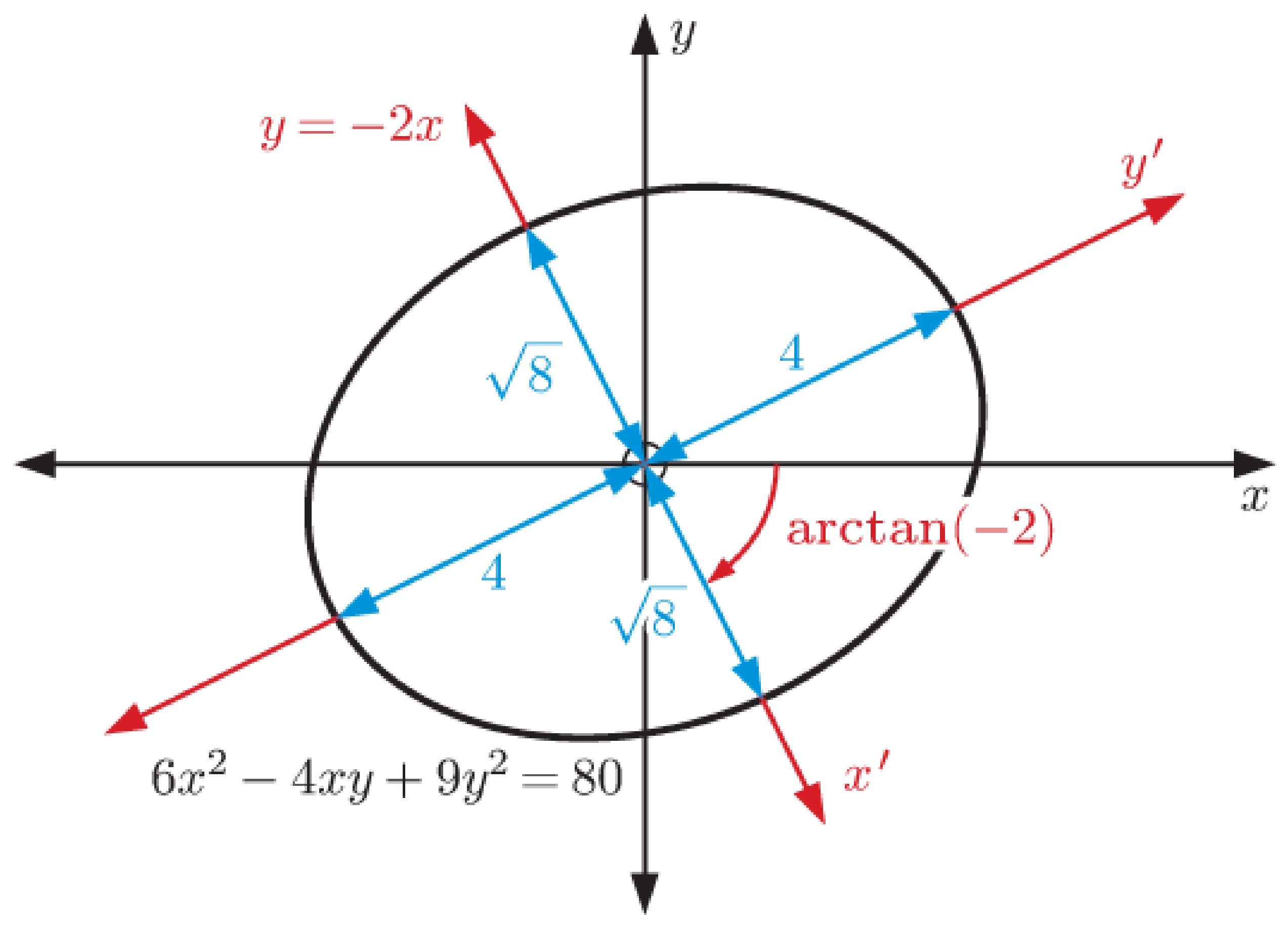
$$\therefore\ \mathbf{x}'^{\,T}\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{x}' = 80$$

$$\therefore\ \mathbf{x}'^{\,T}\begin{pmatrix}10 & 0\\ 0 & 5\end{pmatrix}\mathbf{x}' = 80$$

$$\therefore\ 10x'^{\,2} + 5y'^{\,2} = 80$$

$$\therefore\ \frac{x'^{\,2}}{8} + \frac{y'^{\,2}}{16} = 1 \quad \text{which is an } \textbf{ellipse}$$

∴ the conic is an ellipse with centre $(0, 0)$, rotated through $\arctan(-2)$.



b $8x^2 + 28xy - 13y^2 + 40 = 0$ can be written as

$$\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}8 & 14\\ 14 & -13\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = -40$$

which has the form $\mathbf{x}^T\mathbf{A}\mathbf{x} = -40$

$$\text{Now}\quad |\,\lambda\mathbf{I} - \mathbf{A}\,| = \begin{vmatrix}\lambda - 8 & -14\\ -14 & \lambda + 13\end{vmatrix}$$

$$= \lambda^2 + 5\lambda - 300$$

$$= (\lambda - 15)(\lambda + 20)$$

∴ $\lambda = 15, -20$ are the eigenvalues of $\mathbf{A}$.

When $\lambda = 15$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore\ \begin{pmatrix}7 & -14\\ -14 & 28\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\therefore\ 7x - 14y = 0 \quad (\text{or } x = 2y)$$

$$\therefore\ \mathbf{x} = \begin{pmatrix}2\\ 1\end{pmatrix}t, \quad t \in \mathbb{R}$$

When $\lambda = -20$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore\ \begin{pmatrix}-28 & -14\\ -14 & -7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\therefore\ 14x + 7y = 0 \quad (\text{or } y = -2x)$$

$$\therefore\ \mathbf{x} = \begin{pmatrix}1\\ -2\end{pmatrix}t, \quad t \in \mathbb{R}$$

The normalised eigenvectors are $\frac{1}{\sqrt{5}}\begin{pmatrix}2\\ 1\end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\ -2\end{pmatrix}$.

Let $\mathbf{P} = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$ where $|\,\mathbf{P}\,| = 1$

{since $|\,\mathbf{P}\,| = 1$ for a rotation}

Now, $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$

so $\tan\theta = -2$

From the column order we have $\lambda_1 = -20$ and $\lambda_2 = 15$.

Let $\mathbf{x} = \mathbf{P}\mathbf{x}'$

$$\therefore\ (\mathbf{P}\mathbf{x}')^T\mathbf{A}(\mathbf{P}\mathbf{x}') = -40$$

$$\therefore\ \mathbf{x}'^{\,T}\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{x}' = -40$$

$$\therefore\ \mathbf{x}'^{\,T}\begin{pmatrix}-20 & 0\\ 0 & 15\end{pmatrix}\mathbf{x}' = -40$$

$$\therefore\ -20x'^{\,2} + 15y'^{\,2} = -40$$

$$\therefore\ \frac{x'^{\,2}}{2} - \frac{3y'^{\,2}}{8} = 1$$

$$\therefore\ \frac{x'^{\,2}}{2} - \frac{y'^{\,2}}{\left(\frac{8}{3}\right)} = 1$$

∴ the conic is a hyperbola.

The graph cuts the x'-axis when $y' = 0$

∴ $x'^{\,2} = 2$, $x' = \pm\sqrt{2}$

∴ the vertices for the rotated graph are

$$\left(\frac{\sqrt{2}}{\sqrt{5}}, -\frac{2\sqrt{2}}{\sqrt{5}}\right) \quad \text{and} \quad \left(-\frac{\sqrt{2}}{\sqrt{5}}, \frac{2\sqrt{2}}{\sqrt{5}}\right).$$

The graph cuts the x-axis when $y = 0$

$$\therefore\ 8x^2 = -40$$

$$\therefore\ x^2 = -5$$

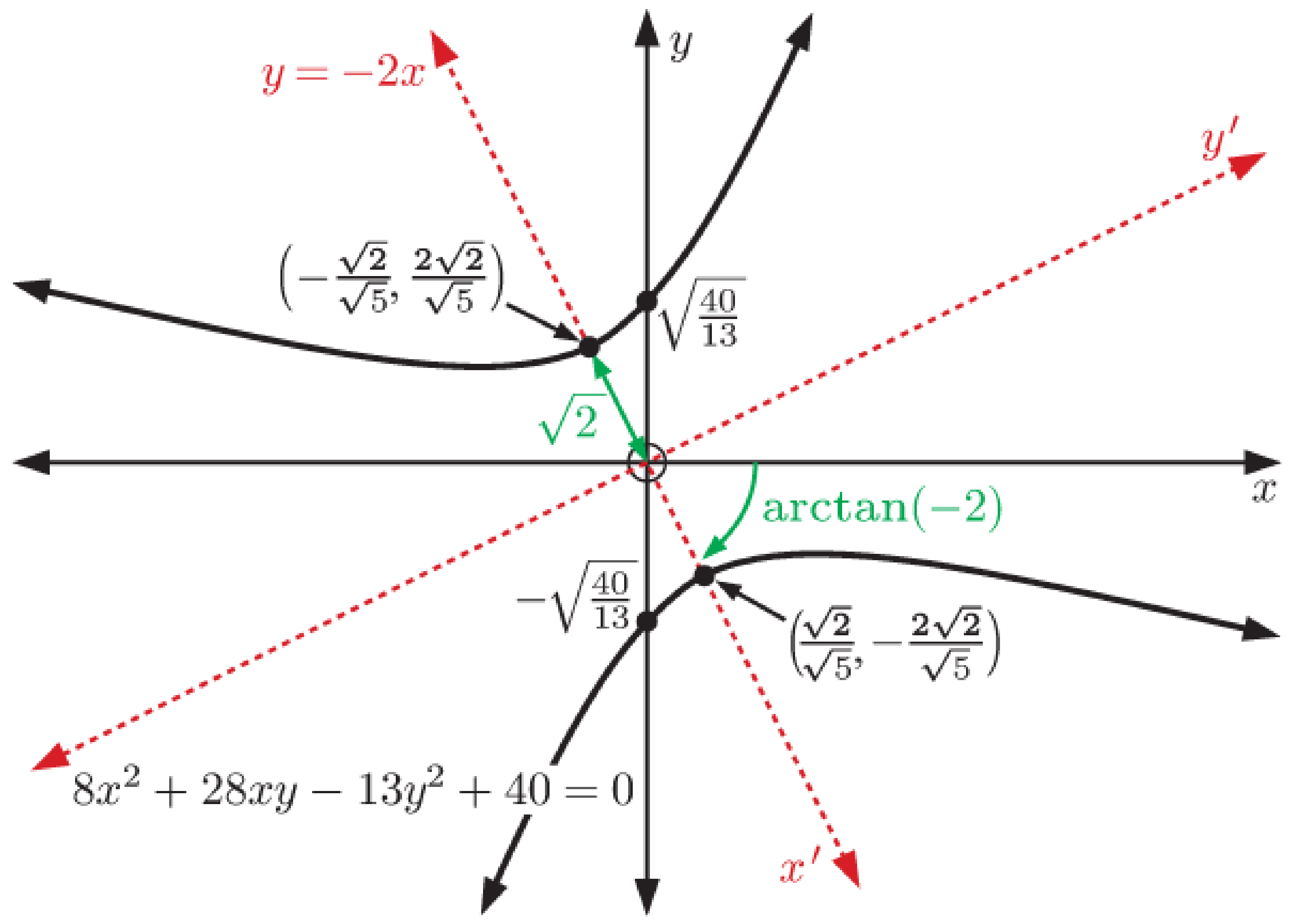
∴ the graph does not cut the x-axis.

The graph cuts the y-axis when $x = 0$

$$\therefore\ -13y^2 = -40$$

$$\therefore\ y^2 = \tfrac{40}{13}$$

$$\therefore\ y = \pm\sqrt{\tfrac{40}{13}}$$


$y = -2x$
y
y'
$\left(-\frac{\sqrt{2}}{\sqrt{5}}, \frac{2\sqrt{2}}{\sqrt{5}}\right)$
$\sqrt{\frac{40}{13}}$
$\sqrt{2}$
arctan(−2)
x
$-\sqrt{\frac{40}{13}}$
$\left(\frac{\sqrt{2}}{\sqrt{5}}, -\frac{2\sqrt{2}}{\sqrt{5}}\right)$
$8x^2 + 28xy - 13y^2 + 40 = 0$
x'

2 **a** $x^2 - xy + y^2 - 2x + y - 3 = 0$

To avoid working with fractions, we consider $2x^2 - 2xy + 2y^2 - 4x + 2y = 6$.

This equation can be written in the form $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{v}^T\mathbf{x} = 6$ where

$$\mathbf{x} = \begin{pmatrix}x\\ y\end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{pmatrix}-4\\ 2\end{pmatrix}.$$

$$\text{Now}\quad |\,\lambda\mathbf{I} - \mathbf{A}\,| = \begin{vmatrix}\lambda - 2 & 1\\ 1 & \lambda - 2\end{vmatrix}$$

$$= \lambda^2 - 4\lambda + 3$$

$$= (\lambda - 1)(\lambda - 3)$$

∴ $\lambda = 1, 3$ are the eigenvalues of $\mathbf{A}$.

When $\lambda = 1$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore x - y = 0$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t, \quad t \in \mathbb{R}$$

When $\lambda = 3$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore x + y = 0$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t, \quad t \in \mathbb{R}$$

The normalised eigenvectors are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Let $\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ where $|\,\mathbf{P}\,| = 1$.

Now, $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$, so $\theta = -\frac{\pi}{4}$.

From the column order we have $\lambda_1 = 3$ and $\lambda_2 = 1$.

Let $\mathbf{x} = \mathbf{P}\mathbf{x}'$

$$\therefore (\mathbf{P}\mathbf{x}')^T\mathbf{A}(\mathbf{P}\mathbf{x}') + \begin{pmatrix} -4 & 2 \end{pmatrix}\mathbf{P}\mathbf{x}' = 6$$

$$\therefore \mathbf{x}'^{\,T}\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{x}' + \begin{pmatrix} -4 & 2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 6$$

$$\therefore \mathbf{x}'^{\,T}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}' + \frac{1}{\sqrt{2}}\begin{pmatrix} -6 & -2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 6$$

$$\therefore 3x'^{\,2} + y'^{\,2} + \begin{pmatrix} -3\sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 6$$

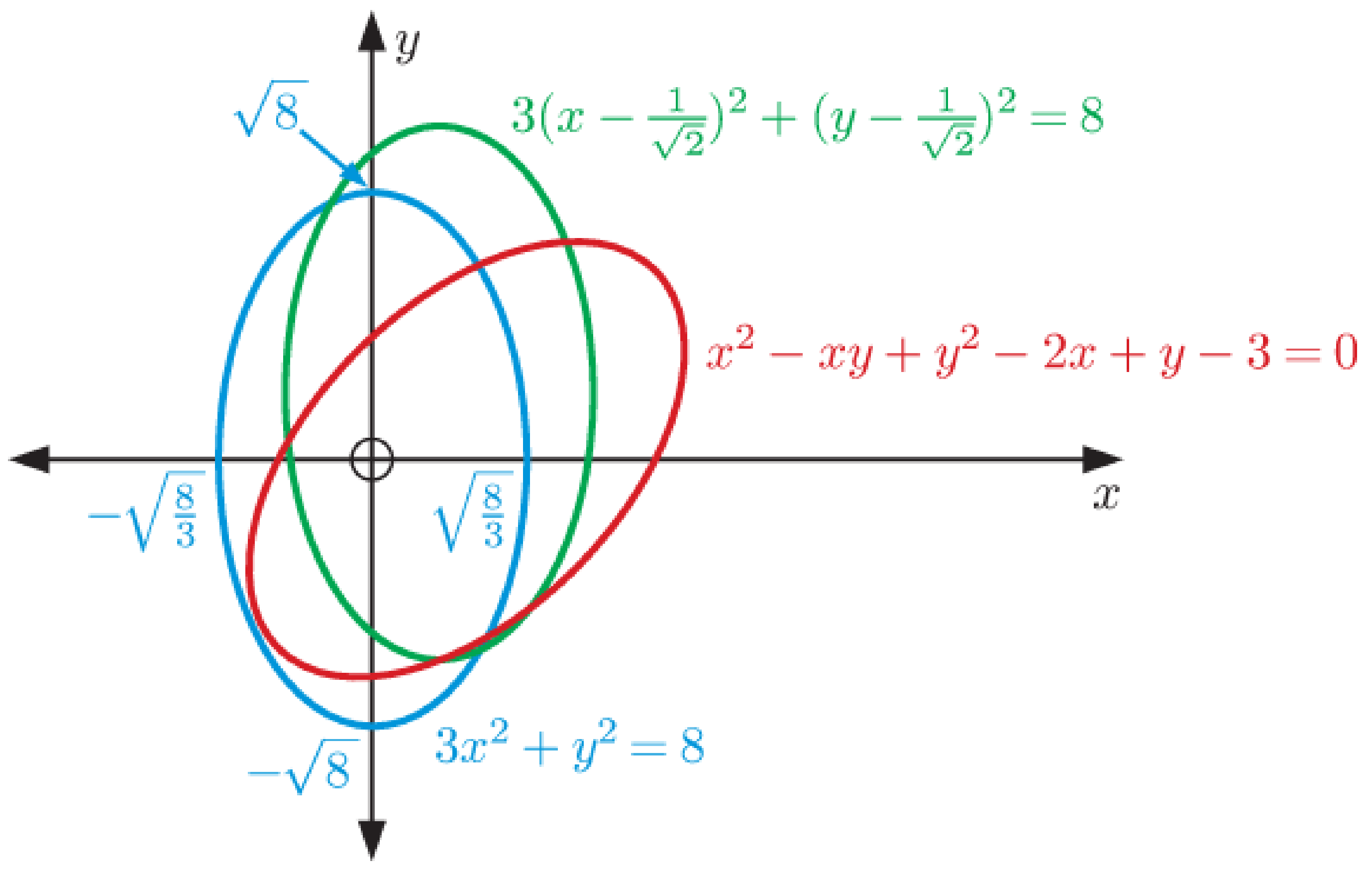
$$\therefore 3x'^{\,2} + y'^{\,2} - 3\sqrt{2}x' - \sqrt{2}y' = 6$$

$$\therefore \quad 3(x'^{\,2} - \sqrt{2}x' + (\tfrac{\sqrt{2}}{2})^2) + (y'^{\,2} - \sqrt{2}y' + (\tfrac{\sqrt{2}}{2})^2)$$
$$= 6 + 3(\tfrac{2}{4}) + \tfrac{2}{4}$$
$$= 8$$

$$\therefore 3(x' - \tfrac{1}{\sqrt{2}})^2 + (y' - \tfrac{1}{\sqrt{2}})^2 = 8$$

which is $3X^2 + Y^2 = 8$ translated $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

$\therefore$ the original conic is an ellipse.



b $x^2 + 4xy - 2y^2 + 2\sqrt{5}x - \sqrt{5}y - 5 = 0$

$\therefore \quad x^2 + 4xy - 2y^2 + 2\sqrt{5}x - \sqrt{5}y = 5$

This equation can be written in the form

$\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{v}^T\mathbf{x} = 5$ where

$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} 2\sqrt{5} \\ -\sqrt{5} \end{pmatrix}$.

Now $|\,\lambda\mathbf{I} - \mathbf{A}\,| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda + 2 \end{vmatrix}$

$$= \lambda^2 + \lambda - 6$$
$$= (\lambda - 2)(\lambda + 3)$$

$\therefore$ $\lambda = 2, -3$ are the eigenvalues of $\mathbf{A}$.

When $\lambda = 2$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore x - 2y = 0$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t, \quad t \in \mathbb{R}$$

When $\lambda = -3$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -2x - y = 0$$

$$\therefore y = -2x$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t, \quad t \in \mathbb{R}$$

The normalised eigenvectors are $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Let $\mathbf{P} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ where $|\,\mathbf{P}\,| = 1$ for a rotation.

$\lambda_1 = -3$ and $\lambda_2 = 2$.

$\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$, so $\tan\theta = -2$.

Let $\mathbf{x} = \mathbf{P}\mathbf{x}'$

$$\therefore (\mathbf{P}\mathbf{x}')^T\mathbf{A}(\mathbf{P}\mathbf{x}') + \mathbf{v}^T\mathbf{P}\mathbf{x}' = 5$$

$$\therefore \mathbf{x}'^{\,T}\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{x}' + \begin{pmatrix} 2\sqrt{5} & -\sqrt{5} \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 5$$

$$\therefore \mathbf{x}'^{\,T}\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x}' + \begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 5$$

$$\therefore \mathbf{x}'^{\,T}\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 5$$

$$\therefore -3x'^{\,2} + 2y'^{\,2} + 4x' + 3y' = 5$$

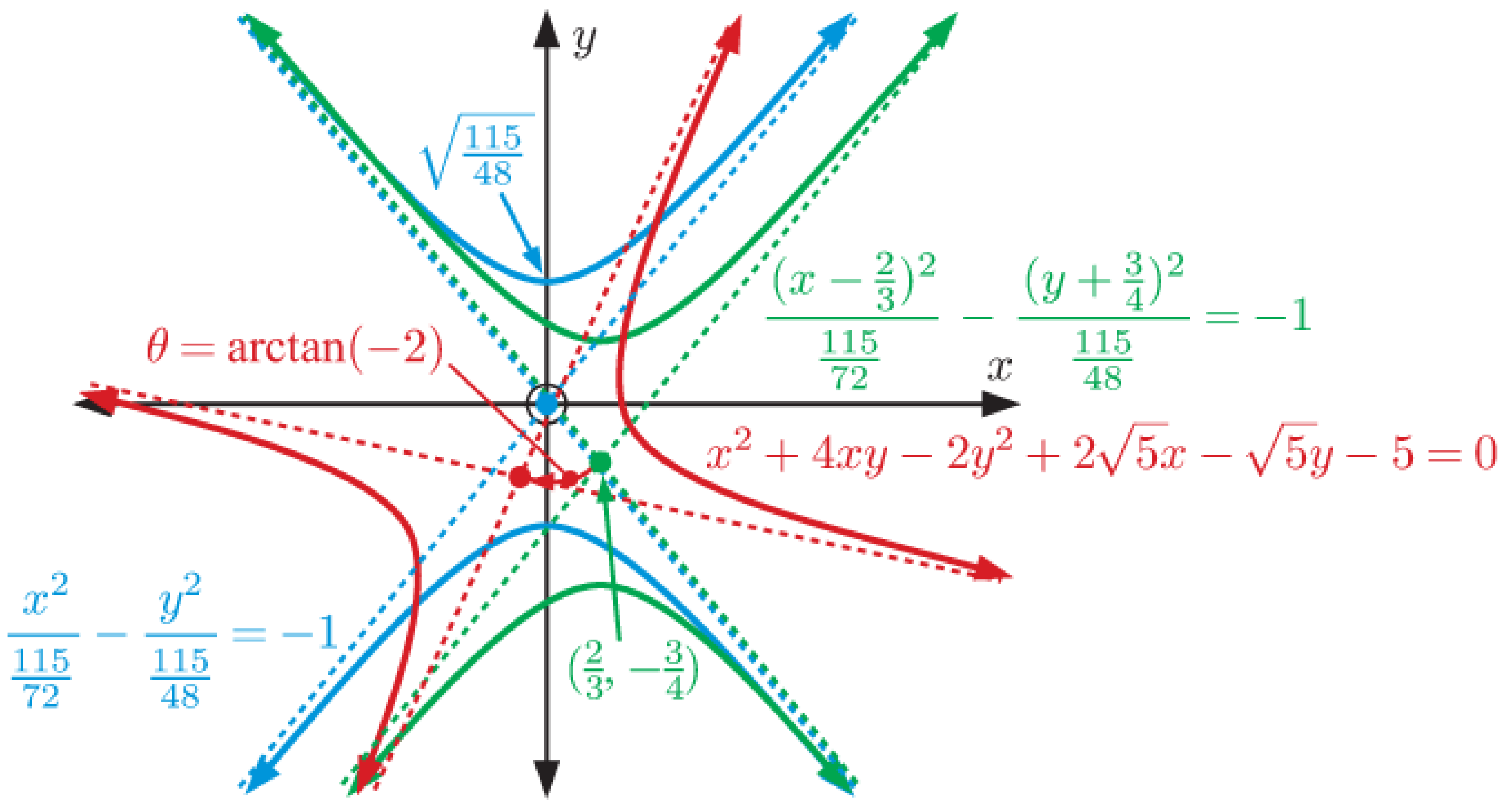
$$\therefore \quad -3(x'^{\,2} - \tfrac{4}{3}x' + (\tfrac{2}{3})^2) + 2(y'^{\,2} + \tfrac{3}{2}y' + (\tfrac{3}{4})^2)$$
$$= 5 - 3(\tfrac{4}{9}) + 2(\tfrac{9}{16})$$
$$= \tfrac{115}{24}$$

$$\therefore -3(x' - \tfrac{2}{3})^2 + 2(y' + \tfrac{3}{4})^2 = \tfrac{115}{24}$$

$$\therefore \frac{(x' - \frac{2}{3})^2}{(\frac{115}{72})} - \frac{(y' + \frac{3}{4})^2}{(\frac{115}{48})} = -1$$

which is $\dfrac{X^2}{(\frac{115}{72})} - \dfrac{Y^2}{(\frac{115}{48})} = -1$ translated $\begin{pmatrix} \frac{2}{3} \\ -\frac{3}{4} \end{pmatrix}$.

$\therefore$ the original conic is a hyperbola.



c $3x^2 - 6xy - 5y^2 + 3x + 9y = 10$

This equation can be written in the form

$\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{v}^T\mathbf{x} = 10$ where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 3 & -3 \\ -3 & -5 \end{pmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}.$$

$$\text{Now} \quad |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 3 & 3 \\ 3 & \lambda + 5 \end{vmatrix}$$
$$= \lambda^2 + 2\lambda - 24$$
$$= (\lambda - 4)(\lambda + 6)$$

$\therefore \quad \lambda = 4, -6$ are the eigenvalues of $\mathbf{A}$.

When $\lambda = 4$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \quad \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \quad x + 3y = 0$$
$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}t, \quad t \in \mathbb{R}$$

When $\lambda = -6$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \quad \begin{pmatrix} -9 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \quad -3x + y = 0$$
$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}t, \quad t \in \mathbb{R}$$

The normalised eigenvectors are $\frac{1}{\sqrt{10}}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Let $\mathbf{P} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$ where $|\mathbf{P}| = 1$ for a rotation.

$\lambda_1 = -6$ and $\lambda_2 = 4$

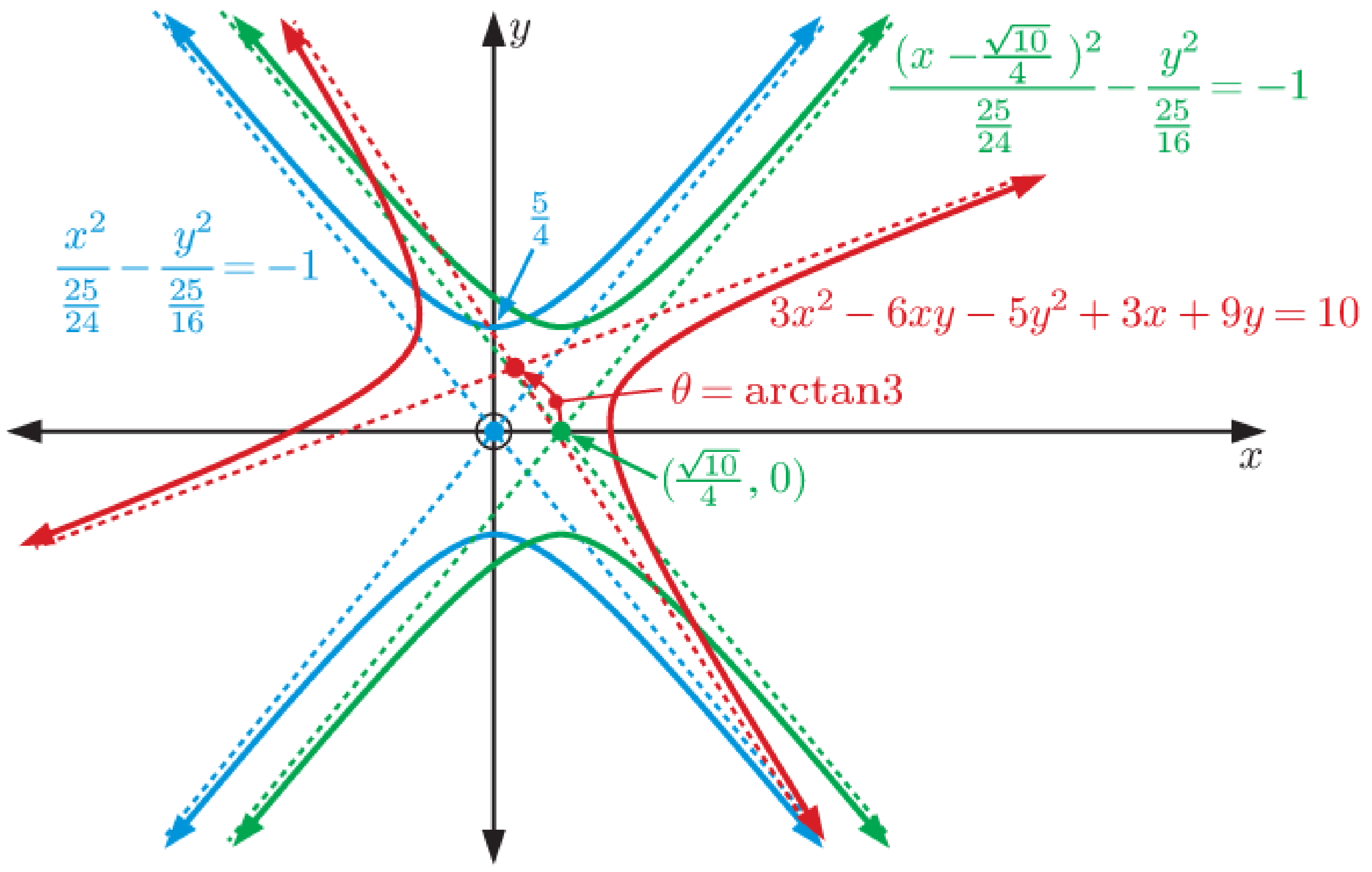
$\cos\theta = \frac{1}{\sqrt{10}}$ and $\sin\theta = \frac{3}{\sqrt{10}}$, so $\tan\theta = 3$

Let $\mathbf{x} = \mathbf{P}\mathbf{x}'$

$$\therefore \quad (\mathbf{P}\mathbf{x}')^T\mathbf{A}(\mathbf{P}\mathbf{x}') + \mathbf{v}^T\mathbf{P}\mathbf{x}' = 10$$
$$\therefore \quad \mathbf{x}'^{\,T}\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{x}' + \begin{pmatrix} 3 & 9 \end{pmatrix}\frac{1}{\sqrt{10}}\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 10$$
$$\therefore \quad \mathbf{x}'^{\,T}\begin{pmatrix} -6 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{x}' + \frac{1}{\sqrt{10}}\begin{pmatrix} 30 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 10$$
$$\therefore \quad -6x'^{\,2} + 4y'^{\,2} + 3\sqrt{10}x' = 10$$
$$\therefore \quad 6x'^{\,2} - 4y'^{\,2} - 3\sqrt{10}x' = -10$$
$$\therefore \quad 6(x'^{\,2} - \tfrac{\sqrt{10}}{2}x' + (\tfrac{\sqrt{10}}{4})^2) - 4y'^{\,2} = -10 + 6(\tfrac{\sqrt{10}}{4})^2$$
$$\therefore \quad 6(x' - \tfrac{\sqrt{10}}{4})^2 - 4y'^{\,2} = -\tfrac{25}{4}$$
$$\therefore \quad \frac{(x' - \frac{\sqrt{10}}{4})^2}{(\frac{25}{24})} - \frac{y'^{\,2}}{(\frac{25}{16})} = -1$$

which is $\dfrac{X^2}{(\frac{25}{24})} - \dfrac{Y^2}{(\frac{25}{16})} = -1$ translated $\begin{pmatrix} \frac{\sqrt{10}}{4} \\ 0 \end{pmatrix}$.

$\therefore$ the original conic is a hyperbola.



d $2x^2 - 4xy + 5y^2 + 4x - 2y = 1$ can be written in the form

$\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{v}^T\mathbf{x} = 1$ where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

$$\text{Now} \quad |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & 2 \\ 2 & \lambda - 5 \end{vmatrix}$$
$$= \lambda^2 - 7\lambda + 6$$
$$= (\lambda - 1)(\lambda - 6)$$

$\therefore \quad \lambda = 1, 6$ are the eigenvalues of $\mathbf{A}$.

When $\lambda = 1$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \quad \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \quad -x + 2y = 0$$
$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}t, \quad t \in \mathbb{R}$$

When $\lambda = 6$, $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

$$\therefore \quad \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\therefore \quad 2x + y = 0$$
$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}t, \quad t \in \mathbb{R}$$

The normalised eigenvectors are $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Let $\mathbf{P} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ where $|\mathbf{P}| = 1$.

$\lambda_1 = 6$ and $\lambda_2 = 1$.

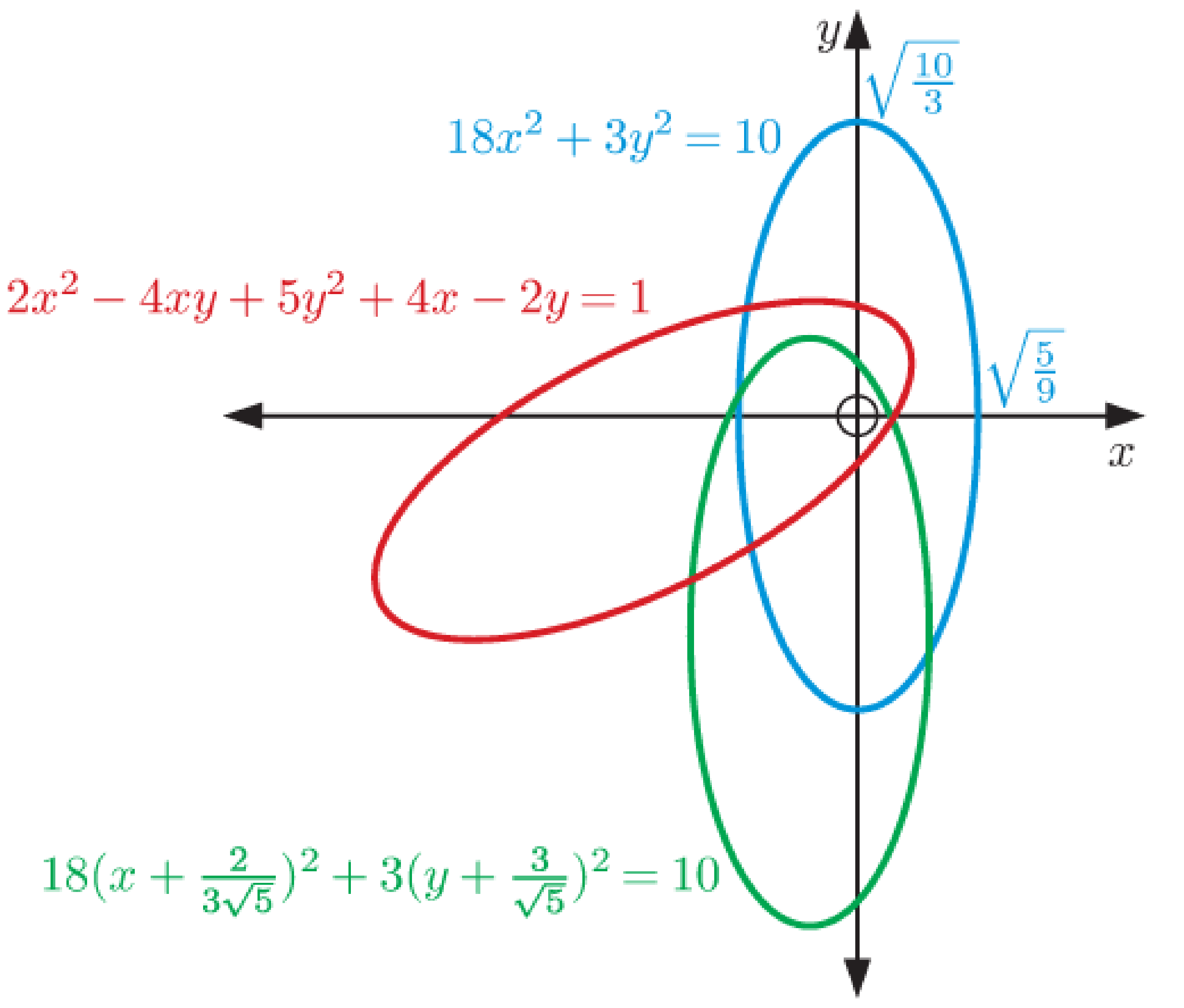
$\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$, so $\tan\theta = -2$.

Let $\mathbf{x} = \mathbf{P}\mathbf{x}'$

$$\therefore \quad (\mathbf{P}\mathbf{x}')^T\mathbf{A}(\mathbf{P}\mathbf{x}') + \begin{pmatrix} 4 & -2 \end{pmatrix}\mathbf{P}\mathbf{x}' = 1$$
$$\therefore \quad \mathbf{x}'^{\,T}\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{x}' + \begin{pmatrix} 4 & -2 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 1$$
$$\therefore \quad \mathbf{x}'^{\,T}\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}' + \frac{1}{\sqrt{5}}\begin{pmatrix} 8 & 6 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 1$$
$$\therefore \quad 6x'^{\,2} + y'^{\,2} + \tfrac{8}{\sqrt{5}}x' + \tfrac{6}{\sqrt{5}}y' = 1$$
$$\therefore \quad 6(x'^{\,2} + \tfrac{4}{3\sqrt{5}}x' + (\tfrac{2}{3\sqrt{5}})^2) + (y'^{\,2} + \tfrac{6}{\sqrt{5}}y' + (\tfrac{3}{\sqrt{5}})^2)$$
$$= 1 + 6(\tfrac{4}{45}) + \tfrac{9}{5}$$
$$\therefore \quad 6(x' + \tfrac{2}{3\sqrt{5}})^2 + (y' + \tfrac{3}{\sqrt{5}})^2 = \tfrac{10}{3}$$

which is $18X^2 + 3Y^2 = 10$ translated $\begin{pmatrix} -\frac{2}{3\sqrt{5}} \\ -\frac{3}{\sqrt{5}} \end{pmatrix}$.

$\therefore$ the original conic is an ellipse.



3 **a**

Qu.	$a+c$	λ_1 *and* λ_2	$\lambda_1+\lambda_2$	*Sign of* $\lambda_1\lambda_2$	*Type of conic*
1 a	15	5, 10	15	>0	ellipse
1 b	-5	15, -20	-5	<0	hyperbola
2 a	4	1, 3	4	>0	ellipse
2 b	-1	2, -3	-1	<0	hyperbola
2 c	-2	-6, 4	-2	<0	hyperbola
2 d	7	1, 6	7	>0	ellipse

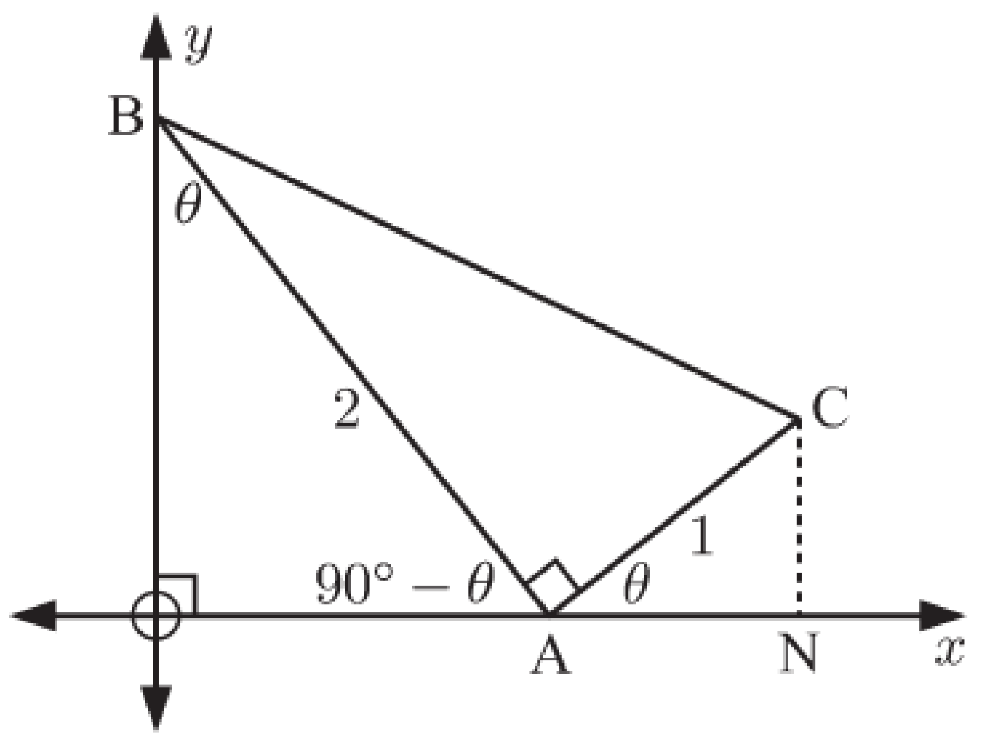
Note: In **2 a**, the eigenvalues were found from the conic $2x^2-2xy+2y^2-4x+2y-6=0$.

b **Conjectures:**

- $\lambda_1+\lambda_2=a+c$
- if $\lambda_1\lambda_2>0$ the conic is an ellipse and if $\lambda_1\lambda_2<0$ the conic is a hyperbola.

These conjectures are true, but proof is left to the reader.

4 **a**



In $\triangle$ACN,

$$\cos\theta=\frac{\text{AN}}{1}$$
$$\therefore\ \text{AN}=\cos\theta$$

and $$\sin\theta=\frac{\text{CN}}{1}$$
$$\therefore\ \text{CN}=\sin\theta$$

In $\triangle$AOB, $\sin\theta=\dfrac{\text{OA}}{2}$

$$\therefore\ \text{OA}=2\sin\theta$$

Thus $\text{ON}=\text{OA}+\text{AN}$

$$=2\sin\theta+\cos\theta$$

$\therefore$ C is at $(2\sin\theta+\cos\theta,\ \sin\theta)$.

b Let $x=2\sin\theta+\cos\theta,\quad y=\sin\theta$

$$\therefore\ x=2y+\sqrt{1-y^2}\quad\{\cos\theta=\sqrt{1-\sin^2\theta}\}$$
$$\therefore\ x-2y=\sqrt{1-y^2}$$
$$\therefore\ (x-2y)^2=1-y^2$$
$$\therefore\ x^2-4xy+4y^2+y^2=1$$
$$\therefore\ x^2-4xy+5y^2=1$$

c $x^2-4xy+5y^2=1$

$$\mathbf{A}=\begin{pmatrix}1 & -2\\ -2 & 5\end{pmatrix}$$

$$\therefore\ |\lambda\mathbf{I}-\mathbf{A}|=\begin{vmatrix}\lambda-1 & 2\\ 2 & \lambda-5\end{vmatrix}$$
$$=\lambda^2-6\lambda+1$$

If $\lambda^2-6\lambda+1=0$,

then $$\lambda=\frac{6\pm\sqrt{36-4(1)(1)}}{2}$$
$$\therefore\ \lambda=\frac{6\pm4\sqrt{2}}{2}$$
$$\therefore\ \lambda=3\pm2\sqrt{2}$$
$$\therefore\ \lambda_1=3+2\sqrt{2},\quad \lambda_2=3-2\sqrt{2}\quad\text{or vice versa}$$

$\therefore$ under a suitable rotation, $x^2-4xy+5y^2=1$ becomes either

$$(3+2\sqrt{2})x'^{\,2}+(3-2\sqrt{2})y'^{\,2}=1\quad\text{or}$$
$$(3-2\sqrt{2})x'^{\,2}+(3+2\sqrt{2})y'^{\,2}=1$$

d As $3+2\sqrt{2}>0$ and $3-2\sqrt{2}>0$, $\lambda_1\lambda_2>0$, so we have an ellipse.

Check: $a+c=1+5=6$

$$\lambda_1+\lambda_2=3+2\sqrt{2}+3-2\sqrt{2}=6\quad ✓$$

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